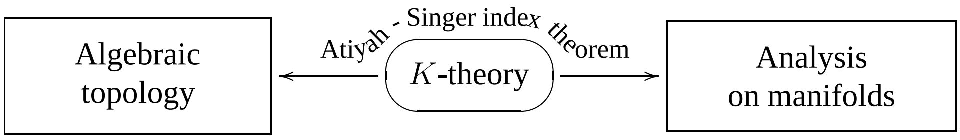
Section 2

a. State-of-the-art and objectives

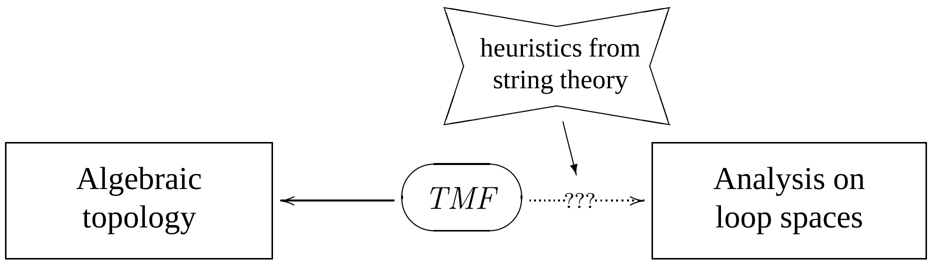
Introduction

***K-theory and TMF*:** The Atiyah Singer index theorem is a celebrated result that predicts the number of solutions of certain differential equations. Its authors got the Abel prize “for their discovery [...] bringing together topology, geometry and analysis, and their outstanding role in building new bridges between mathematics and theoretical physics”. The mathematical theory called K-theory is at the base of this result:



A recent theory strongly related to K-theory is that of topological modular forms (TMF). The latter is a notoriously difficult subject. The existence of TMF was announced in the nineties, but its construction was so intricate that the foundational papers [HMi], [HMa] never got finished. For a long time, the only available references were the background papers [La], [Wi], [Se2], [LRS]. Nevertheless, the results of Hopkins, Mahowald, and Miller attracted a lot of attention, and many astounding results were announced [AHS], [Ho], [Be]. Lurie’s novel approach to TMF [Lu1] also allowed for remarkable applications [BeL].

We expect TMF to have an impact comparable to that of K-theory and of the index theorem, but now in the context of loop spaces. Witten’s anomaly computations [Wi] and the occurrence of modular forms in string theory form strong evidence to that effect:



In short, TMF should establish a strong link between algebraic topology and analysis on loop spaces, with same impact as the index theorem.

***Overall aim*:** Unlike K-theory, the current definitions of TMF are of algebraic nature [HMi], [Lu1]. Many tried to find a geometric model of that theory [BDR], [HK], [ST1]. But even though progress has been made, none of the attempts were completely successful. Our ultimate goal is to provide such a model:

Main goal: Find the first geometric definition of TMF

An attempt towards that goal (probably due to Graeme Segal) would be to define TMF to be the $\mathit{Diff}(S^1)$-equivariant K-theory of the loop space:

$$TMF^*\big(M\big) := K^*_{\mathit{Diff}(S^1)}\big(LM\big).$$

Unfortunately, that particular variant of K-theory remains ill-defined.

Our main innovation in order to achieve the above mentioned goal is to use conformal nets. Initially developed in order to describe quantum field theory in four dimensions, conformal nets turned up most useful in the context of two dimensional conformal field theory. In the same way as Clifford algebras are used for defining K-theory, we expect conformal nets to yield a description of TMF. More specifically, the n-th power of the free fermion conformal net should correspond to the n-th Clifford algebra $\mathit{Cliff}(n)$. The structure in which conformal nets organize, a 3-category, appears to be quite remarkable by itself. Therefore, we also plan to study conformal nets for their own sake.

***Key objectives*:**

- Finish the description of the tricategory of conformal nets [§1].
- Investigate our new notion of equivalence between conformal nets [§4].
- Prove our conjecture about the equivalence of $Fer(n)$ and $Fer(n+576)$ [§5].
- Compute the action of $\nu \in \pi_3^{\text{stable}}(S^0)$ on the invertible net $Fer(1)$ [§7].

Our most ambitious goal is to find a TMF-analog of the Atiyah Singer index theorem. In other words:

- Develop a theory of analytic pushforward in TMF cohomology [§12].

To limit the risks associated with this ambitious project, we have added some side-goals, whose accomplishment are not dependent on the successful completion of other tasks. Each one of them is suitable for a PhD project:

- Study string connections, and develop Chern-Weil theory for string bundles [§9],
- Define and study conformal blocks for conformal nets [§10],
- Extend Chern-Simons theory down to points [§11].

