A proof of Bott periodicity via Clifford algebras

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The purpose of this note is to present a proof of the Bott periodicity theorem that is based on the periodicity of Clifford algebras. Such a proof was first predicted in [2], and then constructed in [8] and in [3]. Here, we give another proof along the same lines as [8], but based on a different model of K-theory.

In order to simplify the notation, we only present the periodicity for KO theory. The arguments apply without difficulty to the case of complex K-theory.

1 Clifford algebras

In this paper, the Clifford algebras are considered as $\mathbb{Z}/2$-graded $*$-algebras, defined over the reals. They are given by

$$Cl(1) := \langle e \,|\, e \text{ is odd}, \, e^2 = 1, \, e^* = e \rangle,$$
$$Cl(-1) := \langle f \,|\, f \text{ is odd}, \, f^2 = -1, \, f^* = -f \rangle,$$
$$Cl(n) := Cl(1)^{\otimes n}, \quad Cl(-n) := Cl(-1)^{\otimes n},$$

where the tensor product of $\mathbb{Z}/2$-graded $*$-algebras has multiplication given by

$$(a \otimes b)(c \otimes d) := (-1)^{|b||c|} ac \otimes bd,$$

and involution given by

$$(a \otimes b)^* := (-1)^{|a||b|} a^* \otimes b^*.$$

See [4, Section 14] for more background about $\mathbb{Z}/2$-graded operator algebras. These algebras are equipped with a trace $tr : Cl(n) \to \mathbb{R}$, given by

$$tr(1) := 1, \qquad tr(e) := 0, \qquad tr(f) := 0$$

on $Cl(1)$ and $Cl(-1)$, and extended via the formula $tr(a \otimes b) := tr(a)tr(b)$. It satisfies $tr(ab) = tr(ba)$, $tr(1) = 1$, $tr(a^*) = tr(a)$, $tr(a) > 0$ for $a > 0$, and $tr(a) = 0$ for a odd.

The Clifford algebras are actually von Neumann algebras[1], meaning that they admit faithful $*$-representations on Hilbert spaces. Let us adopt the following

[1]Since our algebras are finite dimensional, there is no difference between von Neumann algebras and C^*-algebras. However, the formula (1) is better understood within the theory of von Neumann algebras.

Convention. *All modules shall be finite dimensional, and shall be equipped with Hilbert space structures.*

If A is an algebra with a trace as above, then the scalar product $\langle a, b\rangle := tr(ab^*)$ equips it with a Hilbert space structure, thus making it a module over itself. Let $\{a_i\}$ be an orthonormal basis of A with respect to that inner product. The tensor product $M \otimes_A N$ of a right module M with a left module N is again a Hilbert space. Its scalar product is given by the formula

$$\langle m \otimes n, m' \otimes n'\rangle := \sum_i \langle m a_i, m'\rangle\langle n, a_i n'\rangle. \tag{1}$$

Definition 1. *Let A, B be finite dimensional $\mathbb{Z}/2$-graded von Neumann algebras. Then A and B are called Morita equivalent if there exist bimodules ${}_AM_B$ and ${}_BN_A$ such that ${}_AM \otimes_B N_A \simeq {}_AA_A$ and ${}_BN \otimes_A M_B \simeq {}_BB_B$. We shall denote this relation by $A \simeq_M B$.*

If A and B are Morita equivalent, then the functors $N \otimes_A -$ and $M \otimes_B -$ implement an equivalence of categories between the category of A-modules, and that of B-modules.

Lemma 2. *One has* [cl1-1]

$$\mathbb{R} \ \simeq_M \ Cl(1) \otimes Cl(-1). \tag{2}$$

Proof. The algebra $Cl(1) \otimes Cl(-1)$ is isomorphic to $\mathrm{End}(\mathbb{R}^{1|1})$ via the map

$$e \otimes 1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad 1 \otimes f \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and the latter is Morita equivalent to $\mathbb{R}$ via the bimodules ${}_{\mathrm{End}(\mathbb{R}^{1|1})}\,\mathbb{R}^{1|1}{}_{\mathbb{R}}$ and ${}_{\mathbb{R}}\,\mathbb{R}^{1|1}{}_{\mathrm{End}(\mathbb{R}^{1|1})}$. Here, the first $\mathbb{R}^{1|1}$ should be thought of as column vectors, while the second $\mathbb{R}^{1|1}$ should be thought of as row vectors. □

By the above lemma, we then get [cl+]

$$Cl(n+m) \ \simeq_M \ Cl(n) \otimes Cl(m) \tag{3}$$

for all integers n and m. Let [Dnm]

$$_{Cl(n+m)}\big(D_{n,m}\big)_{Cl(n)\otimes Cl(m)} \tag{4}$$

be a bimodule implementing the Morita equivalence (3). In the appendix, we will show how to chose the bimodules (4) so that they satisfy certain nice compatibility properties.

Let $\mathbb{H}$ be the algebra of quaternions, put in even degree, and with involution $i^* := -i$, $j^* := -j$, and $k^* := -k$. Letting $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ denote the generators of $Cl(n)$ and $Cl(-n)$, we then have isomorphisms [cl3]

$$\begin{array}{ccc} Cl(3) \simeq \mathbb{H} \otimes Cl(-1), & \qquad & Cl(-3) \simeq \mathbb{H} \otimes Cl(1). \\ e_1 \mapsto i \otimes f & & f_1 \mapsto i \otimes e \\ e_2 \mapsto j \otimes f & & f_2 \mapsto j \otimes e \\ e_3 \mapsto k \otimes f & & f_3 \mapsto k \otimes e \end{array} \tag{5}$$

Putting together the above computations, one obtains the following periodicity theorem.

Theorem 3 (periodicity of Clifford algebras)**.** [PerClif] *One has*

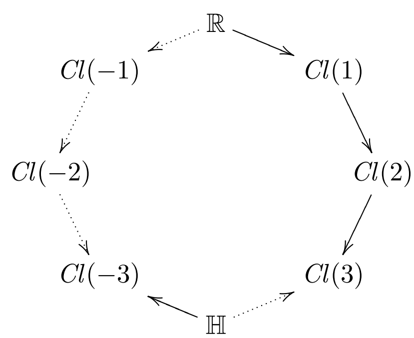
$$Cl(n) \simeq_M Cl(n+8).$$

Proof. In view of (3), it is enough to show the result for a given value of n. We shall take $n = -4$. By (5), we then have isomorphisms

$$Cl(-4) = Cl(-1) \otimes Cl(-3) \simeq Cl(-1) \otimes \mathbb{H} \otimes Cl(1) \simeq Cl(3) \otimes Cl(1) = Cl(4).$$

□

Denoting by a solid arrow the operation $- \otimes Cl(1)$, and by a dotted arrow the operation $- \otimes Cl(-1)$, we can summarize the above computations in the following small diagram:



2 Quasi-bundles

Thereafter, we shall assume that all our base spaces are paracompact, namely, that any open cover can be refined to a locally finite one. This condition is equivalent to the existence of enough partitions of unity [9], and is satisfied by all reasonnable topological spaces. In particular, it is satisfied by CW-complexes [10].

Let X be a space, and $\{\mathcal{U}_i\}$ an open cover that is closed under taking intersections. Suppose that we are given a finite dimensional vector bundle V_i over each $\mathcal{U}_i$, and inclusions $\varphi_{ij} : V_i|_{\mathcal{U}_j} \hookrightarrow V_j$ for $\mathcal{U}_j \subset \mathcal{U}_i$, subject to the cocycle condition $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$. Then we can form the total space $V := \coprod V_i / \sim$, where the equivalence relation $\sim$ is generated by $v \sim \varphi_{ij}(v)$. Such an object is an example of a *quasi-bundle*. So, informally speaking, a quasi-bundle is a vector bundle, where the dimension of the fiber can jump.

Example 4. Given an open subspace $\mathcal{U} \subset X$ and a vector bundle $V \to \mathcal{U}$, the *extension by zero* $V \cup_{\mathcal{U}} X$ is a quasi-bundle over X.

Definition 5. *A vector space object over X consists of a space $V \to X$, and three continuous maps*

$$+ : V \times_X V \to V, \qquad 0 : X \to V, \qquad \times : \mathbb{R} \times V \to V,$$

equipping each fiber of $V \to X$ with the structure of a vector space.

Given a point $x \in X$, a *germ of vector bundle* around x consist of a pair $\mathbf{V} = (\mathcal{U}, V)$, where $\mathcal{U}$ is a neighborhood of x, and V is a vector bundle over $\mathcal{U}$. If $\mathcal{U}' \subset \mathcal{U}$ is a smaller neighborhood, we wish to identify $(\mathcal{U}, V)$ with $(\mathcal{U}', V|_{\mathcal{U}'})$. The correct way to do this is to form a category $\mathrm{Germs}(X, x)$, whose objects are pairs $(\mathcal{U}, V)$ as above, and whose morphisms are given by

$$\hom\big(\mathbf{V}_1, \mathbf{V}_2\big) := \operatorname*{colim}_{\mathcal{U}' \subset \mathcal{U}_1 \cap \mathcal{U}_2} \hom\big(V_1|_{\mathcal{U}'}, V_2|_{\mathcal{U}'}\big), \tag{6}$$

where the colimit is taken over all neighborhoods $\mathcal{U}'$ of x. The objects $(\mathcal{U}, V)$ and $(\mathcal{U}', V|_{\mathcal{U}'})$ are then canonically isomorphic in that category. Similarly, we have the notion of germ of vector space object.

We shall refer to an element of (6) as a map, and write it $f : \mathbf{V}_1 \to \mathbf{V}_2$. Such a map is called injective, or inclusion, if it admits a representative $V_1|_{\mathcal{U}'} \to V_2|_{\mathcal{U}'}$ that is injective. Given a vector bundle V (or vector space object), and a point $x \in X$, we denote by $V_{\langle x \rangle} := (X, V) \in \mathrm{Germs}(X, x)$ the germ of V at x.

