Letter on "Geometric construction of modular functors from conformal field theory" and "Construction of the Reshetikhin-Turaev TQFT from conformal field theory" by Jørgen Andersen and Kenji Ueno

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By the present letter, I wish to point out a subtlety in the definition of modular functor which appears in the papers [AU07, AU15] of Jørgen Andersen and Kenji Ueno. This affects the main result of [AU15] because the authors assume (in [AU15, Def. 4.1]) that their definition is equivalent to the one in [Tur94][1], whereas this is most likely not the case. After adopting a better definition, I expect the proofs in [AU07, AU15] to survive the necessary modifications, and to remain mostly unchanged.

Andersen and Ueno attribute their definition to Kevin Walker [Wal91]. But the two definitions are actually not equivalent: one of the axioms which is implicitly present in [Wal91] has been dropped from [AU07, AU15]. I will argue that this makes the Andersen–Ueno definition too weak.

On the other hand, the definition in [Wal91] is known to be too strong (it does not allow for anti-symmetrically self-dual objects, a fact pointed out in e.g. [Wan10, Chapt. 5]). The WZW modular functors $\mathcal{V}_{N,K}^{\dagger}$ constructed in [AU07][2] are therefore *not* modular functors in the sense of [Wal91]. The problem is therefore quite subtle.

I propose an alternative definition of modular functor that lies *strictly in between* those of Andersen–Ueno and of Walker, but which retains the same spirit. It is obtained by replacing the axiom (1) of [Wal91] by the modified axiom (5). It is very likely that, after including that modified axiom, the notion becomes equivalent to the definition in [BK01, Def. 5.7.10].[3] If this is true, then

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[1] After recieving Andersen and Ueno's answer, I wish to slightly modify my claim. The thing which they wrongly assume is that there exists a well-defined construction

$$\left\{ \text{Modular tensor categories} \right\} \xrightarrow{\substack{\text{"Reshetikhin−Turaev} \\ \text{construction"}}} \left\{ \text{Andersen-Ueno modular functors} \right\}.$$

[2] The notation $\mathcal{V}_{N,K}^{\dagger}$ is from [AU15].

this would lead to a sequence of three *inequivalent* notions:

$$\boxed{\begin{array}{c}\text{Andersen–Ueno}\\\text{modular functor}\end{array}} < \boxed{\begin{array}{c}\text{Bakalov–Kirillov}\\\text{modular functor}\end{array}} < \boxed{\begin{array}{c}\text{Walker}\\\text{modular functor}\end{array}} .$$

In my opinion, the better notion is the one in the middle.

The subtlety is in the formulation of the gluing axiom, which I reproduce here for the convenience of the reader:

MF2. Glueing axiom: Let Σ and Σ_c be marked surfaces such that Σ_c is obtained from Σ by glueing at an ordered pair of points and projective tangent vectors with respect to the glueing map c. Then there is an isomorphism

$$V(\Sigma_c, \lambda) \cong \bigoplus_{\mu \in \Lambda} V(\Sigma, \mu, \mu^\dagger, \lambda)$$

which is associative, compatible with glueing of morphisms, disjoint unions and is independent of the choice of glueing map.

Let us write $g_{p,q}$ for the gluing isomorphism

$$g_{p,q} : V(\Sigma_c, \lambda) \to \bigoplus_{\mu \in \Lambda} V(\Sigma, \mu, \mu^\dagger, \lambda),$$

where p and q are the points (with projective tangent vectors) of Σ that are glued together.

In [AU07, AU15], the gluing isomorphism depends on an *ordered* pair of points, and no relation is imposed between $g_{p,q}$ and $g_{q,p}$. On the contrary, in [Wal91], the gluing isomorphism depends on an *unordered* pair. Reformulated in terms of ordered pairs, this is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc}
 & \bigoplus_\mu V(\Sigma, \mu, \mu^\dagger, \lambda) & \\
\nearrow^{g_{p,q}} & \downarrow \text{\scriptsize permute the direct summands} & \\
V(\Sigma^c, \lambda) & & \quad(1) \\
\searrow_{g_{q,p}} & \bigoplus_\mu V(\Sigma, \mu^\dagger, \mu, \lambda) &
\end{array}$$

This axiom was not included in [AU07], and for a good reason: the diagram (1) does *not* commute for the WZW modular functors.