§1. A 3-category of conformal nets

In [BDH1,2], we plan to show that conformal nets form a 3-category. The existence of such a 3-category had been conjectured by Stolz and Teichner:

Conjecture. (S. Stolz, P. Teichner) *There exists an interesting 3-category $\mathcal{C}$ such that $Hom_{\mathcal{C}}(1_{\mathcal{C}}, 1_{\mathcal{C}})$ is equivalent to the 2-category of von Neumann algebras and bimodules. Here, $\mathcal{C}$ is assumed to be symmetric monoidal, with unit object $1_{\mathcal{C}}$.*

Their hope was that a good answer to that question would help them complete their project, and provide a field theoretical definition of TMF [ST1]. The 3-categorical nature of conformal nets (and thus of conformal field theories) is an extremely interesting feature and its applications will certainly reach beyond TMF. So far, it appears to have escaped the attention of physicists. We borrow terminology from conformal field theory to name the objects, arrows, 2-morphisms and 3-morphisms of our 3-category:

The 3-category $CN3$	
Objects	Conformal nets
Arrows $\mathcal{A} \to \mathcal{B}$	Defects between the nets $\mathcal{A}$ and $\mathcal{B}$
Arrows from $\mathcal{A}$ to the unit object 1	Boundary conditions for the net $\mathcal{A}$
2-morphisms between arrows $\mathcal{A} \underset{\to}{\overset{\to}{\Downarrow}} \mathcal{B}$	Sectors between $\mathcal{A}$-$\mathcal{B}$-defects
2-morphisms from $1_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ to itself	Sectors of the net $\mathcal{A}$
3-morphisms between 2-morphisms	Homomorphisms of sectors

There are many formalisms for doing conformal field theory aside from conformal nets: vertex algebras [FBZ], chiral algebras [BD], algebras over (partial) operads [Se1], [Hu], etc.

Some authors have already considered 2-categories as an appropriate framework for studying conformal field theories [FRS]. But we would like to emphasize that replacing 2-categories by 3-categories is much more than a change of terminology. To our knowledge, the only formalism that exhibits this 3-categorical nature is that of conformal nets. Exporting our ideas to other areas of conformal field theory could be very exciting and will be the subject of future research projects.

The definition. For people who haven't heard of conformal nets, we include a definition. We refer the reader to [Ka] for a short survey article, and to [Lon] for a more extensive treatment. Here, H denotes a Hilbert space, and $B(H)$ its algebra of bounded operators.

Definition. *A conformal net $\mathcal{A}$ consists of a Hilbert space H, a projective representation u of $Diff(S^1)$ on H, a vector $\Omega \in H$, and an assignment*

$$\mathcal{A} : \big\{\text{subintervals of } S^1\big\} \;\longrightarrow\; \big\{\text{subalgebras of } B(H)\big\}.$$

These are subject to the following axioms:
- The $\mathcal{A}(I)$ are von Neumann algebras, i.e. closed in the weak topology.
- If $I \subset J$, then $\mathcal{A}(I) \subset \mathcal{A}(J)$.
- If I is the complement of J, then $\mathcal{A}(I)$ is the commutant of $\mathcal{A}(J)$.
- The algebras $\mathcal{A}(I)$ generate $B(H)$.
- The representation of $Diff(S^1)$ on H is of positive energy.
- Ω is invariant under the action of $SL_2(\mathbb{R}) \subset Diff(S^1)$.
- Given $\varphi \in Diff(S^1)$, then one has $u(\varphi)\mathcal{A}(I)u(\varphi)^{-1} = \mathcal{A}(\varphi(I))$.
- If φ fixes I pointwise, then $\mathrm{Ad}(u(\varphi))$ fixes $\mathcal{A}(I)$ pointwise.

§2. A "coordinate free" approach to conformal nets

In its usual definitions [GF], [Ka], [Lon], a conformal net is given by a Hilbert space H, and an assignment

$$\mathcal{A} : \{\text{subintervals of } S^1\} \longrightarrow \{\text{subalgebras of } B(H)\}.$$

But in order to define the tricategory $\mathit{CN3}$ of conformal nets, we find it is useful to take a more coordinate free point of view. Such an approach was already suggested in the context of quantum field theory on curved 4-dimensional space time [BFV].

Here, by 'coordinate free', we mean that instead of concentrating on subintervals of S^1, one should take $\mathcal{A}$ to be a functor defined on the following larger category:

Alternative definition (Sketch) *A conformal net is a functor*

$$\mathcal{A} : \{\text{1-manifolds}\} \longrightarrow \{\text{von Neumann algebras}\}$$

from the category of all compact one dimensional manifolds (possibly with boundary) and embeddings to the category of all von Neumann algebras, subject to certain axioms.

