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75 open problems on bicommutant categories

Outline of a research programme[1]

André Henriques

Foundations

Bicommutant categories are categorifed von Neumann algebras. The latter admit both a concrete definition (as sub-$*$-algebras $A \subset B(H)$ satisfying $A'' = A$), and an abstract definition (then referred to as 'W^*-algebras'). There similalry exist two competing definition of bicommutant categories: one concrete, and one abstract. It is evident that every abstract bicommutant category is an instance of a concrete bicommutant category, but the converse is currently not clear.
Objective 1: Find a suitable set of extra conditions on concrete bicommutant categories so as to make the definition exactly equivalent to abstract bicommutant categories.

Modules over von Neumann algebras satisfy some remarkable properties, not shared by modules over other types of algebras. We expect module categories over bicommutant categories to exhibit similar properties. If H is a module over some von Neumann algebra A, then A decomposes as $A = A_1 \oplus A_2$, with A_1 the kernel, and A_2 acting faithfully on H.
Objective 2: Prove that if C is a module category for a bicommutant category T, then T decomposes as $T_1 \oplus T_2$, with T_1 acting by zero on C, and T_2 acting faithfully.
If H and K are faithful modules over a von Neumann algebra A, then $\mathrm{Hom}_A(H, K) \neq 0$.
Objective 3: Prove that if C_1 and C_2 are module categories for a bicommutant category T, with T acting faithfully, then $Func_T(C_1, C_2) \neq 0$.

If a W^*-algebra A acts faithfully on a Hilbert space, then its commutant A' is again a W^*-algebra, and $A'' = A$. The corresponding statement for bicommutant categories is non-trivial (e.g., it involves checking that the commutant category has absorbing objects).
Objective 4: Prove that if a bicommutant category T acts faithfully on a module category C, then its commutant $T' := \mathrm{End}_T(C)$ admits absorbing objects.
Objective 5: In the above setup, prove T' is again a bicommutant category, and $T'' = T$.

W^*-algebras are C^*-algebras that admit preduals. One way to describe the predual A_* of a W^*-algebra is $A_* = L^2A \otimes_A L^2A$ (algebraic tensor product), where L^2A denotes the standard form of A. If T is a bicommutant category, the analog of the the standard form is the ideal of absorbing objects $T_{abs} \subset T$. Let $T_* := T_{abs} \otimes_T T_{abs}$, where the tensor product is the relative version of the maximal tensor product of C^*-categories.
Objective 6: Prove that if T is a bicommutant category, then T_* is a predual in the sense that $Func_{ln}(T_*, \mathrm{Hilb}) = T$, were $Func_{ln}$ denotes the set of locally normal functors.

The ideal of absorbing objects $T_{abs} \subset T$ plays a prominent role in the theory of bicommutant categories, it is therefore important to understand when $T_{abs} = T$
Objective 7: Prove that if 1_T is simple and $T_{abs} = T$ (equivalently $1_T \in T_{abs}$), then T is the unitary ind-completion of a rigid C^* tensor category.
Objective 8: Classify bicommutant categories that satisfy $T_{abs} = T$.

[1] I will not update this document to record partial progress towards completion of this research programme. For information, contact me at *andre.g.henriques@gmail.com*.

Expected examples

So far, few examples of bicommutant categories that have been established. These are the categories of solitonic representations of rational conformal nets, and the unitary ind-completions of unitary fusion categories. We expect there are many more examples.

Objective 9: Prove that the following are examples of bicommutant categories:
- The category $(\mathrm{Bim}(A), \boxtimes_A)$ of bimodules over a von Neumann algebra A.
- The category of measurable bundles of Hilbert spaces over some measure space X.
- The category G-equivariant measurable bundles of Hilbert spaces, for some $G \circlearrowright X$.
- The category of representations of (= equivariant bundles over) a measurable groupoid.
- The category of unitary representations of a discrete group / locally compact group.
- The category $\mathrm{Hilb}[G]$ of G-graded Hilbert spaces, for G a discrete group.
- The category $\mathrm{Hilb}^\omega[G]$ of G-graded Hilbert spaces with associator twisted by a 3-cocycle.
- The category $\mathrm{Hilb}[G] := C_0(G)\text{-Mod}$, for G a locally compact group.
- The category $\mathrm{Hilb}^\omega[G]$ for G locally compact, where ω is a multipicative gerbe on G.
- The category $\mathrm{Hilb}^\omega_H[G]$, for H a locally compact group acting on G and preserving ω.
- The category of representation of a locally compact quantum group.
- The unitary ind-completion of a rigid C^* tensor category.
- The category of representations of the tube algebra of a rigid C^* tensor category.
- The category of representations of a conformal net $\mathcal{A}$.
- The category of solitonic representation of a conformal net $\mathcal{A}$.
- The category of $\mathcal{B}$-topological solitonic representation of $\mathcal{A}$, for $\mathcal{B} \subset \mathcal{A}$ a subnet.
- The category of unitary topological line defects in a unitary $d = 2$ QFT (or even $d \geq 3$?)