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[3]I an earlier version of this document, I had suggested that my modification of Walker's definition might be equivalent to Turaev's weak rational modular functor [Tur94, Chapt. V]. It was pointed out to me by Chris Schommer-Pries that the proof of such an equivalence is likely to be a very difficult: roughly as hard as the claim that for a semisimple category with finitely many simples weakly rigid [BK01, §5.3] implies rigid. The latter is a well known open problem in the theory of fusion categories. Turaev imposes rigidity by an extra axiom (axiom 1.5.8 on page 245 of his book), but no such condition is present in the other definitions of modular functors.

Even if we were to accept the definition in [AU07] as our working definition (which I do not recommend), there would still be a problem. Recall that one of the ingredients that enters the construction [AU07] of the WZW modular functors $\mathcal{V}^\dagger_{N,K}$ is the choice of non-degenerate $\mathfrak{g}$-invariant pairings

$$(\ , \ )_\mu : V_\mu \otimes V_{\mu^\dagger} \to \mathbb{C}. \tag{2}$$

Such pairings are only well defined up to a scalar and, as we will see later, the modular functor $\mathcal{V}^\dagger_{N,K}$ *depends* on these normalizations. To show that the dependence is genuine, we first need some invariants of modular functors: the failure of the diagram (1) to commute will provide the invariants that we need. Specifically, given a modular functor as defined in [AU07, AU15], and marked surfaces Σ and Σ_c as above, one may consider the following self-map

$$V(\Sigma, \mu, \mu^\dagger, \lambda) \hookrightarrow \bigoplus_{\mu \in \Lambda} V(\Sigma, \mu, \mu^\dagger, \lambda) \xrightarrow{g^{-1}_{p,q}} V(\Sigma_c, \lambda)$$
$$\xrightarrow{g_{q,R}} \bigoplus_{\mu \in \Lambda} V(\Sigma, \mu, \mu^\dagger, \lambda) \twoheadrightarrow V(\Sigma, \mu, \mu^\dagger, \lambda) \tag{3}$$

of the vector spaces $V(\Sigma, \mu, \mu^\dagger, \lambda)$. The eigenvalues of those self-maps are invariants of the modular functor.

Now, as far as I can tell, no restriction is ever imposed in [AU07] on the choice of invariant pairings (2). In particular, no relation is postulated between $(\ , \ )_\mu$ and $(\ , \ )_{\mu^\dagger}$. By rescaling $(\ , \ )_\mu$ while keeping $(\ , \ )_{\mu^\dagger}$ fixed, one can change the maps (3) and in particular their eigenvalues. The Andersen–Ueno modular functors $\mathcal{V}^\dagger_{N,K}$ are therefore not uniquely defined: by varying the choice of the invariant pairings, one gets a continuum of non-isomorphic modular functors. This is a problem for the construction (we would like it to be uniquely defined!), but it is mostly a problem for the notion of modular functor used in [AU07]: modular functors are not supposed to admit continuous deformations.

This is also raises a philosophical problem about the validity of the main theorem of Andersen and Ueno. According to [AU15, Thm. 1.1], the WZW modular functor $\mathcal{V}^\dagger_{N,K}$ is equivalent to some other, combinatorially defined, modular functor $\mathcal{V}^{SU(N)}_K$. But if the WZW modular functor is not a unique thing... then how could "it" be (how could they all be) isomorphic to $\mathcal{V}^{SU(N)}_K$? We will came back to this issue towards the end of this note.

Let us first examine the question of whether there is any reasonable way of picking invariant pairings (2), so as to make the modular functors (in the sense of [AU07]) $\mathcal{V}^\dagger_{N,K}$ well defined. As we have seen, the problem is that we may rescale $(\ , \ )_\mu$ while keeping $(\ , \ )_{\mu^\dagger}$ fixed. So we must impose a condition that determines one of them in terms of the other. One might try the following:

$$(\ , \ )_\mu = (\ , \ )_{\mu^\dagger} \circ \mathrm{flip}. \tag{4}$$

This is however not an option, as there exist Lie algebra representations for which this equation cannot be satisfied (the simplest one being the two-dimensional

representation of $\mathfrak{su}(2)$). One possibility is to only impose the relation (4) when $\mu \neq \mu^\dagger$, and to not impose anything when $\mu = \mu^\dagger$. This would make $\mathcal{V}^\dagger_{N,K}$ uniquely defined, but is arguably rather artificial. In the next section, I will present a better solution, which treats self-dual and non-self-dual objects on an equal footing.