Note that the Hilbert space H and the chosen vector Ω have now disappeared from the definition. We know that a conformal net (as defined in [§1]) induces a coordinate free conformal net (as defined above). However, we still wonder under which circumstances the converse can be made to hold.

§3. A close analogy between K-theory and TMF

Both in the homotopy theoretical approach [HMi] and in the field theoretic approach [ST1], there is a strong parallel between real K-theory (KO) and of TMF. Therefore, we also expect such an analogy in our analytic context.

One of the most important ingredient in the definition of analytic KO-theory is provided by the Clifford algebras $\mathit{Cliff}(n)$. Finding the analogs of Clifford algebras for TMF was a long outstanding question to which we claim to have an answer: the free fermion conformal net $\mathit{Fer}(n)$. Physically, the free fermion describes n massless particles with no interaction. The associated chiral conformal field theory is undoubtedly among the simplest ones, and the same holds for the corresponding conformal net. Nevertheless, the free fermion has a lot of very interesting mathematical properties, almost identical to those of Clifford algebras:

Clifford algebra $\mathit{Cliff}(n)$	The free fermion $\mathit{Fer}(n)$
$\mathit{Cliff}(n)$ has an action of $O(n)$	$\mathit{Fer}(n)$ has an action of $O(n)$
Cliff is a multiplicative functor: $\mathit{Cliff}(V \oplus W) = \mathit{Cliff}(V) \otimes \mathit{Cliff}(W)$	Fer is a multiplicative functor: $\mathit{Fer}(V \oplus W) = \mathit{Fer}(V) \otimes \mathit{Fer}(W)$
$\mathit{Cliff}(n)$ can be used to define $\mathit{Spin}(n)$	$\mathit{Fer}(n)$ can be used to define $\mathit{String}(n)$

The use of Clifford algebras in the definition of real K-theory goes as follows:

Roughly speaking, a class in $KO^n(X)$ is represented by a bundle of $\mathit{Cliff}(n)$-modules over X. Those modules can be viewed as the elements of $\mathit{Hom}(\mathit{Cliff}(n), 1)$ in an appropriate 2-category. It is therefore natural to try to replace them with elements of $\mathit{Hom}(\mathit{Fer}(n), 1)$ in the tricategory $\mathit{CN3}$ of conformal nets. These are the boundary conditions for $\mathit{Fer}(n)$. So we can now present our first tentative definition of TMF:

Cohomology theory	KO^*	TMF^*
The cohomological degree is controlled by	The Clifford algebras $\mathit{Cliff}(n)$	The Free Fermion conformal nets $\mathit{Fer}(n)$
Cohomology classes of degree n are represented by	Bundles of $\mathit{Cliff}(n)$-modules	Bundles of $\mathit{Fer}(n)$-boundary conditions

Of course, saying that $KO^*(X)$ is given by bundles of $\mathit{Cliff}(n)$-modules is only a caricature. The actual definition involves actions of $\mathit{Cliff}(n)$ on bundles of Hilbert spaces, and fiberwise Fredholm operators. On the TMF side of the story, we expect that similar modifications will be needed. What these modifiations should be is still something that needs to be determined. But the work of Stolz and Teichner [ST2] contains rather clear indications (having to do with moduli of supersurfaces) about the direction one should be looking in.

Defects and boundary conditions are well established in the CFT literature. They have been studied in many contexts (e.g. [FRS]) and, among others, in the context of conformal nets in 1+1 dimensions [LR]. We would like to emphasize that, so far, they have never been considered for *chiral* conformal field theories (our free fermions $\mathit{Fer}(n)$ are chiral). We have thus introduced a novel, mathematically precise, definition of boundary condition, whose properties we are currently working on.

§4. A new notion of equivalence for conformal field theories

Given two rings A and B, there exist two distinct notions of equivalence: ring isomorphism and Morita equivalence. This comes from the fact that there are two different ways of making rings into a category. If we view A and B as objects of the category of rings and ring homomorphisms, we get the notion of ring isomorphism. On the other hand, viewing A and B as objects of the 2-category of rings, bimodules, and bimodule homomorphisms, we get the notion of Morita equivalence.

Similarly, the fact that conformal nets form the objects of a 3-category yields a new notion of equivalence. We call it $\mathit{CN3}$-equivalence. We present some evidence that this notion is worth wile studying. First of all, we know that if two nets $\mathcal{A}$ and $\mathcal{B}$ have different representation categories, then they cannot be $\mathit{CN3}$-equivalent: this shows that the notion of $\mathit{CN3}$-equivalence is at least non-trivial. On the other hand, the notion of $\mathit{CN3}$-equivalence is strictly weaker that the notion of isomorphism. Here is a concrete example of something that we plan to prove:

Claim. *If a conformal net $\mathcal{A}$ has a trivial representation category, then there exists another conformal net $\mathcal{B}$ such that their tensor product $\mathcal{A} \otimes \mathcal{B}$ is $CN3$-equivalent to the unit object of $CN3$.*