Definition 6. [defQ] *A quasi-bundle V over X is a vector space object over X. It comes equipped with a germ of vector bundle $\mathbf{V}_x$ around each point $x \in X$, and an inclusion $\iota_x : \mathbf{V}_x \hookrightarrow V_{\langle x \rangle}$ subject to the following three conditions:*

- *The maps ι_x induce isomorphisms $\mathbf{V}_x|_{\{x\}} \simeq V|_{\{x\}}$.*
- *For each $x \in X$, there is a representative $V_x|_{\mathcal{U}'} \to V|_{\mathcal{U}'}$ of ι_x, such that for all $y \in \mathcal{U}'$, the map $(V_x)_{\langle y \rangle} \to V_{\langle y \rangle}$ factors through $\mathbf{V}_y$.*
- *The topology on V is the weakest one making (representatives of) the maps ι_x continuous.*

A morphism of quasi-bundles is a continuous map $F : V \to W$ that commutes with the projection to X, that is linear in each fiber, and that sends $\mathbf{V}_x$ into $\mathbf{W}_x$ for each $x \in X$.

Remark. If X is a CW-complex, the condition $F(\mathbf{V}_x) \subset \mathbf{W}_x$ is a consequence of the continuity of F. In such case, the underlying vector space object of a quasi-bundle contains all the information.

Remark. The weakest topology on V is independent of the choice of representatives for ι_x.

Most constructions[2] with vector bundles have well defined extensions to quasi-bundles. For example, we have pullbacks, direct sums and tensor products.

[2] To be precise, we need the construction to be functorial with respect to monomorphisms of vector bundles. This excludes contravariant things, such as taking the dual.

Given a map $f : X \to Y$, and a quasi-bundle $V \to Y$, the pullback $W := V \times_X Y$ is a vector space object. It comes with germs $\mathbf{W}_x := f^*(\mathbf{V}_{f(x)})_{\langle x \rangle}$, and inclusions $\mathbf{W}_x \hookrightarrow W_{\langle x \rangle}$ satisfying the first two condition of Definition 6. But the third condition need not be satisfied. The pullback quasi-bundle f^*V is defined by retopologizing W.

Like for vector bundles, we define a scalar product on V to be a continuous, fiberwise bilinear map $\langle\, , \rangle : V \times_X V \to \mathbb{R}$ that is positive definite on each fiber. Given a quasi-bundle with positive definite scalar product, we say that an operator is self-adjoint, respectively positive, if it is so fiberwise. If F is a self-adjoint operator, we recall that its absolute value is given by $|F| := \sqrt{F^2}$.

Lemma 7. [Tle] *(a). Let V, W be two quasi-bundles, and let $F : V \to W$ be a morphism that is invertible on each fiber. Then $F^{-1} : W \to V$ is a morphism of quasi-bundles.*

(b). Let V be a quasi-bundle with positive definite scalar product, and let $F : V \to V$ be a selfadjoint operator. Then $|F| : V \to V$ is morphism of quasi-bundles.

Proof. Given a point $x \in X$, let $\mathbf{V}_x = (\mathcal{U}, V_x)$ and $\mathbf{W}_x = (\mathcal{U}', W_x)$ be the corresponding germs of vector bundles. Since $F(\mathbf{V}_x) \subset \mathbf{W}_x$, there exists an open $\mathcal{V} \subset \mathcal{U} \cap \mathcal{U}'$ such that $F(V_x|_\mathcal{V}) \subset W_x|_\mathcal{V}$.

(a). Since $F(V_x|_\mathcal{V}) \subset W_x|_\mathcal{V}$ and since V_x and W_x have same rank, we also have that $F^{-1}\big(W_x|_\mathcal{V}\big) \subset V_x|_\mathcal{V}$. The map $F^{-1} : W_x|_\mathcal{V} \to V_x|_\mathcal{V}$ is clearly continuous. The topology on W begin the colimit of its subspaces $W_x|_\mathcal{V}$, it follows that F^{-1} is continuous, and thus a morphism of quasi-bundles.

(b). We now assume that $W = V$ has a scalar product. Since $F(V_x|_\mathcal{V}) \subset V_x|_\mathcal{V}$ and F is self-adjoint, we have $|F|\big(V_x|_\mathcal{V}\big) \subset V_x|_\mathcal{V}$. By the same argument as above, this implies the continuity of $|F|$. □

Remark. The adjoint F^* of a morphism F is not always a morphism.

The following lemma is key to a lot of our arguments. The details of defintion 6 are tuned so as to make its proof go through.

Lemma 8. [KLem] *Let V be a $\mathbb{Z}/2$-graded quasi-bundle with scalar product, and let $E, F : V \to V$ be odd self-adjoint operators that graded-commute. Then if E is invertible, so is $E + F$.*

Proof. By Lemma 7.a, and because self-adjointness is defined fiberwise, it is enough to treat the case when V is a vector space.

Since E and F are self-adjoint, their squares are positive operators. Moreover, since E is invertible, we have $E^2 > 0$. It follows that

$$(E + F)^2 = E^2 + EF + FE + F^2 = E^2 + F^2 > 0,$$

and in particular that $(E + F)^2$ is invertible. Hence so is $E + F$. □

The construction described in the beginning of this section provides examples of quasi-bundles. In fact, all quasi-bundles are of that form.

Lemma 9. [AoE] *Let $V \to X$ be a quasi bundle. Then there exists an open cover $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of X, and rank n vector bundles $W_n \subset V|_{\mathcal{U}_n}$ such that $V = \bigcup W_n$, and such that* [wugu]

$$W_n\big|_{\mathcal{U}_n\cap\mathcal{U}_m} \subset W_m\big|_{\mathcal{U}_n\cap\mathcal{U}_m} \tag{7}$$

for all $n < m$.

Proof. Let F_n be the subset of X over which V has rank n. The rank being a lower semi-continuous function, F_n is closed in $X \setminus (F_0 \cup \ldots \cup F_{n-1})$. We shall construct open subsets $\mathcal{U}_n, \hat{\mathcal{U}}_n \subset X$ satisfying

$$F_n \subset \mathcal{U}_n \subset \overline{\mathcal{U}_n} \subset \hat{\mathcal{U}}_n \subset X \setminus (F_0 \cup \ldots \cup F_{n-1}),$$

and rank n vector bundles W_n over $\hat{\mathcal{U}}_n$ satisfying (7). Here, $\overline{\mathcal{U}_n}$ refers to the closure of $\mathcal{U}_n$ inside of $X \setminus (F_0 \cup \ldots \cup F_{n-1})$.

Assume by induction that $\mathcal{U}_n, \hat{\mathcal{U}}_n$ W_n have been constructed for all $n < m$. Given $x \in F_m$, we may pick a representative $f : V_x|_{\mathcal{V}_x} \to V|_{\mathcal{V}_x}$ of ι_x subject to the following condition. Let $Z_x := f(V_x|_{\mathcal{V}_x})$. If $n < m$ is such that $x \in \hat{\mathcal{U}}_n$, then we require that $\mathcal{V}_x \subset \hat{\mathcal{U}}_n$ and that $Z_x \subset W_n|_{\mathcal{V}_x}$. Otherwise, we ask that $\mathcal{V}_x \cap \mathcal{U}_n$ be empty. In that way, we get an open cover $\{\mathcal{V}_x\}_{x\in F_m}$ of F_m, and rank n sub-bundles $Z_x \subset V$.

Pick a locally finite refinement $\{\mathcal{V}_i\}_{i\in I}$ of $\{\mathcal{V}_x\}$, and let $Z_i \to \mathcal{V}_i$ be the vector bundles induced by the Z_x. The inclusions $Z_i \hookrightarrow X$ being morphisms of quasi-bundles, there exists an open neighborhood $\hat{\mathcal{U}}_m$ of F_m such that

$$Z_i\big|_{\mathcal{V}_i\cap\mathcal{V}_j\cap\hat{\mathcal{U}}_m} = Z_j\big|_{\mathcal{V}_i\cap\mathcal{V}_j\cap\hat{\mathcal{U}}_m}$$

for all $i, j \in I$. The Z_i then assemble to a vector bundle W_m over $\hat{\mathcal{U}}_m$ satisfying

$$W_n\big|_{\mathcal{U}_n\cap\hat{\mathcal{U}}_m} \subset W_m\big|_{\mathcal{U}_n\cap\hat{\mathcal{U}}_m}$$

for all $n < m$. We finish the induction step by picking a neighborhood $\mathcal{U}_m$ of F_m whose closure is contained in $\hat{\mathcal{U}}_m$. □

Lemma 10. [KerBot] *Let V be a quasi-bundle equipped with a scalar product, and let $F : V \to V$ be a positive operator. Then $W := \ker(F)^\perp$ is naturally a quasi-bundle.*

Proof. Given a point $x \in X$, we must construct the corresponding germ $\mathbf{W}_x \subset W_{\langle x\rangle}$. Let $\mathbf{V}_x = (\mathcal{U}, V_x)$ be the germ corresponding to V, and let us take $\mathcal{U}$ small enough so that $F(V_x) \subset V_x$. Since $F|_{V_x}$ is a positive operator, its eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\dim(V_x)} \geq 0,$$

are continuous functions on $\mathcal{U}$ with values in $\mathbb{R}_{\geq 0}$. Letting $r := \dim(W|_{\{x\}})$, we then have

$$\lambda_1(x) \geq \lambda_2(x) \geq \ldots \geq \lambda_r(x) > 0 = \lambda_{r+1}(x) = \ldots = \lambda_{\dim(V_x)}(x).$$

Let $\mathcal{U}' \subset \mathcal{U}$ be the open subset defined by the equation $\lambda_r > \lambda_{r+1}$. Over $\mathcal{U}'$ we can then split V_x as

$$V_x|_{\mathcal{U}'} = Y \oplus Z,$$

where Y is the r dimensional subbundle spanned by the eigenspaces corresponding to $\lambda_1, \ldots, \lambda_r$, and Z is its orthogonal complement. We have $Y|_{\{x\}} = W|_{\{x\}}$, and $\mathbf{W}_x := (\mathcal{U}', Y)$ is our desired germ of vector bundle.