I should emphasize that the issue in [AU07] is really just with their *definition* of modular functor, and that once this is fixed the modifications to the construction of $\mathcal{V}^\dagger_{N,K}$ will be minimal.

The trouble with the Andersen–Ueno definition of modular functor is that there is no axiom that relates the two gluing maps $g_{p,q}$ and $g_{q,p}$. As mentioned already, adding the axiom (1) does not work, because the WZW modular functors will not satisfy this (this can be traced back to the fact that invariant pairings do not satisfy (4), and was presumably the reason which led Andersen and Ueno to drop that axiom, i.e., to phrase their definition in terms of *ordered* pairs).

I will present a modification of the definitions of Andersen–Ueno and of Walker which addresses the problem, and in which the axiom (1) gets replaced by the weaker axiom (5). However, in order to formulate that axiom, we will first need to replace the *set* of labels Λ by a *category* $\mathcal{C}$. In order to stay as close as possible to the spirit of Walker's definition, one use the following notion:

Definition 1 *A* category of simple objects *is a linear category in which every object is simple (its endomorphism algebra is $\mathbb{C}$) and every non-zero map is an isomorphism.*

Next, we need a notion of involution on such categories:

Definition 2 *An* involution *on a category of simple objects $\mathcal{C}$ is a linear contravariant functor $* : \mathcal{C} \to \mathcal{C}$ that squares to the identity, where "squaring to the identity" may be interpreted in any of the following two ways:*

- *if one requires that the equation $x^{**} = x$ holds on the nose (for both objects and morphisms), then one gets the notion of a strict involution.*

- *if one requires the data of a natural isomorphisms $\varphi_x : x \to x^{**}$, subject to the coherence $\varphi^*_x = \varphi^{-1}_{x^*}$, then this is called a weak involution.*

Weak involutions can be strictified, and, for all practical purposes, it does not matter whether one works with strict or with weak involutions.

One may restrict to skeletal categories if one so wishes. However, it is usually impossible to simultaneously render a category skeletal and its involution strict.

Let us abbreviate 'category of simple objects with involution' by cosowi. The relevant cosowi for the case of the WZW modular functor is the category $IRep(\mathfrak{g})$ of irreducible representations of the simple Lie algebra $\mathfrak{g}$, equipped with the involution that sends a representation V to its dual $V^* := \mathrm{Hom}(V, \mathbb{C})$ (or

rather, to be precise, the variant thereof where highest weights are required to lie in some appropriately scaled Weyl alcove).

There is an obvious construction that takes as input a set with involution and produces a cosowi, but not every cosowi is of that form. Cosowi's can actually be classified: a cosowi is equivalent to the data of a set Λ (the isomorphism classes of objects), an involution $\dagger$ on Λ (induced by the involution on $\mathcal{C}$), and a partition of $\Lambda^{\text{self-dual}} := \{\lambda : \lambda = \lambda^\dagger\}$ into two pieces:

$$\Lambda^{\text{self-dual}} = \Lambda^{\text{symmetrically self-dual}} \cup \Lambda^{\text{anti-symmetrically self-dual}}.$$

An object x is called symmetrically self-dual if $x \simeq x^*$ and the involution acts trivially on the one-dimensional vector space $\text{Hom}(x, x^*)$. Conversely, an object is called anti-symmetrically self-dual if $x \simeq x^*$ and the involution acts by -1 on $\text{Hom}(x, x^*)$.[4]

If one starts with a cosowi $\mathcal{C}$ instead of a set with involution, then the definition of modular functor must be modified in the following ways:

- The vector spaces $V(\Sigma, \lambda_1, \ldots, \lambda_n)$ should be declared to depend functorially and multilinearly on the $\lambda_i \in \mathcal{C}$.