The above claim can be reformulated by saying that the conformal net $\mathcal{A}$ is invertible, with inverse $\mathcal{B}$. In the celebrated paper [KLM], it was shown that a conformal net $\mathcal{A}$ has trivial representation category if and only if a certain numerical invariant $\mu(\mathcal{A})$ is equal to one – the so called μ-index of $\mathcal{A}$. Thus, we could rephrase the above claim by saying that a conformal net $\mathcal{A}$ is invertible if and only if its μ-index is equal to one. We also believe that the following is true:

Claim. *If two conformal nets $\mathcal{A}$ and $\mathcal{B}$ are $CN3$-equivalent, then $\mu(\mathcal{A}) = \mu(\mathcal{B})$.*

Finding other invariants that can distinguish non-$CN3$-equivalent conformal nets with same representation category and same μ-index is a central problem on which we hope to concentrate our efforts.

At this moment, it is still difficult for us to establish that two conformal nets are $CN3$-equivalent. Here are some open questions:

Conjecture. *Let L_1, L_2 be even unimodular lattices of same rank, and let $\mathcal{A}_{L_1}$, $\mathcal{A}_{L_2}$ be the corresponding conformal nets [Stas]. Then $\mathcal{A}_{L_1}$ and $\mathcal{A}_{L_2}$ are $CN3$-equivalent.*

We would also like to know if the moonshine net (on which the monster group acts [KL]) is $CN3$-equivalent to the one associated to a unimodular lattice of rank 24.

§5. Periodicity of the free fermions

Clifford algebras over the reals exhibit an 8-fold periodicity that is intimately related to the Bott periodicity of KO-theory. Namely, there exist Morita equivalences

$$\mathit{Cliff}(n) \;\simeq\; \mathit{Cliff}(n+8)$$

for every natural number n. We expect the relationship between the free fermion net $\mathit{Fer}(n)$ and TMF to be analogous to the relationship between $\mathit{Cliff}(n)$ and KO. Since the cohomology theory TMF is 576-periodic [HMa], [Ba], it is natural for us to make the following conjecture:

Conjecture. *For every n, there exists a $CN3$-equivalence between the conformal nets $\mathit{Fer}(n)$ and $\mathit{Fer}(n+576)$.*

This conjecture is quite mysterious, even from the point of view of physics. The theoretical physicists to which we showed the conjecture had no idea of where the number 576 should come from. If true, this conjecture would be a rare instance of a situation where mathematicians tell theoretical physicists something that they didn't already know.

§6. Symmetric monoidal 3-categories

One major difficulty in answering Stolz and Teichner's question is that the very notion of symmetric monoidal 3-category has never been described in its entirety. The existing definition of 3-category [GPS] is already quite bulky, and adding the words 'symmetric monoidal' only makes things worse. Very probably, the situation is about to change with the groundbreaking work of Jacob Lurie [Lu2]. But even like this, checking that $CN3$ satisfies all the axioms appears to be very difficult.

So instead of working with fully weak symmetric monoidal 3-categories, we have decided to describe the structure $CN3$ best fits in. It is a notion of symmetric monoidal 3-category in which some of the coherences are made strict, analogous to Shulman's framed bicategories [Shu]. That notion will be the main subject of our paper [BDH1].

One further goal, which could be a collaboration with P. Teichner's team in Bonn, is to finish the project started in [ST1], and use $CN3$ to construct TMF. But for that purpose, we would also need to understand fully weak symmetric monoidal 3-functors, a notion that we have not developed yet. Our current plan is to establish a comparison theorem that would connect our notion to the one sketched in [Lu2].

§7. The free fermion and the 3rd stable homotopy group of the sphere

Given a symmetric monoidal 3-category C, an object $\mathcal{A}$ is called invertible if there exists another object $\mathcal{B}$, such that $\mathcal{A} \otimes \mathcal{B}$ is equivalent to the unit object $1 \in C$. Let us denote by $C^\times$ the groupoid of invertible objects, invertible arrows, invertible 2-morphisms, and invertible 3-morphisms of C. That groupoid being equipped with a symmetric monoidal tensor product, we expect its geometric realization $|C^\times|$ to have the structure of a spectrum (in the sense of stable homotopy theory). The homotopy groups of that spectrum should then be given by:

$\pi_0\big(|C^\times|\big) =$ equivalence classes of invertible objects of C,
$\pi_1\big(|C^\times|\big) =$ equivalence classes of invertible arrows from 1 to itself,
$\pi_2\big(|C^\times|\big) =$ equiv. classes of invertible 2-morphisms from the trivial arrow to itself,
$\pi_3\big(|C^\times|\big) =$ invertible 3-morphisms from the trivial 2-morphism to itself,
$\pi_n\big(|C^\times|\big) = 0 \quad \text{for} \quad n \geq 4.$

Let $\mathbb{S}$ be the sphere spectrum, and recall that any spectrum is an $\mathbb{S}$-module. The homotopy groups of $|C^\times|$ are thus a module over $\pi_*(\mathbb{S})$, the ring of stable homotopy groups of spheres. The most interesting structure that this action provides is a map

$$\nu : \pi_0\big(|C^\times|\big) \longrightarrow \pi_3\big(|C^\times|\big),$$

given by the action of the generator ν of $\pi_3(\mathbb{S}) = \mathbb{Z}/24$.