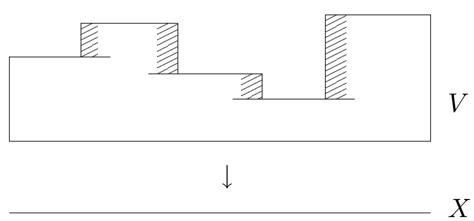
The subspace topology fails the last condition in Definition 6, so we retopologize W, and then it becomes a quasi-bundle. □

3 K-theory

The following definition was inspired by the notion of perfect complex (used in algebraic K-theory of schemes [11]), by that of Kasparov cocycle (used in K-theory of C^*-algebras [7]), and by Furuta's notion of "vectorial bundle" [5] (see also [6]).

Definition 11. [defV] *A K-cocycle is a pair (V, F), where V is a $\mathbb{Z}/2$-graded quasi-bundle equipped with a scalar product, and F is an odd self adjoint operator on V. Moreover, one should be able to write (V, F) locally as an orthogonal direct sum $(V', F') \oplus (V'', F'')$, where V' is a vector bundle, and F'' is an invertible operator.*

Here is an impressionistic picture of a K-cocycle. The shaded areas represent the pieces where the operator F is required to be invertible.



There is an obvious extension of Definition 11 that incorporates Clifford algebra actions. Namely, one requires that V be equipped with a $Cl(n)$ action, that F graded-commutes with the operators coming from the Clifford action, and that the local splittings $V \simeq V' \oplus V''$ be splittings of $Cl(n)$-modules. We shall call such an object a $Cl(n)$-linear K-cocycle.

In the sequel, we will often abuse notation and denote a K-cocycle simply by V instead of (V, F). Given two K-cocycles V_0, V_1 on X, we say that V_0 and V_1 are *homotopic* if there exists a K-cocycle W over $X \times [0, 1]$ such that $V_i \simeq W|_{X \times \{i\}}$. Given a K-cocycle V on X and a subspace $A \subset X$, we say that V is *trivial over A* if the operator F is invertible on $V|_A$. Given two K-cocycles V_0, V_1 on X that are trivial over A, we say that they are *homotopic relatively to A* if the homotopy W can be chosen so that it is trivial over $A \times [0, 1]$.

Definition 12. [defKO] *The n-th real K-theory group $KO^n(X)$ of a topological space X is the set of homotopy classes of $Cl(-n)$-linear K-cocycles over X.*

If A is a subspace of X, the corresponding relative group $KO^n(X,A)$ is the set of equivalence classes of $Cl(-n)$-linear K-cocycles over X that are trivial over A, where two K-cocycles are declared equivalent if they are homotopic relatively to A.

Remark. Note that by definition, we have $KO^n(X) = KO^n(X,\emptyset)$.

There is an obvious map [VtoK]

$$\begin{array}{ccc} \big\{\text{Vector bundles on } X\big\} & \to & KO^0(X) \\ V & \mapsto & (V,0) \end{array} \tag{8}$$

given by picking a scalar product on V, and putting it in even degree. That map is well defined because all scalar products are homotopic. We will show in Section 8 that if X is compact, then (8) induces an isomorphism after group completion.

Remark. Unlike for $n = 0$, the natural map

$$\big\{\mathbb{Z}/2\text{-graded vector bundles with } Cl(-n) \text{ action}\big\} \quad \to \quad KO^n(X)$$

is typically not surjective, even if X is compact. This can be seen most easily in the case of complex K-theory for $n = 1$, and $X = S^1$.

With the above definition, Bott periodicity is an essentially trivial consequence of Theorem 3.

Theorem 13 (Bott Periodicity)**.** *We have natural isomorphisms $KO^n(X) \simeq KO^{n+8}(X)$ and $KO^n(X,A) \simeq KO^{n+8}(X,A)$.*

Proof. Let ${}_{Cl(-n-8)}M_{Cl(-n)}$ be a bimodule implementing the Morita equivalence between $Cl(-n-8)$ and $Cl(-n)$. The functor $M \otimes_{Cl(-n)} -$ is then an equivalence between the categories of $Cl(-n)$-linear and $Cl(-n-8)$-linear K-cocycles over X. That equivalence respects the notion of homotopy, and that of being trivial over A. So it descends to an isomorphism of K groups. □

What remains to be done, is to identify the theory of Definition 12 with the usual definition of real K-theory via vector bundles. Let us write KO^*_{Atiyah} for the theory defined in [1].

First of all, we will show that for X compact, $KO^0(X)$ is isomorphic to $KO^0_{\text{Atiyah}}(X)$, namely to the group completion of the monoid of isomorphisms classes of vector bundles over X. If X has a base point, we will then show that $KO^0(X,*)$ is isomorphic to

$$\widetilde{KO}^0_{\text{Atiyah}}(X) := \ker\big(KO^0_{\text{Atiyah}}(X) \to KO^0_{\text{Atiyah}}(*)\big).$$

Then, we will prove that for $n \le 0$, there is an isomorphism

$$KO^n(X,*) \simeq \widetilde{KO}^n_{\text{Atiyah}}(X) := \widetilde{KO}^0_{\text{Atiyah}}(\Sigma^{-n}X).$$

Finally, we will show that KO^n satisfies excision, which will then imply that $KO^n(X,A) \simeq KO^n(X/A,*)$, and thus that

$$KO^n(X,A) \simeq KO^n_{\text{Atiyah}}(X,A) := \widetilde{KO}^n_{\text{Atiyah}}(X/A).$$

These results will be proved in Theorem 28, Lemma 21, Theorem 25, and Lemma 18 respectively. The isomorphism $KO^n(X) \simeq KO^n_{\text{Atiyah}}(X) := \widetilde{KO}^n_{\text{Atiyah}}(X \sqcup *)$ will then follow from the following rather trivial special case of excision

$$KO^n(X) \simeq KO^n(X \sqcup *, *).$$

4 Elementary properties

In this section, we derive some elementary properties of the functor KO.

Lemma 14. *[Linc] Let (V,F) be a K-cocycle, and let $(W,G) \subset (V,F)$ be a subcocycle, such that F is invertible on the orthogonal complement of W. Then V and W represent the same element in K-theory.*

Proof. The homotopy between V and W is given by $V \times [0,1) \cup_{W \times [0,1)} W \times [0,1]$. □

Lemma 15 (Homotopy)**.** *[L:hom] If $f, g : X \to Y$ are homotopic maps, then $f^* = g^* : KO^*(Y) \to KO^*(X)$.*

Proof. Let V be a K-cocycle over Y, and let $h : X \times [0,1] \to Y$ be a homotopy between f and g. The pull back of h^*V is then a homotopy between f^*V and g^*V. □

The obvious analogs of Lemmas 14 and 15 also hold for pairs of spaces.

Corollary 16. *Homotopy equivalent pairs have isomorphic K-groups.* □

Let $\text{Rep}(Cl(n))$ be the semigroup of isomorphism classes of representations of $Cl(n)$. And let $\text{Rep}^\circ(Cl(n)) \subset \text{Rep}(Cl(n))$ denote those representations that admit an extra $Cl(1)$-action, graded-commuting with the existing $Cl(n)$-action.

Proposition 17 (Coefficients)**.** *[Coef] There is a canonical isomorphism*

$$KO^{-n}(*) \simeq \text{Rep}(Cl(n)) \big/ \text{Rep}^\circ(Cl(n)).$$

Proof. Let $\phi : \text{Rep}(Cl(n)) \to KO^n(*)$ be the map given by $\phi([V]) := [(V,0)]$. If $[V] \in \text{Rep}(Cl(n))$ has an extra $Cl(1)$-action $e : V \to V$, then the K-cocycle

$$\left(V \times [0,1) \cup_{[0,1)} [0,1], \;\; F_t := \begin{cases} 0 & \text{if } t = 1 \\ te & \text{if } t < 1 \end{cases} \right)$$

provides a homotopy between $(V,0)$ and zero. It follows that $\phi([V]) = 0$ for $[V] \in \text{Rep}^\circ(Cl(n))$, and so we get an induced map

$$\bar{\phi} : \text{Rep}(Cl(n)) \big/ \text{Rep}^\circ(Cl(n)) \to KO^{-n}(*).$$

The map $\bar{\phi}$ is surjective since any (V,F) is homotopic to $(V,0)$.

To see that $\bar{\phi}$ is injective, consider $[V]$ such that $\bar{\phi}([V])=0$. Pick a homotopy (W,G) between $(V,0)$ and 0, an open cover $\{\mathcal{U}_i\}$ of $[0,1]$, and decompositions

$$(W|_{\mathcal{U}_i}, F|_{\mathcal{U}_i}) = (W_i', F_i') \oplus (W_i'', F_i''),$$

where W_i' are vector bundles, and F_i'' invertible. By compactness, we may assume that $[t_i, t_{i+1}] \subset \mathcal{U}_i$, for some $0=t_0<t_1\ldots<t_n=1$. Replacing F_i'' by $F_i''/|F_i''|$, we see that $W_{i-1}''|_{\{t_i\}}$ and $W_i''|_{\{t_i\}}$ are in $\mathrm{Rep}^\circ(Cl(n))$. It follows that

$$W_{i-1}'|_{\{t_i\}} \equiv W_{i-1}'|_{\{t_i\}} \oplus W_{i-1}''|_{\{t_i\}} = W_i'|_{\{t_i\}} \oplus W_i''|_{\{t_i\}} \equiv W_i'|_{\{t_i\}}$$

in the quotient $\mathrm{Rep}(Cl(n))/\mathrm{Rep}^\circ(Cl(n))$. Upon trivializing $W_i'|_{[t_i,t_{i+1}]}$, we may identify $W_i'|_{\{t_i\}}$ with $W_i'|_{\{t_{i+1}\}}$. So we get

$$V = W_0'|_{\{t_0\}} \simeq W_0'|_{\{t_1\}} \equiv W_1'|_{\{t_1\}} \simeq W_1'|_{\{t_2\}} \equiv W_2'|_{\{t_2\}} \cdots \simeq W_n'|_{\{t_n\}} = 0.$$

□

Remark. The groups $\mathrm{Rep}(Cl(n))/\mathrm{Rep}^\circ(Cl(n))$ are computed in [2] using elementary methods. They are given by:

$n \mod 8$	0	1	2	3	4	5	6	7
$\mathrm{Rep}(Cl(n))/\mathrm{Rep}^\circ(Cl(n))$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0

By computation of the relevant semigroups, one also sees that a class $[V]$ is zero in $\mathrm{Rep}(Cl(n))/\mathrm{Rep}^\circ(Cl(n))$ if and only if V belongs to $\mathrm{Rep}^\circ(Cl(n))$.

