

0.1 $\mathbf{Bim}(M)$ is bicommutant

Let M be a von Neumann algebra. Let $\mathbf{Bim}(M)$ be the tensor category of M - M -bimodules, with monoidal product provided by Connes fusion, and let us write $\mathbf{Bim}(M)_{\text{abs}} \subset \mathbf{Bim}(M)$ for its absorbing ideal, consisting of those bimodules that come from $M \bar{\otimes} M^{\text{op}}$ -modules. Let ℓ be the left action of $\mathbf{Bim}(M)$ on $\mathbf{Bim}(M)_{\text{abs}}$, and r the right action. They are given by:

$$\ell(V) = (H \mapsto V \boxtimes_M H) \quad \text{and} \quad r(V) = (H \mapsto H \boxtimes_M V),$$

where ${}_M V_M \in \mathbf{Bim}(M)$, and ${}_M H_M \in \mathbf{Bim}(M)_{\text{abs}} = \mathbf{Mod}(M \bar{\otimes} M^{\text{op}})$. Since $\text{End}(\mathbf{Mod}(M \bar{\otimes} M^{\text{op}})) = \mathbf{Bim}(M \bar{\otimes} M^{\text{op}})$, these actions can be written as a tensor functors

$$\mathbf{Bim}(M) \xrightarrow{\ell} \mathbf{Bim}(M \bar{\otimes} M^{\text{op}}) \xleftarrow{r} \mathbf{Bim}(M)^{\text{mp}}. \quad (1)$$

The goal of this section is to prove that the two actions (1) are each other's commutant:

$$(\mathbf{Bim}(M), \ell) = (\mathbf{Bim}(M)^{\text{mp}}, r)' \quad (\mathbf{Bim}(M)^{\text{mp}}, r) = (\mathbf{Bim}(M), \ell)'. \quad (2)$$

More generally, we will prove:

Theorem 0.1. *Let M and N be non-zero von Neumann algebras. Then the obvious actions of $\mathbf{Bim}(M)$ and $\mathbf{Bim}(N)$ on $\mathbf{Mod}(M \bar{\otimes} N)$ are each other's commutants.*

Setting $N = M^{\text{op}}$ and using $\mathbf{Bim}(M)^{\text{mp}} \simeq \mathbf{Bim}(M^{\text{op}})$, we recover (2) as a special case.

Proof. Using $\text{End}(\mathbf{Mod}(M \bar{\otimes} N)) = \mathbf{Bim}(M \bar{\otimes} N)$, the actions of $\mathbf{Bim}(M)$ and $\mathbf{Bim}(N)$ on $\mathbf{Mod}(M \bar{\otimes} N)$ are given by:

$${}_M V_M \mapsto {}_M V_M \otimes {}_N L^2 N_N \quad \text{and} \quad {}_N W_N \mapsto {}_M L^2 M_M \otimes {}_N W_N.$$

The half-braiding, exhibiting those two actions as commuting with each other, is the obvious composite of interchangers and unitors:

$$\begin{aligned} & ({}_M V_M \otimes {}_N L^2 N_N) \boxtimes_{M \bar{\otimes} N} ({}_M L^2 M_M \otimes {}_N W_N) \\ & \rightarrow ({}_M V \boxtimes_M L^2 M_M) \otimes ({}_N L^2 N \boxtimes_N W_N) \rightarrow ({}_M V_M) \otimes ({}_N W_N) \\ & \rightarrow ({}_N W_N) \otimes ({}_M V_M) \rightarrow ({}_N W \boxtimes_N L^2 N_N) \otimes ({}_M L^2 M \boxtimes_M V_M) \\ & \rightarrow ({}_M L^2 M_M \otimes {}_N W_N) \boxtimes_{M \bar{\otimes} N} ({}_M V_M \otimes {}_N L^2 N_N). \end{aligned}$$

We thus get two functors

$$\mathbf{Bim}(M) \rightarrow \mathbf{Bim}(N)' \quad \text{and} \quad \mathbf{Bim}(N) \rightarrow \mathbf{Bim}(M)', \quad (3)$$

where the commutant is in $\mathbf{Bim}(M \bar{\otimes} N)$. Our goal is to show that these are equivalences of categories. We focus on the second one, as the situation with the first one is entirely symmetric.

An object of $\mathbf{Bim}(M)'$ consists of a bimodule ${}_{M\bar{\otimes}N}H_{M\bar{\otimes}N} \in \mathbf{Bim}(M\bar{\otimes}N)$, and a half-braiding with every bimodule ${}_M V_M \otimes {}_N L^2 N_N$ coming from $\mathbf{Bim}(M)$. By Lemma [[,]] such an object is equivalently determined by its half-braiding with $({}_M L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N)$. In view of the isomorphisms

$$\begin{aligned} ({}_{M\bar{\otimes}N}H_{M\bar{\otimes}N}) \boxtimes_{M\bar{\otimes}N} (({}_M L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N)) &\cong ({}_{M\bar{\otimes}N}H_N) \otimes (L^2 M_M) \\ (({}_M L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N)) \boxtimes_{M\bar{\otimes}N} ({}_{M\bar{\otimes}N}H_{M\bar{\otimes}N}) &\cong ({}_M L^2 M) \otimes ({}_N H_{M\bar{\otimes}N}), \end{aligned}$$

this is a unitary

$$c_H : ({}_{M\bar{\otimes}N}H_N) \otimes (L^2 M_M) \rightarrow ({}_M L^2 M) \otimes ({}_N H_{M\bar{\otimes}N}).$$

This unitary must satisfy:

- left $(M\bar{\otimes}N)$ -linearity: $c_H : ({}_{M\bar{\otimes}N}H_N) \otimes (L^2 M_M) \rightarrow ({}_M L^2 M) \otimes ({}_N H_{M\bar{\otimes}N})$
- right $(M\bar{\otimes}N)$ -linearity: $c_H : ({}_{M\bar{\otimes}N}H_N) \otimes (L^2 M_M) \rightarrow ({}_M L^2 M) \otimes ({}_N H_{M\bar{\otimes}N})$
- naturality w.r.t. $\text{End}({}_M L^2 M \otimes L^2 M_M) = M\bar{\otimes}M^{\text{op}}$, which upgrades c_H to a tri-module map:

$$c_H : ({}_{M\bar{\otimes}N}H_{M\bar{\otimes}N}) \otimes ({}_M L^2 M_M) \rightarrow ({}_M L^2 M_M) \otimes ({}_{M\bar{\otimes}N}H_{M\bar{\otimes}N}). \quad (4)$$

- monoidality:

$$\begin{aligned} &({}_{M\bar{\otimes}N}H_{M\bar{\otimes}N}) \boxtimes_{M\bar{\otimes}N} (({}_M L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N)) \boxtimes_{M\bar{\otimes}N} (({}_M L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N)) \\ &\quad \downarrow c_H \boxtimes \text{id} \\ &(({}_M L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N)) \boxtimes_{M\bar{\otimes}N} ({}_{M\bar{\otimes}N}H_{M\bar{\otimes}N}) \boxtimes_{M\bar{\otimes}N} (({}_M L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N)) \\ &\quad \downarrow \text{id} \boxtimes c_H \\ &(({}_M L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N)) \boxtimes_{M\bar{\otimes}N} (({}_M L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N)) \boxtimes_{M\bar{\otimes}N} ({}_{M\bar{\otimes}N}H_{M\bar{\otimes}N}) \end{aligned}$$

should be equal to

$$\begin{aligned}
& \left({}_{M\bar{\otimes}N}H_{M\bar{\otimes}N} \right) \boxtimes_{M\bar{\otimes}N} \left(({}_M L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N) \right) \boxtimes_{M\bar{\otimes}N} \left(({}_M L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N) \right) \\
& \quad \downarrow \text{id} \boxtimes \phi \\
& \left({}_{M\bar{\otimes}N}H_{M\bar{\otimes}N} \right) \boxtimes_{M\bar{\otimes}N} \left(({}_M L^2 M \otimes L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N) \right) \\
& \quad \downarrow \cong \\
& L^2 M \otimes \left[\left({}_{M\bar{\otimes}N}H_{M\bar{\otimes}N} \right) \boxtimes_{M\bar{\otimes}N} \left(({}_M L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N) \right) \right] \\
& \quad \downarrow \text{id} \otimes c_H \\
& L^2 M \otimes \left[\left(({}_M L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N) \right) \boxtimes_{M\bar{\otimes}N} \left({}_{M\bar{\otimes}N}H_{M\bar{\otimes}N} \right) \right] \\
& \quad \downarrow \cong \\
& \left(({}_M L^2 M \otimes L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N) \right) \boxtimes_{M\bar{\otimes}N} \left({}_{M\bar{\otimes}N}H_{M\bar{\otimes}N} \right) \\
& \quad \downarrow \text{id} \boxtimes \phi^{-1} \\
& \left(({}_M L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N) \right) \boxtimes_{M\bar{\otimes}N} \left(({}_M L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N) \right) \boxtimes_{M\bar{\otimes}N} \left({}_{M\bar{\otimes}N}H_{M\bar{\otimes}N} \right),
\end{aligned}$$

where

$$\begin{aligned}
\phi &: \left(({}_M L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N) \right) \boxtimes_{M\bar{\otimes}N} \left(({}_M L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N) \right) \\
&\rightarrow \left(({}_M L^2 M \otimes L^2 M_M) \boxtimes_M ({}_M L^2 M \otimes L^2 M_M) \right) \otimes \left(({}_N L^2 N_N) \boxtimes_N ({}_N L^2 N_N) \right) \\
&\rightarrow ({}_M L^2 M \otimes L^2 M \otimes L^2 M_M) \otimes ({}_N L^2 N_N)
\end{aligned}$$

is the obvious composite of interchangers and unitors. After various simplifications, the above requirement is the condition that the following diagram commutes:

$$\begin{array}{ccc}
{}_{M\bar{\otimes}N}H_N \otimes L^2 M \otimes L^2 M_M & \xrightarrow{\sigma \otimes \text{id}} & L^2 M \otimes [{}_{M\bar{\otimes}N}H_N \otimes L^2 M_M] \\
\downarrow c_H \otimes \text{id} & & \downarrow \text{id} \otimes c_H \\
{}_M L^2 M \otimes {}_N H_N \otimes L^2 M_M & & \\
\downarrow \text{id} \otimes c_H & & \\
{}_M L^2 M \otimes L^2 M \otimes {}_N H_{M\bar{\otimes}N} & \xrightarrow{\sigma \otimes \text{id}} & L^2 M \otimes [{}_M L^2 M \otimes {}_N H_{M\bar{\otimes}N}]
\end{array} \tag{5}$$

(where σ denotes the switch map).