Let us now specialize to the case $C = CN3$. By chance, it turns out that the third homotopy group of $|CN3^\times|$ is easy to describe, namely, $\pi_3(|CN3^\times|) = S^1$. So we get a map

$$\nu : \{\text{Invertible conformal nets}\} \to S^1.$$

Moreover, since the class ν has order 24 in the abelian group $\pi_3(\mathbb{S}) = \mathbb{Z}/24$, the above map necessarily lands in the set of 24^{th} roots of unity.

One test of whether $CN3$ is a good answer to Stolz and Teichner's question is whether it contains enough interesting invertible objects. More precisely, whether there exists an invertible net $\mathcal{A}$ such that $\nu(\mathcal{A})$ is a primitive 24^{th} roots of unity. We believe that the free fermion net $Fer(1)$ satisfies that condition.

Claim. *The image of $Fer(1)$ under the map ν is a primitive 24^{th} root of unity.*

At this moment, we certainly don't know which primitive root of unity $\nu\big(Fer(1)\big)$ is. To answer that question, one would need to better understand the geometry of the free fermions, and how they behave in bundles.

§8. Geometric string structures

As our first concrete application of the free fermions, we plan to construct an explicit, geometric model of the String group [DH].

Recall that the String group $String(n)$ is the 3-connected cover of $O(n)$. More generally, starting from the orthogonal group $O(n)$, one encounters the following groups by inductively killing their lowest homotopy group:

$$O(n) \underset{\substack{\uparrow \\ \text{kill } \pi_0}}{\longleftarrow} SO(n) \underset{\substack{\uparrow \\ \text{kill } \pi_1}}{\longleftarrow} Spin(n) \underset{\substack{\uparrow \\ \text{kill } \pi_3}}{\longleftarrow} String(n).$$

The first three are Lie groups while the last one is a topological group, well defined up to homotopy. The String group can been realized in many different ways, and various models have already appeared in the literature [ST1], [BCSS]. But rather than studying the topological groups, it is better to focus on the corresponding structures on manifolds: orientability, spin structures, and string structures.

These structures on manifolds play important roles for the homology theories $H(\text{-},\mathbb{R})$, KO, and TMF respectively. Namely, whereas all manifolds have a fundamental class in mod-two homology, only those which are oriented have a fundamental class in $H(\text{-},\mathbb{R})$. Similarly, only the manifolds that are spin have fundamental classes in KO, and only the manifolds that are string have fundamental classes in TMF [AHR]. Thus, given the intimate connection between string structures and TMF, it is important to have a good geometric understanding of the former.

As mentioned above, the group $String(n)$ has already been constructed in many different ways. But finding a notion of string structure that is suitable for doing analysis remains a non-trivial task. Before explaining our proposed answer, we recall some background on spin structures.

If V is an n dimensional vector space equipped with an inner product, then one can consider its Clifford algebra $Cliff(V)$. A good geometric model for a spin structure on V

is then provided by the choice of an invertible $\mathit{Cliff}(V)$-$\mathit{Cliff}(n)$ bimodule i.e., a Morita equivalence between those algebras. Similarly, we expect a string structure on V to be encoded by an invertible $\mathit{Fer}(V)$-$\mathit{Fer}(n)$ defect, namely, a *CN3*-equivalence between $\mathit{Fer}(V)$ and $\mathit{Fer}(n)$. Showing that this is indeed the case is the goal of our paper [DH].

§9. Connections on string bundles

The analytic construction of pushforwards in KO-theory makes crucial use of connections on spin bundles. To construct an analytic pushforward in TMF-theory, one therefore expects connections on string bundles to be necessary.

One of the existing models of the string group is a Lie 2-group [BCSS]. This is a 2-categorical analog of a Lie group. Vector bundles with string structures therefore also inherit this 2-categorical feature. For example, one can describe a string bundle as a pair $(P, \mathcal{G})$ where P is a $Spin(n)$ principal bundle, and $\mathcal{G}$ is a gerbe on P. Using the above formalism, Waldorf introduced a satisfactory notion of connection on string bundles [Wal]. However, Waldorf's notion is specific to the 2-group model of the string group. Adapting his ideas to the notion of string structure defined in [DH] should be an interesting and non-trivial task. This would require combining the theory of von Neumann algebras (which are the building blocks of conformal nets) with differential geometry, two subjects that don't look very compatible, at least at first glance.