Lemma 18 (Excision). [L:exc] *Let (X,A) be a pair of spaces, and let U be a subspace of A with the property that there exist disjoint opens $\mathcal{U}_1, \mathcal{U}_2 \subset X$ such that $U \subset \mathcal{U}_1$ and $\mathcal{U}_2 \cup A = X$. Then the restriction map*

$$r : KO^n(X,A) \to KO^n(X \setminus U, A \setminus U)$$

is an isomorphism.

Proof. The inverse of r is given by extension by zero: it sends a K-cocycle V over $X \setminus U$ to the K-cocycle

$$s(V) := V|_{X\setminus \bar{U}} \underset{X\setminus\bar{U}}{\cup} X,$$

where $\bar{U}$ denotes the closure of U. The equation $r \circ s = 1$ is clear. The equation $s \circ r = 1$ follows from Lemma 14. □

Corollary 19. [x/a] *Let X be a space, and let $A \subset X$ be a neighborhood deformation retract. Then $KO^n(X,A) \simeq KO^n(X/A,*)$.*

Proof. Let CA be the cone on A. Applying excision and then homotopy invariance, we get $KO^n(X,A) \simeq KO^n(X\cup_A CA, CA) \simeq KO^n(X/A,*)$. □

Lemma 20 (Group structure)**.** *The operation of direct sum equips $KO^n(X)$ and $KO^n(X, A)$ with the structure of abelian groups.*

Proof. It is quite clear that direct sum descends to K-theory, and so that $KO^n(X)$ and $KO^n(X, A)$ are abelian monoids. We must show the existence of inverses.

Given a K-cocycle (V, F), its inverse in K-theory is given by $(\bar{V}, F)$, where $\bar{V} := V \otimes \mathbb{R}^{0|1}$ denotes the bundle with reversed $\mathbb{Z}/2$-grading. The homotopy (W, G) between $(V \oplus \bar{V}, F \oplus F)$ and the zero bundle is given by

$$W := (V \oplus \bar{V}) \times [0,1) \underset{X \times [0,1)}{\cup} X \times [0,1] \;\to\; X \times [0,1],$$

and the action of G on the fiber $W_{(x,t)}$ is given by

$$G_{(x,t)} := \begin{cases} \quad 0 & \text{if} \quad t = 1, \\ \begin{pmatrix} F_x & t\gamma \\ t\gamma & F_x \end{pmatrix} & \text{if} \quad t < 1, \end{cases}$$

where γ denotes the grading involution.

To see that W is indeed a K-cocycle, we note that by Lemma 8, the operator

$$G_{(x,t)} = \begin{pmatrix} 0 & t\gamma \\ t\gamma & 0 \end{pmatrix} + \begin{pmatrix} F_x & 0 \\ 0 & F_x \end{pmatrix}$$

is invertible as soon as $t > 0$. Over the subspace $X \times (0, 1]$, the pair (W, G) is a K-cocycle because G is invertible. And over $X \times [0, 1)$, it is a K-cocycle because (V, F) was one. □

Remark. The inverse K-cocycle $(\bar{V}, F)$ can be rewritten more suggestively as $(V \otimes \mathbb{R}^{0|1}, F \otimes 1 + 1 \otimes 0)$, see Lemma 22 below.

From the above lemma, we see that the map {Vector bundles on X} $\to$ $KO^0(X)$ factors through $KO^0_{\text{Atiyah}}(X)$. In section 8, we will show that that map is an isomorphism whenever X is compact.

Lemma 21. [L:bp] *Let X be a space, with base point $\iota : * \to X$. Then the restriction map $KO^n(X, *) \to KO^n(X)$ induces an isomorphism* [kkrl]

$$r : KO^n(X, *) \xrightarrow{\sim} \ker\big(\iota^* : KO^n(X) \to KO^n(*)\big).$$

Proof. If $[(V, F)]$ is in $\ker(\iota^*)$, then by Proposition 17, the $Cl(n)$-module $\iota^* V$ admits an extra $Cl(1)$-action $e : \iota^* V \to \iota^* V$. Pick a neighborhood $\mathcal{U}$ of the base point, and a splitting

$$(V|_{\mathcal{U}}, F|_{\mathcal{U}}) = (V', F') \oplus (V'', F'')$$

with V' a trivial vector bundle, and F'' invertible. Let $\varphi : X \to \mathbb{R}_{\geq 0}$ be a function with support contained in $\mathcal{U}$, and such that $\varphi(*) > \|F'|_{\{*\}}\|$. Then

$$\varphi e \oplus 0 : V' \oplus V'' \to V' \oplus V''$$

extends by zero to an operator $E : V \to V$. Since $(E+F)|_{\{*\}}$ is invertible, $(V, E+F)$ is a cocycle for $KO^n(X,*)$. The cocycles $(V, E+F)$ and (V,F) being homtopic via $(V, tE+F)$, $t \in [0,1]$, this shows that r is surjective.

To see that r is injective, consider a class $[(V,F)] \in KO^n(X,*)$ that maps to zero in $KO^n(X)$. By definition, there is a homotopy (W,G) between (V,F) zero. Our goal is to find a new homotopy $(\tilde{W}, \tilde{G})$ such that $\tilde{G}|_{\{*\}\times[0,1]}$ is invertible. Let $p : X \to *$ be the projection. Since $[p^*\iota^*(V,F)] = 0$, we may as well construct a homotopy between $[(V,F) \oplus p^*\iota^*(V,F)]$ and zero. We set

$$\tilde{W} := [W \oplus p^*\iota^*\bar{W}].$$

Let $\{\mathcal{U}_i\}$ be a finite collection of open subsets of $X \times [0,1]$ covering $\{*\}\times[0,1]$. And let us assume that we have decompositions

$$(W|_{\mathcal{U}_i}, G|_{\mathcal{U}_i}) = (W_i', G_i') \oplus (W_i'', G_i''),$$

where W_i' are trivial vector bundles and G_i'' are invertible. We may assume that $p(\mathcal{U}_i) \subset \mathcal{U}_i$. We then get corresponding decompositions

$$\tilde{W}|_{\mathcal{U}_i} = W_i' \oplus p^*\iota^*\bar{W}_i' \oplus W_i'' \oplus p^*\iota^*\bar{W}_i'',$$

and identifications $W_i' \simeq p^*\iota^*W_i'$. Let $\varphi_i : X \times [0,1] \to \mathbb{R}_{\geq 0}$ be functions with support in $\mathcal{U}_i$, and such that $\sum \varphi_i|_{\{*\}\times[0,1]} > 0$. Let $\gamma : W_i' \to W_i' \simeq p^*\iota^*W_i'$ denote the grading involution. The operator

$$\begin{pmatrix} 0 & \varphi_i\gamma \\ \varphi_i\gamma & 0 \end{pmatrix} \oplus 0 \oplus 0 : \tilde{W}|_{\mathcal{U}_i} \to \tilde{W}|_{\mathcal{U}_i}$$

then extends by zero to an odd operator $E_i : \tilde{W} \to \tilde{W}$. We define

$$\tilde{G} := \big(G \oplus p^*\iota^*G\big) + \sum E_i$$

Given a point $x = (*,t) \in X \times [0,1]$, we now show that $\tilde{G}|_{\{x\}}$ is invertible. For i such that $\varphi_i(*,t) > 0$, let q_i denote the projection of $\tilde{W}|_{\{x\}}$ onto the summand $(W_i' \oplus p^*\iota^*\bar{W}_i')|_{\{x\}} = W_i'|_{\{x\}} \oplus \bar{W}_i'|_{\{x\}}$. We then have

$$\tilde{G}|_{\{x\}} = \begin{pmatrix} G|_{\{x\}} & 0 \\ 0 & G|_{\{x\}} \end{pmatrix} + \begin{pmatrix} 0 & \sum q_i\varphi_i(x)\gamma \\ \sum q_i\varphi_i(x)\gamma & 0 \end{pmatrix}$$

The first summand is invertible on each $\mathrm{im}(q_i)$, and hence on their linear span. The second summand is invertible on the intersection of the $\mathrm{im}(q_i)$. So by Lemma 8, $\tilde{G}|_{\{x\}}$ is invertible on $\tilde{W}|_{\{x\}} = \mathrm{span}\{\mathrm{im}(q_i)\} \oplus \bigcap \mathrm{im}(q_i)$. □

Lemma 22 (Ring structure)**.** [L:R] *The operation* [topro]

$$((V,F),(W,G)) \mapsto (V \otimes W,\ F \otimes 1 + 1 \otimes G) \tag{9}$$

induces an associative, graded-commutative product on $KO^*(X)$. Moreover, if (V,F), (W,G) are classes in $KO^*(X,A)$ and $KO^*(X,B)$ respectively, then their product naturally lives in $KO^*(X, A \cup B)$.

Proof. To see that (9) defines a K-cocycle, write (V,F), (W,G) locally as

$$(V,F)=(V',F')\oplus(V'',F''),$$
$$(W,G)=(W',G')\oplus(W'',G''),$$

where V', W' are vector bundles, and F'', G'' are invertible. We can then decompose $(V\otimes W,\, F\otimes 1+1\otimes G)$ as $(Z',H')\oplus(Z'',H'')$, with

$$Z'=V'\otimes W'$$
$$H''=\big(F'\otimes 1+1\otimes G''\big)\oplus\big(F''\otimes 1+1\otimes G'\big)\oplus\big(F''\otimes 1+1\otimes G''\big).$$

By Lemma 8, each summand of H'' is invertible. Thus so is H''. Since Z' is a vector bundle, (9) is indeed a K-cocycle. If F is trivial over A, and G is trivial over B, Lemma 8 also ensures that $F\otimes 1+1\otimes G$ is trivial over $A\cup B$.