Example 0.2. For ${}_{M\bar{\otimes}N}H_{M\bar{\otimes}N} = {}_M L^2 M_M \otimes {}_N W_N \in \mathbf{Bim}(M\bar{\otimes}N)$ in the image of (3), the half-braiding (4)

$$c_H : ({}_M L^2 M \otimes {}_N W_N) \otimes (L^2 M_M) \rightarrow ({}_M L^2 M) \otimes (L^2 M_M \otimes {}_N W_N) \tag{6}$$

is given by $\xi \otimes w \otimes \eta \mapsto \xi \otimes \eta \otimes w$.

Now suppose given ${}_{M\bar{\otimes}N}H_{M\bar{\otimes}N} \in \mathbf{Bim}(M\bar{\otimes}N)$, and a unitary c_H satisfying all the conditions listed above. The Hilbert space $L^2M \otimes H \otimes L^2M$ in the middle left of (5) admits three actions $\alpha_1, \alpha_2, \alpha_3$ of $B(L^2M)$:

$$\begin{aligned} \alpha_1(x) &:= x \otimes \text{id} \otimes \text{id} & \alpha_3(x) &:= \text{id} \otimes \text{id} \otimes x \\ \alpha_2(x) &:= (c_H \otimes \text{id})(\text{id} \otimes x \otimes \text{id})(c_H^{-1} \otimes \text{id}) = (\text{id} \otimes c_H^{-1})(\text{id} \otimes x \otimes \text{id})(\text{id} \otimes c_H), \end{aligned} \quad (7)$$

where the two formulas for α_2 agree by (5). Since $\alpha_2(x) = \text{id} \otimes (c_H^{-1} \circ (x \otimes \text{id}) \circ c_H)$, the actions α_1 and α_2 commute with each other. And since $\alpha_2(x) = (c_H \circ (\text{id} \otimes x) \circ c_H^{-1}) \otimes \text{id}$, the actions α_2 and α_3 also commute with each other. So all three α_i commute with each other. This implies in particular that the image of α_2 is contained in the commutant of α_1 and α_3 , which is $B(H)$. So we may write

$$\alpha_2(x) = \text{id} \otimes \alpha(x) \otimes \text{id}. \quad (8)$$

for some action α of $B(L^2M)$ on H . Since α commutes with the N - N -bimodule structure on H (as can be seen from any of the two formulas for α_2), we may form

$${}_N V_N := \text{Hom}_{B(L^2M)}(L^2M, {}_N H_N) = L^2M \boxtimes_{B(L^2M)} ({}_N H_N). \quad (9)$$

We claim that the functor $\mathbf{Bim}(M)' \rightarrow \mathbf{Bim}(N) : ({}_{M\bar{\otimes}N}H_{M\bar{\otimes}N}, c_H) \mapsto {}_N V_N$ given by (9) is an inverse to the functor $\mathbf{Bim}(N) \rightarrow \mathbf{Bim}(M)'$ from (3).

The fact that the composite $\mathbf{Bim}(N) \rightarrow \mathbf{Bim}(M)' \rightarrow \mathbf{Bim}(N)$ is naturally isomorphic to the identity is immediate from Example 0.2.

To see that the composite $\mathbf{Bim}(M)' \rightarrow \mathbf{Bim}(N) \rightarrow \mathbf{Bim}(M)'$ is naturally isomorphic to the identity, consider $({}_{M\bar{\otimes}N}H_{M\bar{\otimes}N}, c_H) \in \mathbf{Bim}(M)'$ as above, mapping to ${}_N V_N$ as in (9), mapping to $({}_M L^2 M_M \otimes {}_N V_N, \varsigma)$ as in Example 0.2, with ς as in (6). The desired natural unitary isomorphism

$$\theta_H : {}_M L^2 M_M \otimes {}_N V_N = {}_M L^2 M_M \otimes \text{Hom}_{B(L^2M)}(L^2M, {}_N H_N) \rightarrow {}_{M\bar{\otimes}N}H_{M\bar{\otimes}N}$$

is then given by $\theta_H(\xi \otimes f) = f(\xi)$. It remains to check that θ_H intertwines the two half-braidings; by Lemma [[.]] it suffices to check that it intertwines ς and c_H . Specifically, we must show that the vectors

$$\begin{aligned} (\text{id} \otimes \theta_H)(\varsigma(\xi \otimes f \otimes \eta)) &= (\text{id} \otimes \theta_H)(\xi \otimes \eta \otimes f) = \xi \otimes f(\eta) \\ \text{and } c_H((\theta_H \otimes \text{id})(\xi \otimes f \otimes \eta)) &= c_H(f(\xi) \otimes \eta) \end{aligned}$$

agree for all $\xi, \eta \in L^2M$ and $f \in V$.

By (7) and (8), $c_H : H \otimes L^2M \rightarrow L^2M \otimes H$ is equivariant for the actions $\alpha \otimes \text{id}$ and $\text{id} \otimes \alpha$ of $B(L^2M)^{\otimes 2}$ on $H \otimes L^2M$ and on $L^2M \otimes H$. Since $\theta_H : L^2M \otimes V \rightarrow H$ is $B(L^2M)$ -equivariant, it follows that

$$(\text{id} \otimes (\sigma \circ \theta_H^{-1})) \circ c_H \circ (\theta_H \otimes \text{id}) : L^2M \otimes V \otimes L^2M \rightarrow L^2M \otimes V \otimes L^2M \quad (10)$$

is $B(L^2M)^{\otimes 2}$ -equivariant (where σ denotes the switch map). The map (10) is therefore of the form $\xi \otimes f \otimes \eta \mapsto \xi \otimes u(f) \otimes \eta$ for some unitary $u \in U(V)$. Equivalently,

$$c_H = (\text{id} \otimes (\theta_H \circ \sigma)) \circ (\text{id} \otimes u \otimes \text{id}) \circ (\theta_H^{-1} \otimes \text{id}).$$