As a further step, we could hope to develop Chern-Weil theory for string bundles, and connect it to the 2-Lie algebra [BC] [He] associated to the string group.

§10. Conformal blocks for conformal nets

Quoting [KL], there are "two mathematically rigorous approaches to study chiral conformal field theory using infinite dimensional algebraic systems. One is algebraic quantum field theory where we study [conformal nets], and the other is theory of vertex operator algebras". Given a vertex algebra, one gets bundles of conformal blocks over the moduli spaces of Riemann surfaces [FBZ]. But a similar construction for conformal nets is missing at the moment. As an application of our coordinate free approach to conformal nets, we present a sketch of definition for the conformal blocks.

Given a Riemann surface, one first picks a cellular decomposition. The edges being 1-manifolds, one can apply the functor $\mathcal{A}$ to them. So to each edge e, one can associate a von Neumann algebra $\mathcal{A}(e)$. To every face F one then associates a Hilbert space H_F. If e is an edge between F and F', there are left and right actions of the algebra $\mathcal{A}(e)$ on H_F and $H_{F'}$. It therefore makes sense to take the Connes fusion of $H_F \boxtimes_{\mathcal{A}(e)} H_{F'}$. The space of conformal blocks should then be the "total fusion" of all the Hilbert spaces H_F over all the algebras $\mathcal{A}(e)$.

Unfortunately, the definition of Connes fusion is technical, and it is not obvious how to define the total Connes fusion. We are confident that this approach can be made to work.

§11. Extending Chern-Simons down to points

Let G be a compact Lie group. In its current formulation [Tu], quantum Chern-Simons theory for G is a 1+1+1 dimensional topological field theory. In other words, it assigns algebraic objects to closed one-dimensional manifolds, to two-dimensional manifolds with boundary, and to three-manifolds with corners of codimension two. Refining the definition to a 0+1+1+1 dimensional theory is an interesting and difficult problem. Roughly speaking, it would require having algebraic objects associated to points, intervals, surfaces with corners, and to 3-manifolds with codimension 3 corners. Until recently [Fr], the guess was that the 0+1+1+1 dimensional version of Chern-Simons theory would look roughly as follows:

0-manifolds	$\mapsto$	$\mathbb{C}$-linear 2-categories
1-manifolds	$\mapsto$	$\mathbb{C}$-linear categories
2-manifolds	$\mapsto$	vector spaces
3-manifolds	$\mapsto$	numbers

In other words, it should be a functor $Bord_3 \to 2Cat_{\mathbb{C}}$, from the 3-category $Bord_3$ of zero-, one-, two-, and three-dimensional manifolds (with appropriate extra structure) into the 3-category $2Cat_{\mathbb{C}}$ of $\mathbb{C}$-linear 2-categories.

In their recent preprint [FHLT], Freed, Hopkins, Lurie, Teleman have made significant progress. They replaced $2Cat_{\mathbb{C}}$ with another 3-category, whose objects are tensor categories equipped with extra structure (a central action of a braided category). By using the main result of [Lu2], they could then extend Chern Simons theory down to points in the special case when G is a torus.

But if G is non-abelian, the 3-category used in [FHLT] doesn't seem to be powerful enough. $CN3$ however, seems perfectly suited for the problem. Namely, applying the result of [Lu2] to the loop group conformal nets of [Was] and [GF] gives us for free a 0+1+1+1 dimensional topological field theory. In other words, it provides a functor $Bord_3 \to CN3$, from the 3-category $Bord_3$ of framed 0-, 1-, 2-, and 3-manifolds into the 3-category $CN3$ of conformal nets:

0-manifolds	$\mapsto$	conformal nets
closed 1-manifolds	$\mapsto$	von Neumann algebras
1-manifolds with ∂_{in} and ∂_{out}	$\mapsto$	defects between conformal nets
closed 2-manifolds	$\mapsto$	Hilbert spaces
2-manifolds with ∂_{in} and ∂_{out}	$\mapsto$	bimodules over vN algebras
2-manifolds with corners	$\mapsto$	2-morphisms in the tricategory $CN3$
closed 3-manifolds	$\mapsto$	complex numbers
3-manifolds with ∂_{in} and ∂_{out}	$\mapsto$	maps between Hilbert spaces
3-manifolds with codim 2 corners	$\mapsto$	maps between bimodules
3-manifolds with codim 3 corners	$\mapsto$	3-morphisms in the tricategory $CN3$

The remaining non-trivial question is to identify the above with Chern-Simons theory.