If (V,F) and (W,G) come with $Cl(n)$ and $Cl(m)$ actions, then their product aquires an action of $Cl(n)\otimes Cl(m)$. Let $D=D_{n,m}$ be the bimodule constructed in Lemma 33, implementing the Morita equivalence between $Cl(n+m)$ and $Cl(n)\otimes Cl(m)$. We then get a product [PrK]

$$\begin{gathered}KO^{-n}\big(X\big)\times KO^{-m}\big(X\big)\to KO^{-n-m}\big(X\big)\\ \big[(V,F)\big]\cdot\big[(W,G)\big]:=\Big[\big(D\underset{Cl(n)\otimes Cl(m)}{\otimes}(V\otimes W),\,1_D\otimes(F\otimes 1+1\otimes G)\big)\Big],\end{gathered}\tag{10}$$

and its associativity is garanteed by the first part of Lemma 33.

We now show that this product is graded-commutative, i.e. that it satisfies

$$[(V,F)]\cdot[(W,G)]=(-1)^{nm}\,[(W,G)]\cdot[(V,F)].\tag{11}$$

For that purpose, we need to compare the modules $D_{n,m}\otimes(V\otimes W)$ and $D_{m,n}\otimes(W\otimes V)$. Let $\theta: Cl(n)\otimes Cl(m)\to Cl(m)\otimes Cl(n)$ denote the commutor isomorphism, and let $D^{\theta}_{m,n}:=D_{m,n}$ denote the $(Cl(n+m),\, Cl(n)\otimes Cl(m))$-bimodule, whose right action is precomposed by θ. The map $W\otimes V\to V\otimes W$ then induces a $Cl(n+m)$-module isomorphism

$$D_{m,n}\otimes(W\otimes V)\simeq D^{\theta}_{m,n}\otimes(V\otimes W),$$

intertwining the actions of $F\otimes 1+1\otimes G$ and $G\otimes 1+1\otimes F$. The graded commutativity follows from the second part of Lemma 33. □

5 Further properties of K-cocycles

In this section, we list some further properties of K-cocycles, that are of more technical nature. We begin with a sight strengthening of Lemma 14.

Lemma 23. *[Linc2] Let (V,F) be a $Cl(-n)$-linear K-cocycle. Let W be a quasi-bundle contained in V, that is invariant under F and under the action of $Cl(-n)$.*

If the restriction of F is invertible on $W^{\perp}$, then $(W,F|_W)$ is a K-cocycle and represents the same class as (V,F).

Proof. By Lemma 14, the only thing that we need to check is that $(W, F|_W)$ is a K-cocycle. Pick a point x in the base and let $\mathbf{W}_x = (\mathcal{U}_x, W_x)$ and $\mathbf{V}_x = (\mathcal{V}_x, V_x)$ be the corresponding germs of vector bundles. Since (V, F) is a K-cocycle, there is a neighborhood $\mathcal{U}$ of x and a decomposition

$$(V|_{\mathcal{U}}, F|_{\mathcal{U}}) = (V', F') \oplus (V'', F'')$$

with V' is a vector bundle and F'' and invertible operator. We have $V' \subset V_x$ around x. So we may modify V' and assume that $V' = V_x|_{\mathcal{U}}$. We can also assume that that $W_x|_{\mathcal{U}} \subset V_x|_{\mathcal{U}}$.

The operator F is invertible on $(V' \ominus W_x)|_{\{x\}}$. Since $V' \ominus W_x$ is a vector bundle, there is a neighborhood $\mathcal{V} \subset \mathcal{U}$ of x on which $F|_{V' \ominus W_x}$ is invertible. Consider the decomposition

$$(W, F) = (W_x, F|_{W_x}) \oplus (W \ominus W_x,\ F|_{W \ominus W_x})$$

on $\mathcal{V}$. To finish the proof, we need to show that F is invertible on $(W \ominus W_x)|_{\mathcal{V}}$. This is indeed the case since $W \ominus W_x$ is contained in $(V' \ominus W_x) \oplus V''$ and since F is invertible on both $(V' \ominus W_x)|_{\mathcal{V}}$ and $V''|_{\mathcal{V}}$. □

Recall that by Lemma 9, every quasi-bundle can be written as a union of vector bundles, where the union is taken over a coherent system of inclusions. The following extends of this result to K-cocycles.

Lemma 24. [coi] *Let (V, F) be a $Cl(k)$-linear K-cocycle on X. Then there exist an open cover $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ and rank n vector bundles $W_n \subset V$ that are F-invariant, $Cl(k)$-invariant, satisfy $V = \bigcup W_n$, and satisfy $W_n|_{\mathcal{U}_n \cap \mathcal{U}_m} \subset W_m|_{\mathcal{U}_n \cap \mathcal{U}_m}$ for $n < m$. Moreover, $\{\mathcal{U}_n\}$ can be chosen such that given any refinement $\{\mathcal{U}'_i\}$, $\mathcal{U}'_i \subset \mathcal{U}_{n(i)}$, the expression*

$$(W, F|_W), \qquad W := \bigcup W_{n(i)}|_{\mathcal{U}'_i},$$

is a K-cocycle, and represents the same class as (V, F).

Proof. By Lemma 9, there is a cover $\{\mathcal{V}_n\}_{n \in \mathbb{N}}$ of X, and vector bundles $W_n \subset V|_{\mathcal{V}_n}$ such that $\bigcup W_n = V$, and such that $W_n|_{\mathcal{V}_n \cap \mathcal{V}_m} \subset W_m|_{\mathcal{V}_n \cap \mathcal{V}_m}$ whenever $n < m$. Let $\mathcal{U}_n \subset \mathcal{V}_n$ be the biggest open subsets on which W_n is F-invariant, $Cl(k)$-invariant, and such that $F|_{W_n^\perp}$ is invertible. If $x \in X$ is a point over which V has ran n, then x necessarily belongs to $\mathcal{U}_n$. Hence, $V = \bigcup W_n$ as desired.

Now let $\{\mathcal{U}'_i\}$ be a refinement of $\{\mathcal{U}_n\}$. Since $F|_{\mathcal{U}'_i}$ is invertible on $W_{n(i)}^\perp$, and since every point belongs to some $\mathcal{U}'_i$, the result follows from Lemma 23. □

6 The suspension axiom

Let X be a well pointed space. In this section, we shall construct an isomorphism between $KO^{-n}(X, *)$ and $KO^0(\Sigma^n X, *)$. Here, $\Sigma^n X$ denotes the reduced suspension

$$\Sigma^n X := X \times I^n / X \times \partial I^n \cup \{*\} \times I^n.$$

By Lemma 19, we have $KO^0(\Sigma^n X, *) \simeq KO^0(X \times I^n, X \times \partial I^n \cup \{*\} \times I^n)$. So it is enough to prove the following:

Theorem 25. [Sus] *Let (X, A) be a pair of topological spaces, and let $I := [-1, 1]$. Then there exists an isomorphism*

$$KO^{n-m}(X, A) \simeq KO^n(X \times I^m,\ X \times \partial I^m \cup A \times I^m),$$

The following is a useful result about K-cocycles on spaces of the form $X \times I$.

Definition 26. *A K-cocycle in product form on $X \times I$ consists of a pair (W, F), where $W \to X$ is a $\mathbb{Z}/2$-graded quasi-bundle with scalar product, and F is an odd self adjoint operator on $W \times I$. Moreover, around every point of X, there should exist an orthogonal decomposition $W = W' \oplus W''$ with W' a vector bundle, and an invertible operator G on W'' inducing a decomposition*

$$(W \times I, F) = (W' \times I, F') \oplus (W'' \times I, G \times I).$$

A K-cocycle in product form is informally denoted $(W \times I, F)$.

Two K-cocycles in product form $(W_i \times I, F_i)$, $i = 0, 1$, are homotopic in froduct form if there exists a K-cocycle in product form $(\hat{W} \times I, \hat{F})$ over $[0, 1] \times Y$ such that $(\hat{W} \times I, \hat{F})|_{\{i\} \times Y} \simeq (W_i \times I, F_i)$ for $i = 0, 1$.

Lemma 27. [lara] *Let X be a space, and A a subspace of $X \times I$. Then the natural map* [PrF]

$$\left\{ \begin{matrix} Cl(-n)\text{-linear } K\text{-cocycles in prod-} \\ \text{uct form on } X \times I, \text{ trivial on } A \end{matrix} \right\} \Big/ \left\{ \begin{matrix} \text{homotopy in product} \\ \text{form, relatively to } A \end{matrix} \right\} \longrightarrow KO^n(X \times I, A) \tag{12}$$

is an isomorphism.

Proof. We first show that (12) is surjective. Let (V, F) be a K-cocycle on $X \times I$, trivial on A. Pick an open cover $\{\mathcal{U}_n\}$ of $X \times I$ as in Lemma 24, and chose a locally finite refinement of the form $\{\mathcal{V}_i \times (a_i, b_i)\}_{i \in J}$, for some opens $\mathcal{V}_i \subset X$. Let

$$W := \bigcup W_i|_{\mathcal{V}_i \times (a_i, b_i)},$$

where we have abbreviated $W_{n(i)}$ by W_i. By Lemma 24, $(W, F|_W)$ is then a K-cocycle and represents the same class as (V, F).

indexed by an ordered set J, and vector bundles $W_i \subset V|_{\mathcal{U}_i}$ such that $\bigcup W_i = V$, and such that $W_i|_{\mathcal{U}_i \cap \mathcal{U}_j} \subset W_j|_{\mathcal{U}_i \cap \mathcal{U}_j}$ whenever $i < j$.