By (5), $u^2 = u$. Hence $u = 1$. Therefore $c_H = (\text{id} \otimes (\theta_H \circ \sigma)) \circ (\theta_H^{-1} \otimes \text{id})$, and

$$c_H(f(\xi) \otimes \eta) = \xi \otimes f(\eta). \quad \square$$

0.2 $\text{Rep}(G)$ is bicommutant

Let G be a discrete group, and let NG be the von Neumann algebra generated by the left regular action of G on $\ell^2 G$. In this section, we prove that the left action of $\text{Rep}(G)$ on $\text{Rep}(NG)$ is its own commutant:

$$\text{Rep}(G) = \text{Rep}(G)',$$

where the action of $V \in \text{Rep}(G)$ on $H \in \text{Rep}(NG)$ is given by $H \mapsto V \otimes H$.

Since $\text{End}(\text{Rep}(NG)) = \text{Bim}(NG)$, the action of $\text{Rep}(G) \rightarrow \text{End}(\text{Rep}(NG))$ can be written as a tensor functor

$$\text{Rep}(G) \rightarrow \text{Bim}(NG). \quad (11)$$

The bimodule associated to $V \in \text{Rep}(G)$ (i.e. to the functor $V \otimes - : H \mapsto V \otimes H$) has underlying vector space $V \otimes \ell^2 G$, with left action $v \otimes \xi \mapsto gv \otimes g\xi$, and right action $v \otimes \xi \mapsto v \otimes \xi g$. We encode this information by writing

$$V \xrightarrow{(11)} {}_{NG}V \otimes {}_{NG}\ell^2 G {}_{NG}, \quad (12)$$

where the colours indicate the NG -bimodule structure on $V \otimes \ell^2 G$. The action coherence isomorphism

$$({}_{NG}V \otimes {}_{NG}\ell^2 G {}_{NG}) \boxtimes {}_{NG}({}_{NG}W \otimes {}_{NG}\ell^2 G {}_{NG}) \rightarrow (({}_{NG}V \otimes {}_{NG}W) \otimes {}_{NG}\ell^2 G {}_{NG}),$$

ensuring that (12) is indeed a tensor functor, is given by $(v \otimes \delta_g) \boxtimes (w \otimes \delta_h) \mapsto v \otimes gw \otimes \delta_{gh}$.

But the action of $\text{Rep}(G)$ on $\text{Rep}(NG)$ is not just a functor $\text{Rep}(G) \rightarrow \text{End}(\text{Rep}(NG))$. It's a functor

$$\begin{aligned} \text{Rep}(G) &\rightarrow \text{End}_{\text{Rep}(G)}(\text{Rep}(NG)) \\ V &\mapsto ({}_{NG}V \otimes {}_{NG}\ell^2 G {}_{NG}, (\sigma_{V,W})_{W \in \text{Rep}(G)}) \end{aligned} \quad (13)$$

with $\sigma_{V,W}$ is given by

$$\begin{aligned} ({}_{NG}V \otimes {}_{NG}\ell^2 G {}_{NG}) \boxtimes {}_{NG}({}_{NG}W \otimes {}_{NG}\ell^2 G {}_{NG}) &\rightarrow ({}_{NG}W \otimes {}_{NG}\ell^2 G {}_{NG}) \boxtimes {}_{NG}({}_{NG}V \otimes {}_{NG}\ell^2 G {}_{NG}) \\ (v \otimes \delta_g) \boxtimes (w \otimes \delta_h) &\mapsto (gw \otimes \delta_g) \boxtimes (g^{-1}v \otimes \delta_h). \end{aligned} \quad (14)$$

Our goal is to show that the functor (13) : $\text{Rep}(G) \rightarrow \text{Rep}(G)'$ is an equivalence. An object of $\text{Rep}(G)'$ consists of an NG - NG -bimodule H and a unitary half-braiding $\varsigma_{H,\bullet} = (\varsigma_{H,V})_{V \in \text{Rep}(G)}$ with every bimodule ${}_{NG}V \otimes {}_{NG}\ell^2 G_{NG}$ for $V \in \text{Rep}(G)$, natural with respect to endomorphisms coming from $\text{End}_{\text{Rep}(G)}(V)$. By Lemma [???], this is equivalent to having a half-braiding with just the bimodule

$${}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}$$

corresponding to the regular representation $\ell^2 G \in \text{Rep}(G)$. Such a half-braiding is a unitary

$$c_H := \varsigma_{H,\ell^2 G} : ({}_{NG}H_{NG}) \boxtimes_{NG} ({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \longrightarrow ({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \boxtimes_{NG} ({}_{NG}H_{NG})$$

satisfying various coherences. In view of the isomorphisms

$$({}_{NG}H_{NG}) \boxtimes_{NG} ({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \rightarrow {}_{NG}H \otimes \ell^2 G_{NG} : \xi \boxtimes (\delta_g \otimes \delta_h) \mapsto \xi g \otimes \delta_{g^{-1}h} \quad (15)$$

$$({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \boxtimes_{NG} ({}_{NG}H_{NG}) \rightarrow {}_{NG}\ell^2 G \otimes {}_{NG}H_{NG} : (\delta_g \otimes \delta_h) \boxtimes \xi \mapsto \delta_g \otimes h\xi, \quad (16)$$

this is equivalent to an NG - NG bimodule isomorphism

$$c_H : {}_{NG}H \otimes \ell^2 G_{NG} \longrightarrow {}_{NG}\ell^2 G \otimes {}_{NG}H_{NG} \quad (17)$$

again satisfying various coherences. Writing

$$c_H(\xi \otimes \delta_h) = \sum_g \delta_g \otimes c_{g,h}(\xi), \quad (18)$$

we get a matrix of maps $c_{g,h} : H \rightarrow H$ indexed by pairs of group elements. The unitarity of c_H is the claim that $\sum_h c_{g,h} c_{k,h}^* = \delta_{g=k} \text{id}_H$ and $\sum_g c_{g,h}^* c_{g,k} = \delta_{h=k} \text{id}_H$, where the sums are in the strong operator topology. A matrix of maps $(c_{g,h})$ defines a half-braiding if and only if it is unitary, and the following conditions are satisfied:

- left G -linearity: $c_H(k\xi \otimes \delta_h) = \sum_g \delta_g \otimes c_{g,h}(k\xi)$ is equal to $k c_H(\xi \otimes \delta_h) = \sum_g \delta_{kg} \otimes k c_{g,h}(\xi) = \sum_g \delta_g \otimes k c_{k^{-1}g,h}(\xi)$. Equivalently,

$$c_{g,h}(k\xi) = k c_{k^{-1}g,h}(\xi)$$

for all $g, h, k \in G$.

- right G -linearity: $c_H(\xi \otimes \delta_{hk}) = \sum_g \delta_g \otimes c_{g,hk}(\xi)$ is equal to $c_H(\xi \otimes \delta_h)k = \sum_g \delta_g \otimes c_{g,h}(\xi)k$. Equivalently,

$$c_{g,hk}(\xi) = c_{g,h}(\xi)k$$

for all $g, h, k \in G$.

- naturality for the action of $G \subset \text{End}_{\text{Rep}(G)} \ell^2 G$ on ${}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}$, thus upgrading c_H to a tri-module isomorphism

$$c_H : {}_{NG}H_{NG} \otimes {}_{NG}\ell^2 G_{NG} \longrightarrow {}_{NG}\ell^2 G_{NG} \otimes {}_{NG}H_{NG} :$$

$c_H(\xi k \otimes \delta_{k^{-1}h}) = \sum_g \delta_g \otimes c_{g,k^{-1}h}(\xi k)$ is equal to $\sum_g \delta_{gk} \otimes c_{g,h}(\xi) = \sum_g \delta_g \otimes c_{gk^{-1},h}(\xi)$. Equivalently,

$$c_{g,k^{-1}h}(\xi k) = c_{gk^{-1},h}(\xi)$$

for all $g, h, k \in G$.

- monoidality:

$$\begin{aligned} & ({}_{NG}H_{NG}) \boxtimes_{NG} ({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \boxtimes_{NG} ({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \\ & \quad \downarrow c_H \boxtimes \text{id} \\ & ({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \boxtimes_{NG} ({}_{NG}H_{NG}) \boxtimes_{NG} ({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \\ & \quad \downarrow \text{id} \boxtimes c_H \\ & ({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \boxtimes_{NG} ({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \boxtimes_{NG} ({}_{NG}H_{NG}) \end{aligned}$$

should be equal to

$$\begin{aligned} & ({}_{NG}H_{NG}) \boxtimes_{NG} ({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \boxtimes_{NG} ({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \\ & \quad \downarrow \text{id} \boxtimes \phi \\ & ({}_{NG}H_{NG}) \boxtimes_{NG} ({}_{NG}\ell^2 G \otimes \ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \\ & \quad \downarrow \cong \\ & \bigoplus_G ({}_{NG}H_{NG}) \boxtimes_{NG} ({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \\ & \quad \downarrow \bigoplus_G c_H \\ & \bigoplus_G ({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \boxtimes_{NG} ({}_{NG}H_{NG}) \\ & \quad \downarrow \cong \\ & ({}_{NG}\ell^2 G \otimes \ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \boxtimes_{NG} ({}_{NG}H_{NG}) \\ & \quad \downarrow \phi^{-1} \boxtimes \text{id} \\ & ({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \boxtimes_{NG} ({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \boxtimes_{NG} ({}_{NG}H_{NG}), \end{aligned}$$

where

$$\begin{aligned} \phi : ({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) \boxtimes_{NG} ({}_{NG}\ell^2 G \otimes {}_{NG}\ell^2 G_{NG}) & \rightarrow {}_{NG}\ell^2 G \otimes \ell^2 G \otimes {}_{NG}\ell^2 G_{NG} \\ (\delta_a \otimes \delta_b) \boxtimes (\delta_g \otimes \delta_h) & \mapsto \delta_a \otimes \delta_{a^{-1}bg} \otimes \delta_{bh}. \end{aligned} \quad (19)$$

In view of the isomorphisms

$$(\mathit{NG}H_{\mathit{NG}}) \boxtimes_{\mathit{NG}}(\mathit{NG}\ell^2G \otimes \mathit{NG}\ell^2G_{\mathit{NG}}) \boxtimes_{\mathit{NG}}(\mathit{NG}\ell^2G \otimes \mathit{NG}\ell^2G_{\mathit{NG}}) \rightarrow \mathit{NG}H \otimes \ell^2G \otimes \ell^2G_{\mathit{NG}} \\ \xi \boxtimes (\delta_a \otimes \delta_b) \boxtimes (\delta_g \otimes \delta_h) \mapsto \xi a \otimes \delta_{a^{-1}bg} \otimes \delta_{g^{-1}h} \quad (20)$$