- The vector space $\bigoplus_{\mu \in \Lambda} V(\Sigma, \mu, \mu^\dagger, \lambda)$ that appears in the formulation of the gluing axiom should be replaced by the quotient[5]

$$\int_\mu V(\Sigma, \mu, \mu^*, \lambda) := \bigoplus_{\mu \in Ob(\mathcal{C})} V(\Sigma, \mu, \mu^*, \lambda)/\sim,$$

  where the summands $V(\Sigma, \mu, \mu^*, \lambda)$ and $V(\Sigma, \nu, \nu^*, \lambda)$ are identified in the obvious way whenever μ and ν are isomorphic in $\mathcal{C}$.[6]

- An extra axiom should be added, that requires the following diagram to commute:

$$
\begin{array}{ccc}
& & \int_\mu V(\Sigma, \mu, \mu^*, \lambda) \\
& \nearrow^{g_{p,q}} & \downarrow{\substack{\text{re-index by } \mu^* \\ \text{instead of } \mu}} \\
V(\Sigma^c, \lambda) & & \int_\mu V(\Sigma, \mu^*, \mu^{**}, \lambda) \\
& \searrow_{g_{q,p}} & \uparrow{\int V(\Sigma, \text{id}_{\mu^*}, \varphi_\mu, \text{id}_\lambda)} \\
& & \int_\mu V(\Sigma, \mu^*, \mu, \lambda)
\end{array}
\tag{5}
$$

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[4] The question of whether a self-dual object is symmetrically or anti-symmetrically self-dual can be formulated cohomologically: the relevant cohomology group is $H^2(\mathbb{Z}/2, \mathbb{C}^\times) \cong \mathbb{Z}/2$, where the group $\mathbb{Z}/2$ acts on the coefficient group $\mathbb{C}^\times$ by $z \mapsto z^{-1}$.

[5] Note that one may keep $\bigoplus_{\mu \in Ob(\mathcal{C})} V(\Sigma, \mu, \mu^*, \lambda)$ if one takes $\mathcal{C}$ to be skeletal.

[6] Note that the map $V(\Sigma, f, (f^*)^{-1}, \text{id}_\lambda) : V(\Sigma, \mu, \mu^*, \lambda) \to V(\Sigma, \nu, \nu^*, \lambda)$ is independent of the choice of isomorphism $f : \mu \to \nu$.

It is very likely that the definition of modular functor which I sketched above is equivalent to the one in [BK01, Def. 5.7.10]: the involution which I impose on $\mathcal{C}$ (Definition 2) is equivalent to the datum of a symmetric object $R \in \mathcal{C}^{\boxtimes 2}$ as in [BK01, p.45], and my axiom (5) corresponds to the axiom "symmetry of gluing" [BK01, p.97].

Let us return to the main theorem [AU15, Thm. 1.1], which says that the modular functors $\mathcal{V}_{N,K}^{\dagger}$ and $\mathcal{V}_{K}^{SU(N)}$ are equivalent. We have seen above that $\mathcal{V}_{N,K}^{\dagger}$ is not well defined[7], which raised the question of the validity of the proof. It turns out $\mathcal{V}_{K}^{SU(N)}$ is also not well defined!

The gap lies in the definition [AU15, Def. 4.1] of $\mathcal{V}_{K}^{SU(N)}$, where the authors assume that their notion of modular functor is equivalent to the one used in [Tur94]. The modular functor $\mathcal{V}_{K}^{SU(N)}$ (obtained by applying the Reshetikhin-Turaev construction [Tur94] to a certain the modular tensor category) is well-defined as a modular functor in the sense of [Tur94]. But it is most likely not well-defined as a modular functor in the sense of [AU15]. It is my guess that, in order to specify $\mathcal{V}_{K}^{SU(N)}$, one would need as before to pick perfect pairings between the objects μ and $\mu^{\dagger}$ of the modular tensor category, and that the resulting Andersen–Ueno modular functor $\mathcal{V}_{K}^{SU(N)}$ *depends* on those choices.

Let me finish this letter by saying that the situation is not as dramatic as I make it sound. I believe and hope that, after implementing my suggested modifications (or any other equivalent ones), both $\mathcal{V}_{K}^{SU(N)}$ and $\mathcal{V}_{N,K}^{\dagger}$ will become well-defined, and the proof that they are isomorphic will go through mostly unchanged.

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[7]Here, I use the term "not well defined" to mean that the construction depends on some unspecified choices. 'The' modular functor $\mathcal{V}_{N,K}^{\dagger}$ is therefore not one thing, but rather a multitude of things.