§12. Pushforward along fibrations in TMF cohomology

This subject is by far the most ambitious of all our goals.

We use the analogy with KO-theory do describe the situation that we expect to find. Given a map $f : N \to M$ between spin manifolds, there are two cases in which one can define a pushforward in KO-theory: *a*) If f is an immersion, and *b*) if f is a submersion. In the first case, the construction is geometric and doesn't require to leave the realm of finite dimensional vector bundles. On the other hand, the construction of pushforwards for submersions uses fiberwise Dirac operators, and is thus of analytic nature. The compatibility between those two pushforwards is then given by the family version of the Atiyah Singer index theorem.

Similarly to the case of KO, we expect TMF-pushforwards along immersions to be much simpler than pushforwards along fibrations. The latter should be somehow related to taking the $Diff(S^1)$-equivariant Dirac operator of the fiberwise free loop space [Stac].

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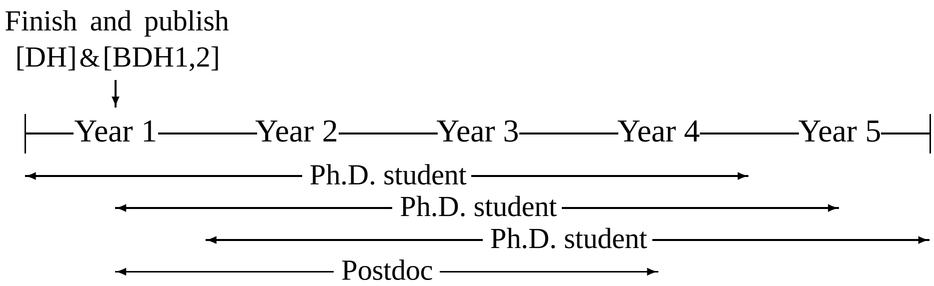
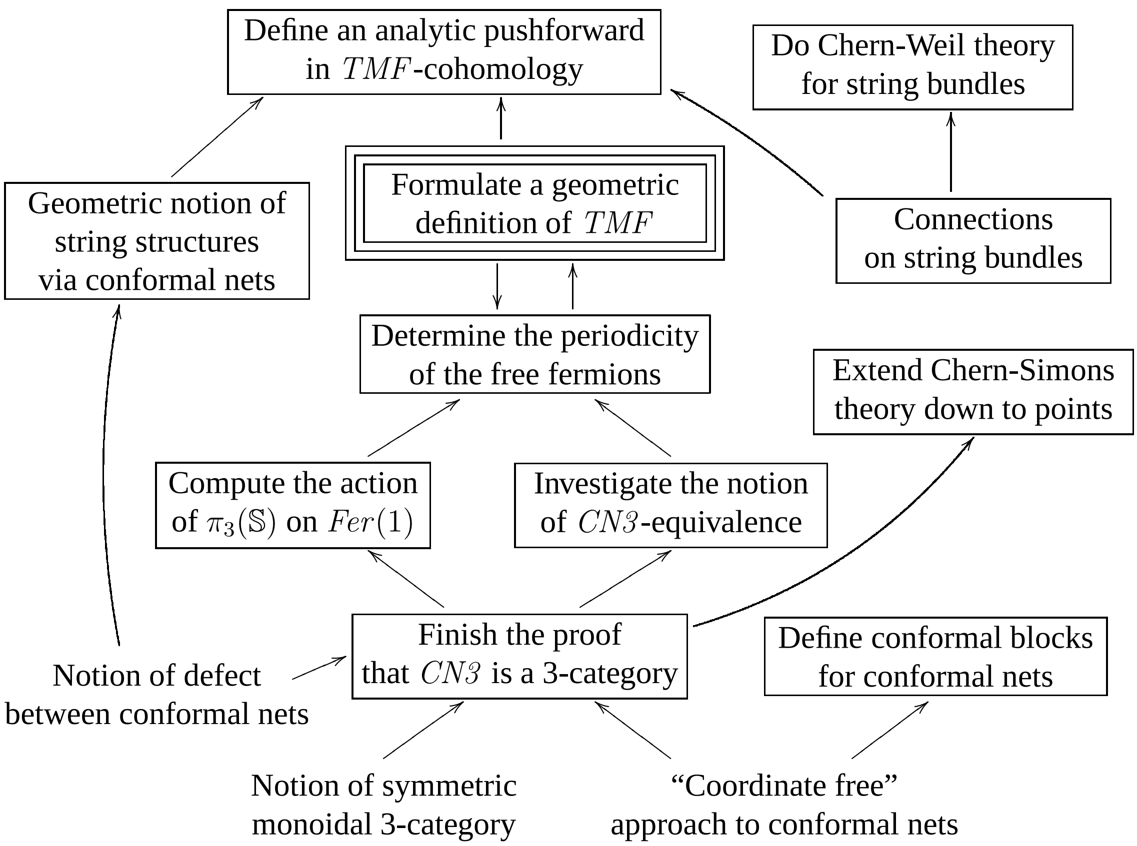
ETHICAL ISSUES TABLE	
Does this project involve human embryos or fetuses ?	No
Does this project involve research on humans?	No
Can this project present any privacy issues?	No
Does this project involve research on animals?	No
Does this project involve developing countries?	No
Can this project be of any military/terrorist use?	No
Are there other ethical issues related to this project?	No