$$(\mathit{NG}\ell^2G \otimes \mathit{NG}\ell^2G_{\mathit{NG}}) \boxtimes_{\mathit{NG}}(\mathit{NG}H_{\mathit{NG}}) \boxtimes_{\mathit{NG}}(\mathit{NG}\ell^2G \otimes \mathit{NG}\ell^2G_{\mathit{NG}}) \rightarrow \mathit{NG}\ell^2G \otimes \mathit{NG}H \otimes \ell^2G_{\mathit{NG}} \\ (\delta_a \otimes \delta_b) \boxtimes \xi \boxtimes (\delta_g \otimes \delta_h) \mapsto \delta_a \otimes b\xi g \otimes \delta_{g^{-1}h} \quad (21)$$

$$(\mathit{NG}\ell^2G \otimes \mathit{NG}\ell^2G_{\mathit{NG}}) \boxtimes_{\mathit{NG}}(\mathit{NG}\ell^2G \otimes \mathit{NG}\ell^2G_{\mathit{NG}}) \boxtimes_{\mathit{NG}}(\mathit{NG}H_{\mathit{NG}}) \rightarrow \mathit{NG}\ell^2G \otimes \mathit{NG}\ell^2G \otimes \mathit{NG}H_{\mathit{NG}} \\ (\delta_a \otimes \delta_b) \boxtimes (\delta_g \otimes \delta_h) \boxtimes \xi \mapsto \delta_a \otimes \delta_{bg} \otimes bh\xi \quad (22)$$

the above requirement translates into the condition that

$$\begin{array}{c} \mathit{NG}H \otimes \ell^2G \otimes \ell^2G_{\mathit{NG}} \\ \downarrow c_H \otimes \text{id}: \xi \otimes \delta_h \otimes \delta_k \mapsto \sum_g \delta_g \otimes c_{g,h}(\xi) \otimes \delta_k \quad [a] \\ \mathit{NG}\ell^2G \otimes \mathit{NG}H \otimes \ell^2G_{\mathit{NG}} \\ \downarrow \text{id} \otimes c_H: \delta_g \otimes \xi \otimes \delta_k \mapsto \sum_h \delta_g \otimes \delta_h \otimes c_{h,k}(\xi) \quad [b] \\ \mathit{NG}\ell^2G \otimes \mathit{NG}\ell^2G \otimes \mathit{NG}H_{\mathit{NG}} \end{array}$$

should be equal to

$$\begin{array}{c} \mathit{NG}H \otimes \ell^2G \otimes \ell^2G_{\mathit{NG}} \\ \downarrow \oplus c_H: \xi \otimes \delta_h \otimes \delta_k \mapsto \sum_g \delta_g \otimes \delta_{gh} \otimes c_{g,hk}(\xi) \quad [c] \\ \mathit{NG}\ell^2G \otimes \mathit{NG}\ell^2G \otimes \mathit{NG}H_{\mathit{NG}} \end{array}$$

i.e., $\sum_{g,h'} \delta_g \otimes \delta_{h'} \otimes c_{h',k}(c_{g,h}(\xi)) = \sum_g \delta_g \otimes \delta_{gh} \otimes c_{g,hk}(\xi)$ for all $\xi \in H$, $h, k \in G$.
Equivalently:

$$\sum_{h'} \delta_{h'} \otimes c_{h',k}(c_{g,h}(\xi)) = \delta_{gh} \otimes c_{g,hk}(\xi)$$

for all $\xi \in H$, $g, h, k \in G$. This means that $c_{h',k}(c_{g,h}(\xi)) = 0$ when $h' \neq gh$, and that $c_{gh,k}(c_{g,h}(\xi)) = c_{g,hk}(\xi)$ for all $\xi \in H$, $g, h \in G$.

The formulas [a], [b], [c] above can be checked by the following diagram chases:

$$\begin{array}{ccccc} (\xi \otimes \delta_b) \boxtimes (\delta_1 \otimes \delta_h) & \xleftarrow{(15) \boxtimes \text{id}} & \xi \boxtimes (\delta_1 \otimes \delta_b) \boxtimes (\delta_1 \otimes \delta_h) & \xrightarrow{(20)} & \xi \otimes \delta_b \otimes \delta_h \\ \downarrow c_H \otimes \text{id} & & & & \downarrow \text{RHS of [a]} \\ (\sum_g \delta_g \otimes c_{g,b}(\xi)) \boxtimes (\delta_1 \otimes \delta_h) & \xleftarrow{(16) \boxtimes \text{id}} & \sum_g (\delta_g \otimes \delta_1) \boxtimes c_{g,b}(\xi) \boxtimes (\delta_1 \otimes \delta_h) & \xrightarrow{(21)} & \sum_g \delta_g \otimes c_{g,b}(\xi) \otimes \delta_h \end{array}$$