Let $\mathcal{U}_i' \subset \mathcal{U}_i$ be the biggest open subsets on which W_i is invariant under F, invariant under the $Cl(-n)$-action, and such that $F|_{W_i^\perp}$ is invertible. Since $\bigcup W_i = V$, the sets $\mathcal{U}_i'$ also form an open cover of $X \times I$. Refine $\{\mathcal{U}_i'\}$ to a locally finite open cover $\{\mathcal{U}_i''\}$ whose elements are of the form $\mathcal{U}_i'' = \mathcal{V}_i \times (a_i, b_i)^*$ for some opens $\mathcal{V}_i \subset X$ and $(a_i, b_i)^* \subset I$. Here, our notation $(a, b)^*$ refers to the interior of $[a, b]$ in I, which is bigger than (a, b) if $a = -1$ or $b = 1$. Let

$$V' := \operatorname{Span}\{W_i|_{\mathcal{V}_i \times (a_i, b_i)}\}.$$

By Lemma 23, the K-cocycles (V, F) and $(V', F|_{V'})$ represent the same class in $KO^n(X \times I, A)$.

□

Given a quasi-bundle V with a scalar product and a self adjoint operator $F : V \to V$, we define $\mathbf{n}F : \ker(F)^\perp \to \ker(F)^\perp$ by

$$\mathbf{n}F := \frac{F}{|F|}.$$

It satisfies $(\mathbf{n}F)^2 = 1$. Note also that $\ker(F)^\perp = \ker(F^2)^\perp$ is a quasi-bundle by Lemma 10.

Proof of Theorem 25. By induction, it is enough to treat the case $m = 1$. An element of $KO^{n-1}(X, A)$ is represented by a $Cl(-n)$-linear K-cocycle (V, F), equipped with an extra $Cl(1)$-action that graded commutes with F

$$e : V \to V, \qquad e^2 = 1, \quad eF = -Fe.$$

Given such a K-cocycle, we can construct a K-cocycle (W, G) on $X \times I$ by letting the underlying $Cl(-n)$-linear quasi-bundle be $W := V \times I$, and letting the operator $G : W \to W$ act on the fiber $W_{x,t} = V_x$ by the formula

$$G_{x,t} := F_x + te_x.$$

That K-cocycle is trivial on $A \times I \cup X \times \partial I$, and thus defines a class in $K^n(X \times I, X \times \partial I \cup A \times I)$.

We now wish to construct the inverse homomorphism

$$K^n(X \times I,\, X \times \partial I \cup A \times I) \to KO^{n-1}(X, A).$$

For technical reasons, it shall be easier to construct a map with values in

$$K^{n-1}(X \times \{1\} \cup A \times [0, 1],\, A \times \{0\}). \tag{13}$$

Given a $Cl(-n)$-linear K-cocycle on $X \times I$ that is trivial on $X \times \partial I \cup A \times I$, then by Lemma 27, we may replace it by an equivalent one (W, G) whose underlying quasi-bundle is a product $W = \tilde{V} \times I$. The corresponding $Cl(-n) \otimes Cl(1)$-linear K-cocycle (V, F) on $X \times \{1\} \cup A \times [0, 1]$ is defined as follows. Its underlying quasi-bundle is given by

$$V_{x,t} := \ker(\mathbf{n}G_{x,t} - \mathbf{n}G_{x,-t})^\perp.$$

The odd self adjoint operator is

$$F_{x,t} := \frac{1}{2}(\mathbf{n}G_{x,t} + \mathbf{n}G_{x,-t}),$$

and the extra $Cl(1)$-action is given by

$$e_{x,t} := \mathbf{n}(\mathbf{n}G_{x,t} - \mathbf{n}G_{x,-t}).$$

We first note that the operators $\mathbf{n}G_{x,t}$ and $\mathbf{n}G_{x,-t}$ are globally defined for all $(x,t) \in X \times \{1\} \cup A \times [0,1]$. So $F_{x,t}$ and $e_{x,t}$ are well defined on $V_{x,t}$. It is then an easy exercise to check that $(\mathbf{n}G_{x,t} + \mathbf{n}G_{x,-t})$ and $(\mathbf{n}G_{x,t} - \mathbf{n}G_{x,-t})$ graded commute, from which it follows that $F_{x,t}$ and $e_{x,t}$ also graded commute. [As constructed, (V,F) is not going to be a K-cocycle. I have some ideas how to fix all that, but it needs more work...] The K-cocycle (V,F) begin trivial over $A \times \{0\}$, it defines a class in (13).

It remains to check that the assignments $(V,F) \mapsto (W,G,e)$ and $(W,G,e) \mapsto (V,F)$ are homotopy inverses. This is done by writing down explicit homotopies. [That whole proff still depends on Lemma 27, so there is no point in writing down all the details...] □

7 The connecting homomorphism

Given an NDR pair, namely a pair of topological spaces $A \subset X$, such that A has neighborhood U in X that deformation retracts back to A, we shall construct a homomorphism $\delta : KO^{n-1}(A) \to KO^n(X,A)$.

8 Comparison with vector bundles

[secVB]

Unlike our theory, KO^*_{Atiyah} is only a cohomology theory when restricted to compact spaces. So one cannot expect the map $KO^0_{\text{Atiyah}}(X) \to KO^0(X)$ to be an isomorphism when X is not compact. In this section, we will prove:

Theorem 28. *[thm:VB] Let X be a compact space. Then the map [comparison]*

$$KO^0_{\text{Atiyah}}(X) \to KO^0(X) \tag{14}$$

induced by (8) is an isomorphism.

For technical reasons, it shall be convenient to work with a slightly stricter notion of K-cocycle.

Definition 29. *If a K-cocycle (V,F) has the property that the operators F'' of Definition 11 are orthogonal operators, then we call it a strict K-cocycle.*

The following lemma says that any K-cocycle can be deformed to a strict cocycle.

Lemma 30. *[lem:ortho]* *Let (V, F) be a $Cl(n)$-linear K-cocycle over a space X. Then $F_0 := F$ can be deformed through a family F_t, $t \in [0,1]$, of odd, self adjoint, $Cl(n)$-linear operators in such a way that the following conditions are satisfied:*

For each point $x \in X$, there is a neighborhod $\mathcal{N}$ of x, and a decomposition $V|_{\mathcal{N}} = V' \oplus V''$, inducing corresponding decompositions [VN']

$$\big(V|_{\mathcal{N}}, F_t|_{\mathcal{N}}\big) = \big(V', F_t'\big) \oplus \big(V'', F_t''\big), \tag{15}$$

such that V' is a vector bundle, F_t'' is invertible for all $t \in [0,1]$, and F_1'' is an orthogonal operator. Moreover, if $F_0''|_{\{y\}}$ was orthogonal for some $y \in N$, then $F_t''|_{\{y\}} = F_0''|_{\{y\}}$ for all t.

Proof. Let $\{\mathcal{U}_i\}_{i\in I}$ be a locally finite open cover of X for which we have decompositions

$$\big(V|_{\mathcal{U}_i}, F|_{\mathcal{U}_i}\big) = \big(W_i', F_i'\big) \oplus \big(W_i'', F_i''\big),$$

with W_i' a vector bundle, and F_i'' an invertible operator. Let $\{\varphi_i : X \to \mathbb{R}_{\geq 0}\}$ be a partition of unity such that φ_i has support in $\mathcal{U}_i$. Let

$$H_i := \begin{cases} 1_{W_i'} \oplus |F_i''|^{-\varphi_i} & \text{over } \mathcal{U}_i \\ 1_V & \text{over } X \setminus \operatorname{supp}(\varphi_i), \end{cases}$$

$$\tilde{F}_t := F \cdot \prod_{i\in I} H_i^t, \qquad F_t := \tfrac{1}{2}(\tilde{F}_t + \tilde{F}_t^*),$$

where we have picked an order on I to make sense of the product. The operator F_t is clearly odd, self adjoint, and $Cl(n)$-linear. The existence of the adjoint $\tilde{F}_t^*$ follows from the special form of $\tilde{F}_t$.

Given a point $x \in X$, we now describe the neighborhood $\mathcal{N}$ of x, and the decomposition (15). Since V is a quasi-bundle, we have a germ $\mathbf{V}_x = (\mathcal{U}_x, V_x)$ around x, and an inclusion $\mathbf{V}_x \hookrightarrow V_{\langle x\rangle}$. Pick a representative $V_x|_{\mathcal{U}} \hookrightarrow V|_{\mathcal{U}}$ of that inclusion, and define

$$\mathcal{N}_i := \big\{ y \in \mathcal{U} : W_i'|_{\{y\}} \subset V_x|_{\{y\}} \big\}$$

for all i such that $x \in \mathcal{U}_i$. The set $\mathcal{N}_i$ is a neighborhood of x because $W_i' \to V|_{\mathcal{U}_i}$ is a morphism of quasi-bundles. Letting $I_x := \{i \in I \mid x \in U_i\}$, we define

$$\mathcal{N} := \bigcap_{i\in I_x} \mathcal{N}_i \cap \bigcap_{i\notin I_x} (X \setminus \operatorname{supp}(\varphi_i)),$$
$$V' := V_x|_{\mathcal{N}}, \qquad V'' := (V')^{\perp}.$$

By the definition of $\mathcal{N}_i$, we have $W_i'|_{\mathcal{N}_i} \subset V'|_{\mathcal{N}_i}$ for all $i \in I_x$. By taking orthogonal complements, it follows that $V'' \subset W_i''|_{\mathcal{N}}$, and hence that $H_i|_{V''} = \big|F|_{V''}\big|^{-\varphi_i}$ for $i \in I_x$. Since $\mathcal{N}$ doesn't intersect the support of φ_i for $i \notin I_x$, we have

$$\prod_{i\in I} H_i|_{V''} = \prod_{i\in I_x} H_i|_{V''} = \prod_{i\in I_x} \big|F|_{V''}\big|^{-\varphi_i} = \big|F|_{V''}\big|^{-\Sigma\varphi_i} = \big|F|_{V''}\big|^{-1}.$$

From the above expression, we see that $F''_t = F_t|_{V''}$ is given by

$$F''_t = \frac{F|_{V''}}{\left|F|_{V''}\right|^t}.$$

This operator is invertible for $t \in [0,1]$, orthogonal for $t = 1$, and independent of t whenever F''_0 is orthogonal. $\square$

Corollary 31. [c:or] *Modifying Definition 12 by only allowing strict K-cocycles does not affect the groups $KO^n(X)$.*

Proof. Let us call KO' the K-theory groups defined using strict K-cocycles. The forgetful map $KO'^n(X) \to KO^n(X)$ is surjective by Lemma 30. To see that it is also injective, consider two strict K-cocycle whose image agrees in $KO^n(X)$. By applying Lemma 30 to the homotopy, we see that their images already agreed in $KO'^n(X)$. $\square$

Given a strict K-cocycle (V, F), we define a *presentation* to be an open cover $\{\mathcal{U}_i\}$, along with a family of orthogonal direct sum decompositions

$$(V|_{\mathcal{U}_i}, F|_{\mathcal{U}_i}) = (V'_i, F'_i) \oplus (V''_i, F''_i),$$

where V'_i are vector bundles, and F''_i are orthogonal operators.