b. Methodology

In the very beginning of the project, I will continue my projects with C. Douglas and A. Bartels, and finish [BDH1,2] and [DH]. These will form the basis on which other people will do their work. I then plan to hire three students and one postdoc, as shown on the following timetable:



The big diagram below displays our various intermediate goals and their interdependences. The boxed items represent our various objectives, listed hierarchically, while an arrow indicates the logical dependence of one project on another one. The central box states our main objective: to develop a geometric definition of TMF using our notion of defect between conformal nets [§3]:



Let us emphasize that, even though they are not published yet, the notions of symmetric monoidal 3-category [§6], of defect between conformal nets [§1], and the coordinate free approach to conformal nets [§2] have been established by us, and constitute a solid ground for further investigations. The first two are posted on my webpage [DH], while the third should appear very soon in [BDH1].

c. Resources

In total, I would like to hire one postdoc and three PhD students. The postdoc should be well acquainted with conformal nets, and, for example, could help investigate the notion of $CN3$-equivalence [§4].

For the three PhD students, I have defined some side-goals whose accomplishment is not dependent on the completion of other intermediate projects: Chern-Weil theory for connections on string bundles [§9], higher genus conformal blocks for conformal net [§10], and extendend Chern-Simons theory [§11]. These should be suitable as PhD projects: they are not too difficult, but will nevertheless constitute very nice results on their own.

Here is an estimation of the overall costs of this proposal:

	Cost category	Year 1	Year 2	Year 3	Year 4	Year 5	Total
Direct Costs	Personnel						
	PI: A. Henriques (1,0 fte)	67373	70690	74103	77612	81087	370866
	Post doc	27797	57770	60956	31558		178080
	PhD student	34034	40448	43195	46147		163824
	PhD student	17017	37241	41821	44671	23509	164259
	PhD student		36241	43050	45962	49085	174339
	Total Personnel	146221	242390	263126	245950	153681	1051368
	Other Direct Costs						
	Comput., software, books	5000	2500	1500	1500	1500	12000
	Travel (2500 pp per year)	7500	15000	15000	13750	6250	57500
	International guests	5000	5000	5000	5000	5000	25000
	Costs of audit certificate			2500		2500	5000
	Total Other Direct Costs	17500	22500	24000	20250	15250	99500
	Total Direct Costs	163721	264890	287126	266200	168931	1150868
Indirect Costs (overheads)		32744	52978	57425	53240	33786	230174
Total costs of project		196465	317868	344552	319440	202717	1381042
Requested grant:		**196465**	**317868**	**344552**	**319440**	**202717**	**1381042**

d. Ethical issues

None. (ethical issues table on p. 13)

Section 3

PI's Host institution.

The research described in this proposal will be carried out at the University of Utrecht. It is the leading university of the Netherlands, and its math department comprises more than 80 staff members (professors, post docs, and PhD students), across all areas of mathematics.

It should provide fertile ground for this project and we have no doubt about its ability to attract bright PhD students and highly qualified post docs. Moreover, the topic of this proposal overlaps with the interests of many Utrecht professors:

• The subject of K-theory [§3] is well represented in Utrecht. W. van der Kallen, J. Stienstra, and J. Strooker have written many articles of the subject ([KS], [KMS], [Str], [SV] to cite just a few).

• Categorical structures is among the specialties of I. Moerdijk [BM] and he has shown much interest for our notion of symmetric monoidal tricategory [§6].

• The string group [§8] was the main motivation for my paper on L_∞-algebras, which was a direct continuation of M. Crainic's work on Lie algebroids [CF].

• E. Looijenga has spent time investigating the properties of conformal blocks [Loo]. A definition of conformal blocks via conformal nets [§10] would nicely complement the work of his Ph.D. student [BoL].

• G. Cornelissen is a number theorist who is well acquainted with modular forms and related mathematical objects [CL], [C].

• U. Schreiber is a recently hired post doc who did extensive work on the applications of higher categorical structures in theoretical physics: [SSW], [Sch], [BCSS], [BS].

• Finally, the UU also has a strong physics department, with numerous mathematical physicists. This could be crucial for the good development of our project given its interdisciplinary nature. One physicist who I might expect to get involved is S. Vandoren, whose research is on superstring theory, supergravity and supersymmetric field theory.

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