$$\begin{array}{ccccc}
(\delta_a \otimes \delta_1) \boxtimes (\xi \otimes \delta_h) & \xleftarrow{\text{id} \boxtimes (15)} & (\delta_a \otimes \delta_1) \boxtimes \xi \boxtimes (\delta_1 \otimes \delta_h) & \xrightarrow{(21)} & \delta_a \otimes \xi \otimes \delta_h \\
\downarrow \text{id} \otimes c_H & & & & \downarrow \text{RHS of [b]} \\
(\delta_a \otimes \delta_1) \boxtimes (\sum_g \delta_g \otimes c_{g,h}(\xi)) & \xleftarrow{\text{id} \boxtimes (16)} & \sum_g (\delta_a \otimes \delta_1) \boxtimes (\delta_g \otimes \delta_1) \boxtimes c_{g,h}(\xi) & \xrightarrow{(22)} & \sum_g \delta_a \otimes \delta_g \otimes c_{g,h}(\xi) \\
\\
\xi \boxtimes (\delta_1 \otimes \delta_b \otimes \delta_{bh}) & \xleftarrow{\text{id} \boxtimes (19)} & \xi \boxtimes (\delta_1 \otimes \delta_b) \boxtimes (\delta_1 \otimes \delta_h) & \xrightarrow{(20)} & \xi \otimes \delta_b \otimes \delta_h \\
\downarrow \bigoplus_{b \in G} c_H & & & & \downarrow \text{RHS of [c]} \\
(\sum_g \delta_g \otimes \delta_b \otimes \delta_1) \boxtimes c_{g,bh}(\xi) & \xleftarrow{(19) \boxtimes \text{id}} & \sum_g (\delta_g \otimes \delta_{gb}) \boxtimes (\delta_1 \otimes \delta_{(gb)^{-1}}) \boxtimes c_{g,bh}(\xi) & \xrightarrow{(22)} & \sum_g \delta_g \otimes \delta_{gb} \otimes c_{g,bh}(\xi)
\end{array}$$

Gathering everything in one place, the conditions that the matrix (c_{gh}) should satisfy are:

$$\sum_h c_{g,h} c_{k,h}^* = \sum_h c_{h,g}^* c_{h,k} = \delta_{g=k} \text{id}_H \quad (23)$$

$$c_{kg,h}(\xi) = k c_{g,h}(k^{-1}\xi) \quad (24)$$

$$c_{g,hk}(\xi) = c_{g,h}(\xi)k \quad (25)$$

$$c_{gk,h}(\xi) = c_{g,kh}(\xi k^{-1}) \quad (26)$$

$$c_{w,k}(c_{g,h}(\xi)) = \delta_{w=gh} c_{g,hk}(\xi) \quad (27)$$

From (24) and (25) above, we see that $c_{g,h}$ is completely determined by $e := c_{1,1} \in B(H)$ via the formula $c_{g,h}(\xi) = ge(g^{-1}\xi)h$. We then also have $c_{g,h}^*(\xi) = ge^*(g^{-1}\xi h^{-1})$.

Example 0.3. For $H = {}_{NG}V \otimes {}_{NG}\ell^2 G {}_{NG}$ in the image of the functor (13), the half-braiding (17) is given by

$$\begin{array}{ccc}
{}_{NG}V \otimes {}_{NG}\ell^2 G \otimes \ell^2 G {}_{NG} & \longrightarrow & {}_{NG}\ell^2 G \otimes {}_{NG}V \otimes {}_{NG}\ell^2 G {}_{NG} \\
v \otimes \delta_g \otimes \delta_h & \mapsto & \delta_g \otimes v \otimes \delta_{gh},
\end{array} \quad (28)$$

as can be seen by pre- and post-composing (14) by the isomorphisms (15) and (16):

$$v \otimes \delta_g \otimes \delta_h \xleftarrow{(15)} (v \otimes \delta_g) \boxtimes (\delta_1 \otimes \delta_h) \xrightarrow{(14)} (\delta_g \otimes \delta_g) \boxtimes (g^{-1}v \otimes \delta_h) \xrightarrow{(16)} \delta_g \otimes v \otimes \delta_{gh}.$$

The maps $c_{g,h}$ are then given by

$$c_{g,h}(v \otimes \delta_k) = \delta_{g=k} v \otimes \delta_{gh}, \quad (29)$$

$e(v \otimes \delta_g) = \delta_{g=1} v \otimes \delta_1$ is the orthogonal projection onto $V = V \otimes \mathbb{C}\delta_1 \subset V \otimes \ell^2 G$, and the conjugation action $\xi \mapsto g\xi g^{-1}$ of G on $H = V \otimes \ell^2 G$ restricts to the given action of G on $V \subset H$.

Lemma 0.4. *The conditions (23), (26) and (27) satisfied by $c_{g,h}$ are equivalent to the following conditions for e :*

$$e = e^2 = e^* \quad (30)$$

$$\sum_g ge(g^{-1}\xi) = \xi \quad (31)$$

$$e(ge(\xi)) = \delta_{g=1} e(\xi) \quad (32)$$

$$ge(\xi)g^{-1} = e(g\xi g^{-1}) \quad (33)$$

Proof. Expressed in terms of e , (23), (26), (27) become

$$\begin{aligned}\sum_h ge(g^{-1}ke^*(k^{-1}\xi h^{-1}))h &= \sum_h he^*(h^{-1}he(h^{-1}\xi)kg^{-1}) = \delta_{g=k}\xi \\ gke((gk)^{-1}\xi)h &= ge(g^{-1}\xi k^{-1})kh \\ we(w^{-1}ge(g^{-1}\xi)h)k &= \delta_{w=gh}ge(g^{-1}\xi)hk,\end{aligned}$$

which readily simplify to:

$$\sum_h ge(g^{-1}ke^*(\xi h^{-1}))h = \delta_{g=k}k\xi \quad (34)$$

$$\sum_h he^*(e(h^{-1}\xi)kg^{-1}) = \delta_{g=k}\xi \quad (35)$$

$$ke(k^{-1}\xi) = e(\xi k^{-1})k \quad (36)$$

$$we(w^{-1}ge(\xi)h) = \delta_{w=gh}ge(\xi)h. \quad (37)$$

Replacing $k^{-1}g$ by g , (34) becomes $\sum_h ge(g^{-1}e^*(\xi h^{-1}))h = \delta_{g=1}\xi$. And replacing w by gh' , (37) becomes $gh'e((gh')^{-1}ge(\xi)h) = \delta_{h'=h}ge(\xi)h$, which simplifies to $h'e((h')^{-1}e(\xi)h) = \delta_{h'=h}e(\xi)h$. Relabeling some variables and simplifying, (34), (35), (36), (37) become:

$$\sum_g e(he^*(\xi g^{-1}))g = \delta_{h=1}\xi \quad (38)$$

$$\sum_g ge^*(e(g^{-1}\xi)h) = \delta_{h=1}\xi \quad (39)$$

$$ge(g^{-1}\xi) = e(\xi g^{-1})g \quad (40)$$

$$ge(g^{-1}e(\xi)h) = \delta_{g=h}e(\xi)h. \quad (41)$$

Using (40), equation (41) is equivalent to $e(e(\xi)hg^{-1})g = \delta_{g=h}e(\xi)h$, which is equivalent to $e(e(\xi)h)g = \delta_{h=1}e(\xi)hg$, which is equivalent to

$$e(e(\xi)h) = \delta_{h=1}e(\xi). \quad (41')$$

Using (40), equation (41') is also equivalent to

$$e(he(\xi)) = \delta_{h=1}e(\xi). \quad (41'')$$

And using (40), the equations (38) and (39) imply $\sum_g ge(e^*(g^{-1}\xi)) = \sum_g ge^*(e(g^{-1}\xi)) = \xi$. Letting $e_g(\xi) := ge(g^{-1}\xi)$, this can be rewritten as $\sum_g e_g e_g^* = \sum_g e_g^* e_g = 1$. It follows that $e_g = e_g e_g^*$, and that e_g is self-adjoint:

$$e_g = \sum_h e_g e_h e_h^* = \sum_h \delta_{g=h} e_h e_h^* = e_g e_g^*.$$

From (41''), we also learn that $e_g e_h = \delta_{g=h} e_g$. So the e_g are orthogonal projections adding up to 1. Equations (38), (39), (40), (41'') are therefore equivalent to

$$\begin{aligned}e &= e^2 = e^* \\ \sum_g ge(e(g^{-1}\xi)) &= \xi \\ ge(g^{-1}\xi) &= e(\xi g^{-1})g \\ e(he(\xi)) &= \delta_{h=1}\xi\end{aligned}$$

The latter are visibly equivalent to (30), (31), (32), (33). □

Letting $e_g(\xi) = ge(g^{-1}\xi)$, the first three conditions (30), (31), (32) say that the e_g are orthogonal projections adding up to 1. The fourth condition (33) says that the image of e is invariant under the conjugation action $\pi_g : \xi \mapsto g\xi g^{-1}$ of G on H . We can use this to construct a functor

$$\text{Rep}(G)' \rightarrow \text{Rep}(G) : H \mapsto (eH, \pi|_{eH}) \quad (42)$$

We claim that this functor is an inverse of the functor (13) : $\text{Rep}(G) \rightarrow \text{Rep}(G)'$. The computations in Example 0.3 show that (42) \circ (13) $\cong \text{id}_{\text{Rep}(G)}$.

To show (13) \circ (42) $\cong \text{id}_{\text{Rep}(G)'}$, take $(H, \varsigma_{H,\bullet}) \in \text{Rep}(G)'$ and consider the image of (eH, π) under the functor (13) : $\text{Rep}(G) \rightarrow \text{Rep}(G)'$, namely:

$$(\text{NG}(eH)_{\text{NG}} \otimes \text{NG}\ell^2 G_{\text{NG}}, (\sigma_{eH,W})_{W \in \text{Rep}(G)}) \in \text{Rep}(G)'.$$

The desired natural isomorphism $\theta : (13) \circ (42) \rightarrow \text{id}$ is given by:

$$\theta_H : eH \otimes \ell^2 G \rightarrow H : \eta \otimes \delta_g \mapsto \eta g.$$

By Lemma[[...]], to prove that θ_H intertwines $\sigma_{eH,\bullet}$ and $\varsigma_{H,\bullet}$, it suffices to verify that it intertwines $\sigma_{eH,\ell^2 G}$ and $c_H = \varsigma_{H,\ell^2 G}$. Recall that $c_{g,h}$ was defined in (18), and let $\sigma_{g,h}$ be given by (29) from Example 0.3: $\sigma_{g,h}(\eta \otimes \delta_k) = \delta_{g=k}\eta \otimes \delta_{gh}$. The isomorphism θ_H intertwines $\sigma_{eH,\ell^2 G}$ and c_H if and only if it intertwines $\sigma_{g,h}$ and $c_{g,h}$. At last, we check that for $\eta \in eH$ and $k \in G$:

$$\begin{aligned} \theta_H(\sigma_{g,h}(\eta \otimes \delta_k)) &= \theta_H(\delta_{g=k}\eta \otimes \delta_{gh}) = \delta_{g=k}\eta gh && \text{agrees with} \\ c_{g,h}(\theta_H(\eta \otimes \delta_k)) &= c_{g,h}(\eta k) = ge(g^{-1}\eta k)h \stackrel{(40)}{=} e(\eta k g^{-1})gh \stackrel{(41')}{=} \delta_{g=k}\eta gh. \end{aligned}$$

We summarize the above results in the following theorem:

Theorem 0.5. *The functor $\text{Rep}(G) \rightarrow \text{End}_{\text{Rep}(G)}(\text{Rep}(NG))$ given by $V \mapsto (V \otimes -)$ is an equivalence of W^* tensor categories.*