We now show that, modulo replacing a strict K-cocycle by an equivalent one, we can always embed it in a vector bundle.

Lemma 32. [lem:emb] *Let X be a compact space, and let (V, F) be a strict K-cocycle over X. Then there exists a strict sub-cocycle $(W, G) \subset (V, F)$ such that $F|_{W^\perp}$ is an orthogonal operator, and such that W is isometrically embeddable in a trivial vector bundle $X \times \mathbb{R}^{n|m}$.*

Proof. Let $(\{\mathcal{U}_i\}, (V'_i, F'_i), (V''_i, F''_i))$ be a presentation of (V, F). Without loss of generality, we may assume that the bundles V'_i are tirvial:

$$V'_i = \mathcal{U}_i \times \mathbb{R}^{n_i|m_i},$$

and that the cover $\{\mathcal{U}_i\}$ is finite. Let $\{\varphi_i : X \to \mathbb{R}_{\geq 0}\}$ be a partition of unity with $\mathrm{supp}(\varphi_i) \subset \mathcal{U}_i$, and let us define operators $H_i : V \to X \times \mathbb{R}^{n_i|m_i}$ by

$$H_i := \begin{cases} \varphi_i \cdot 1_{V'_i} \oplus 0 & \text{over } \mathcal{U}_i \\ 0 & \text{over } X \setminus \mathrm{supp}(\varphi_i). \end{cases}$$

Note that the adjoint $H_i^* : X \times \mathbb{R}^{n_i} \to V$ is also a morphism of quasi-bundles. Adding all the H_i, we get a map

$$H : V \to X \times \mathbb{R}^{\Sigma n_i|\Sigma m_i}$$

which, once again, admits an adjoint.

Let W be the orthogonal complement of $\ker(H) = \ker(H^*H)$; it is a quasi-bundle by Lemma 10. Since H commutes with F, the latter restricts to an operator G on W.

We now verify that (W, G) is a strict K-cocycle, and that $F|_{W^\perp}$ is an orthogonal operator. We check these facts on the opens $\mathcal{V}_i := \varphi_i^{-1}(\mathbb{R}_{>0})$. For the first condition, we have the decomposition

$$W|_{\mathcal{V}_i} = V_i'|_{\mathcal{V}_i} \oplus (V_i'' \cap W)|_{\mathcal{V}_i},$$

where $V_i|_{\mathcal{V}_i}$ is a vector bundle, and where the restriction of $G|_{\mathcal{V}_i}$ to the second summand is orthogonal. For the second condition, we note that

$$W^\perp|_{\mathcal{V}_i} = \ker(H)|_{\mathcal{V}_i} \subset \ker(H_i)|_{\mathcal{V}_i} = V_i''|_{\mathcal{V}_i}$$

and that $F|_{V_i''}$ is an orthogonal operator.

We now show that W can be isometrically embedded in $X \times \mathbb{R}^{\Sigma n_i|\Sigma m_i}$. The operator H is injective on W, but typically not an isometry. However, the restriction of H^*H to W is an isomorphism by Lemma 7.a, and so it makes sense to write

$$H' := H|_W \cdot \big((H^*H)|_W\big)^{-1/2} : W \to X \times \mathbb{R}^{\Sigma n_i|\Sigma m_i}.$$

The latter is an isometric operator. □

Proof of Theorem 28. We first show that the map (14) is surjective. Let (V, F) be a K-cocycle. By Corollary 31, we may assume that (V, F) is a strict K-cocycle, and by Lemmata 32 and 14, we may assume that V embeds isometrically into a trivial vector bundle $X \times \mathbb{R}^{n|m}$. Let us write V and F as

$$V = V_0 \oplus V_1, \qquad F = \begin{pmatrix} 0 & F_1 \\ F_0 & 0 \end{pmatrix},$$

where V_0, V_1 are the even and odd parts of V, and where $F_0 : V_0 \to V_1$, $F_1 : V_1 \to V_0$ are the components of F. Let $(\{\mathcal{U}_i\}, (V_i', F_i'), (V_i'', F_i''))$ be a presentation of (V, F), and let

$$V_i' = V_{i,0}' \oplus V_{i,1}', \quad V_i'' = V_{i,0}'' \oplus V_{i,1}'',$$

be the corresponding decompositions. Let ι denote the embedding $V_0 \hookrightarrow X \times \mathbb{R}^n$, and define

$$W_0 := X \times \mathbb{R}^n = \mathit{Pushout}\Big(V_0 \leftarrow \mathrm{Span}\{V_{i,0}''\} \xrightarrow{\iota} \mathrm{Span}\{\iota(V_{i,0}')^\perp\}\Big)$$
$$W_1 := \mathit{Pushout}\Big(V_1 \xleftarrow{F_0} \mathrm{Span}\{V_{i,0}''\} \xrightarrow{\iota} \mathrm{Span}\{\iota(V_{i,0}')^\perp\}\Big).$$

Clearly, W_0 is a vector bundle; we will soon show that this also holds for W_1.

Let W be the $\mathbb{Z}/2$-graded object with even part W_0 and odd part W_1. Since $F|_{\mathrm{Span}\{V_{i,0}''\}}$ is an orthogonal operator, we have $(F_1 \circ F_0)|_{\mathrm{Span}\{V_{i,0}''\}} = 1$, and so the vertical arrows in

$$\begin{array}{ccccc}
V_0 \oplus V_1 & \xleftarrow{1\oplus F_0} & \mathrm{Span}\{V''_{i,0}\} \oplus \mathrm{Span}\{V''_{i,0}\} & \xrightarrow{\iota\oplus\iota} & \mathrm{Span}\{\iota(V'_{i,0})^\perp\} \oplus \mathrm{Span}\{\iota(V'_{i,0})^\perp\} \\
\Big\downarrow F & & \Big\downarrow \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) & & \Big\downarrow \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \\
V_0 \oplus V_1 & \xleftarrow{1\oplus F_0} & \mathrm{Span}\{V''_{i,0}\} \oplus \mathrm{Span}\{V''_{i,0}\} & \xrightarrow{\iota\oplus\iota} & \mathrm{Span}\{\iota(V'_{i,0})^\perp\} \oplus \mathrm{Span}\{\iota(V'_{i,0})^\perp\}
\end{array}$$

induce a map $G : W \to W$. Consider the decomposition

$$W_1|_{\mathcal{U}_i} = V'_{i,1} \oplus (V'_{i,1})^\perp.$$

Each restriction $G|_{\mathcal{U}_i}$ is an orthogonal operator on $(V'_{i,1})^\perp$, and so we have an isomorphism $(V'_{i,0})^\perp \simeq (V'_{i,1})^\perp$. The former being a vector bundle, so is the latter. It follows that $W_1|_{\mathcal{U}_i} = V'_{i,1} \oplus (V'_{i,1})^\perp$ is a vector bundle. The $\mathcal{U}_i$ form an open cover, hence W_1 is a vector bundle.

We have an obvious embedding $(V, F) \hookrightarrow (W, G)$, and the restriction of G to the complement of V is an orthogonal operator. So by Lemma 14, the two cocycles (W, G) and (V, F) are equal in K-theory. We have thus shown that (V, F) lies in the image of (14).

It remains to show that the map (14) is injective. Let $[V_0] - [V_1]$ be a class in $KO^0_{\text{Atiyah}}(X)$ whose image is zero in $KO^0(X)$. By definition, this means that we have a K-cocycle $(\tilde{V}, \tilde{F})$ over $X \times [0,1]$ whose restriction to $X \times \{0\}$ is $(V_0 \oplus V_1, 0)$ and whose restriction to $X \times \{1\}$ is trivial. As before, we may assume that $(\tilde{V}, \tilde{F})$ is strict and that $\tilde{V}$ embeds in a trivial vector bundle. Applying the same tricks as in the first part of the proof, we construct an embedding of K-cocycles [tel]

$$(\tilde{V}, \tilde{F}) \hookrightarrow (\tilde{W}, \tilde{G}), \tag{16}$$

where $\tilde{W}$ is a vector bundle and $\tilde{G}|_{\tilde{V}^\perp}$ is invertible.

Since $\tilde{W}$ is a vector bundle over $X \times [0,1]$, we can write it as a product $\tilde{W} = W \times [0,1]$, where W is a vector bundle over X. Moreover, since $\tilde{G}|_{X\times\{1\}}$ is invertible, the even and odd parts of W are isomorphic; let us call them Z. Restricting (16) over $X \times \{0\}$, we thus get an embedding

$$\iota : V = V_0 \oplus V_1 \hookrightarrow W = Z \oplus Z.$$

Since the complement of $\iota(V)$ is equipped with as invertible odd operator, we also get an isomorphism between the even and odd parts of $\iota(V)^\perp$; let us call them Y. Thus, we have isomorphisms

$$V_0 \oplus Y \simeq Z, \qquad V_1 \oplus Y \simeq Z.$$

It follows that $[V_0]$ and $[V_1]$ are equal in $KO^0_{\text{Atiyah}}(X)$. $\square$

Appendix

In this appendix, we show that we can pick the bimodules (4), so that they satisfy certain nice compatibility properties.

Lemma 33. [ZCl] *The bimodules $D_{n,m}$ can be chosen so that for any triple $n, m, r \in \mathbb{Z}$, one has bimodule isomorphisms* [Das]

$$D_{n+m,r} \underset{Cl(n+m)\otimes Cl(r)}{\otimes} (D_{n,m} \otimes Cl(r)) \;\simeq\; D_{n,m+r} \underset{Cl(n)\otimes Cl(m+r)}{\otimes} (Cl(n) \otimes D_{m,r}). \tag{17}$$

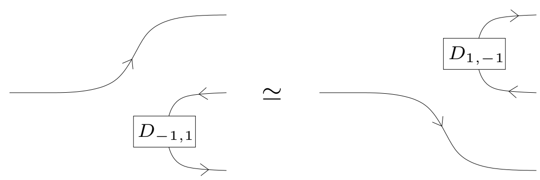
Letting $\theta_{n,m} : Cl(n) \otimes Cl(m) \to Cl(m) \otimes Cl(n)$ denote the commutor isomorphism, and $D^{\theta}_{m,n}$ be the bimodule $D_{m,n}$ with right action precomposed by $\theta_{n,m}$, we then have [nm]

$$D^{\theta}_{m,n} \simeq \begin{cases} D_{n,m} & \text{if } nm \text{ is even,} \\ D_{n,m} \otimes \mathbb{R}^{0|1} & \text{if } nm \text{ is odd.} \end{cases} \tag{18}$$

Proof. If n and m have same sign, we let $D_{n,m} := Cl(n+m)$, with the obvious actions. Pick a bimodule $D_{1,-1}$ implementing (2). The bimodule $D_{-1,1}$ is then uniquely determined by the equation

$${}_{Cl(1)}\big(Cl(1) \otimes D_{-1,1}\big)_{Cl(1)\otimes Cl(-1)\otimes Cl(1)} \simeq {}_{Cl(1)}\big(D_{1,-1} \otimes Cl(1)\big)_{Cl(1)\otimes Cl(-1)\otimes Cl(1)}.$$

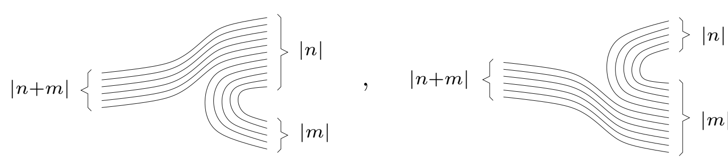
That last equation is best understood graphically: [biY]



(19)

If $n > 0$, $m < 0$, we let $D_{n,m}$ be a tensor product of $Cl(n+m)$ with $\min(n, -m)$ copies of $D_{1,-1}$. And if $n < 0$, $m > 0$, we define it as tensor product of $Cl(n+m)$ with $\min(-n, m)$ copies of $D_{-1,1}$. Graphically, this becomes

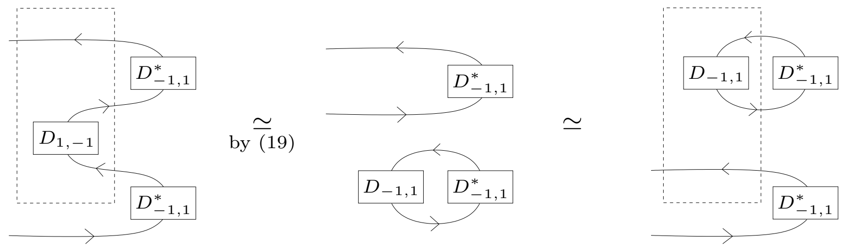
$D_{n,m} :=$



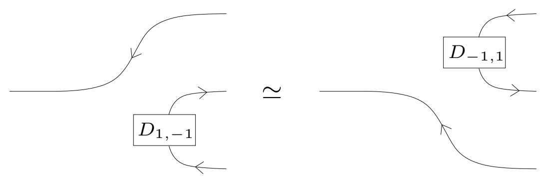
if $|n| \geq |m|$ if $|n| \leq |m|$

where the orientations of the lines depend on the signs of n and m, and the little boxes are implicit.

Let $D^*_{-1,1}$ denote the inverse bimodule of $D_{-1,1}$, with defining equation $D_{-1,1} \otimes_{Cl(-1)\otimes Cl(1)} D^*_{-1,1} \simeq \mathbb{R}$. The graphical computation

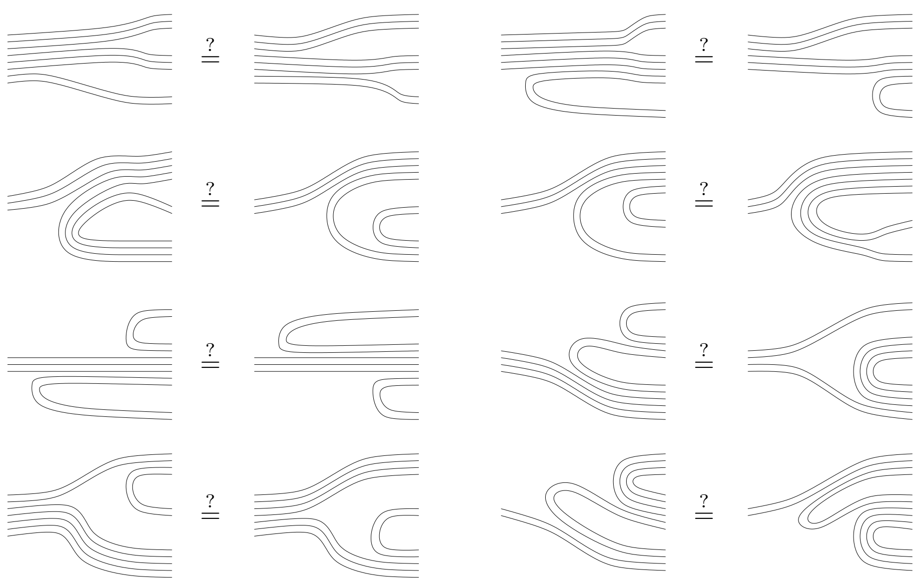


then implies the relation [biY2]



(20)

dual to (19). Armed with (19) and (20), it is now easy to check (17) case by case. Depending on the relative sizes of n, m, and r, the graphical representation of equation (17) is one of the following types (modulo vertical flip):



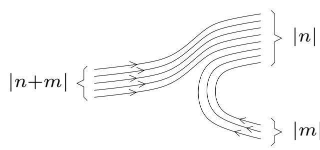
The first five are obviousely true; the last three follow from (19) and (20).

We now proceed to show (18). We use the notation $\bar{V} := V \otimes \mathbb{R}^{0|1}$. If n and m have same sign, then $\theta_{n,m}$ is a composite of nm transpositions, and so it is enough to show (18) for $|n| = |m| = 1$. In that case, the isomorphism can be constructed explicitely as

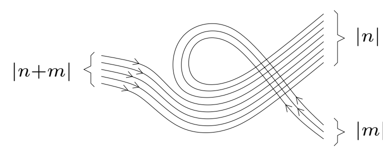
$$D^\theta_{1,1} = Cl(2) \to \bar{D}_{1,1} = \bar{C}l(2) : \begin{array}{cc} 1 \mapsto e_1 + e_2, & e_1 \mapsto 1 + e_1e_2, \\ e_2 \mapsto 1 + e_2e_1, & e_1e_2 \mapsto e_1 - e_2. \end{array}$$

$$D^\theta_{-1,-1} \to \bar{D}_{-1,-1} = \bar{C}l(-2) : \begin{array}{cc} 1 \mapsto f_1 + f_2, & f_1 \mapsto -1 + f_1f_2, \\ f_2 \mapsto -1 + f_2f_1, & f_1f_2 \mapsto f_2 - f_1. \end{array}$$

If n and m have different signs, then $D_{n,m}$ and $D^\theta_{m,n}$ can be represented (modulo vertical flip, and reorientation of the strands) by

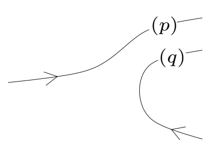


and

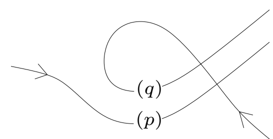


.

Let us simplify the above notation to



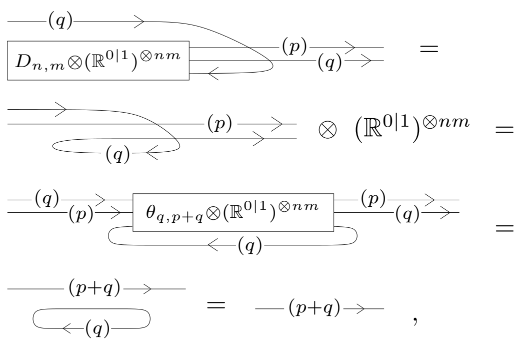
and



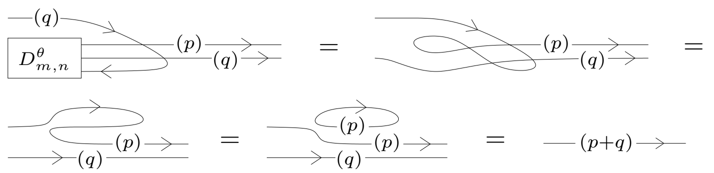
,

where $p = |n + m|$ and $q = |m|$ denote the multiplicities.

Equation (18) follows from the following two graphical computations. We first evaluate



where the third equality follows from our previous computation and the fact that $(-1)^{q(p+q)} = (-1)^{nm}$. We then evaluate



where the third equality is given by p applications of (19). By comparing the above two computation, we deduce that $D^{\theta}_{m,n} = D_{n,m} \otimes (\mathbb{R}^{0|1})^{\otimes nm}$. □

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