In [Hen17], we showed that the category $\mathrm{Sol}(\mathcal{A})$ of solitonic representations of a rational conformal net $\mathcal{A}$ is a bicommutant category, and that its Drinfel'd center is $\mathrm{Rep}(\mathcal{A})$. We conjecture that this result also holds true in the absence of the rationality assumption.

Objective 10: Prove that for every conformal net $\mathcal{A}$ the category $\mathrm{Sol}(\mathcal{A})$ is a bicommutant category, and that $Z(\mathrm{Sol}(\mathcal{A})) = \mathrm{Rep}(\mathcal{A})$.

More generally, given an inclusion of conformal nets $\mathcal{B} \subset \mathcal{A}$, one can consider the category $\mathrm{Sol}_\mathcal{B}(\mathcal{A})$ of $\mathcal{B}$-topological solitons of $\mathcal{A}$.

Objective 11: Prove that $\mathrm{Sol}_\mathcal{B}(\mathcal{A})$ is a bicommutant category, and that it is equivalent to the relative commutant of $\mathrm{Sol}(\mathcal{B})$ inside $\mathrm{Sol}(\mathcal{A})$. Compute its center.

Disintegration

Commutative C^*/W^*-algebras correspond bijectively to locally compact topological spaces/measure spaces by the Gelfand–Naimark theorem. The analogous question for bicommutant categories concerns symmetric (='commutative') bicommutant categories. If $G = (G_1 \rightrightarrows G_0)$ is a measurable groupoid, we expect $\mathrm{Rep}(G)$, the category of measurable Hilbert bundles over G_0 equipped with an action of G_1, to be a bicommutant category.

Objective 12: Show that for G a measurable groupoid $\mathrm{Rep}(G)$ is a bicommutant category.

To go the other way, we must work over super-vector spaces, because $\mathrm{sVec}_\mathbb{C}$ is the algebraic colsure of $\mathrm{Vec}_\mathbb{C}$. (The Gelfand–Naimark theorem is typically stated for algebras over $\mathbb{C}$, as opposed to over $\mathbb{R}$.) In the next two objectives, everything is implicitly 'super':

Objective 13: Prove that for every symmetric bicommutant category T there exists a measurable groupoid G such that $T = \mathrm{Rep}(G)$.

The category $\mathrm{Rep}(G)$ admits a fiber functor to $L^\infty(G_0)\text{-Mod}$. A first step would thus be:

Objective 14: Prove that every symmetric bicommutant category T admits a fiber functor $T \to L^\infty(X)\text{-Mod}$, for some measure space X.

If A is a von Neumann algebra, X is a measure space, and $L^\infty(X) \to A$ is a map that lands in the center of A, then one can construct a measurable bundle of von Neumann algebra $\{A_x\}_{x\in X}$ such that $A = \int_{x\in X}^{\oplus} A_x$. The following is a simultaneous generalization of disintegration, and of de-equivariantization:
Objective 15: Prove that if T is a bicommutant category, $G = (G_1 \rightrightarrows G_0)$ is a measurable groupoid, and $\mathrm{Rep}(G) \to Z_1(T)$ is a braided functor, then one can form a 'measurable bundle' of bicommutant categories over G_0 along with an action of G_1 such that T is equivalent to the category of measurable sections equipped with an action of G_1.

Continuous gradings, continuous actions

G-gradings and G-actions are fundamental in the theory of fusion categories. When applied to bicommutant categories, we expect these two notions can be generalised to the case when G is a locally compact group.
Objective 16: Define what it means for a locally compact group G to act on a bicommutant category. Prove that the category of equivariant objects is again a bicommutant category.
Objective 17: Define what it means for a bicommutant category T to be graded by a locally compact group G. Show that if $\varphi : H \to G$ is a continuous group homomorphism, then we can pull back T along φ to obtain an H-graded bicommutant category.

The notions of G-action and G-grading (where G is a locally compact group) should be special cases of actions of locally compact quantum groups on bicommutant categories.
Objective 18: Define what it means for a locally compact quantum group $\mathbb{G}$ to act on a bicommutant category, and show it encompases the notions of G-actions and G-gradings.

If $\mathbb{G}$ is a locally compact quantum group, then $\mathrm{Rep}(\mathbb{G})$ acts on Hilb via the fiber functor. We expect that the commutant category $\mathrm{End}_{\mathrm{Rep}(\mathbb{G})}(\mathrm{Hilb})$ can be naturally identified with the category $\mathrm{Rep}(\hat{\mathbb{G}})$ of representations of the dual locally compact quantum group $\hat{\mathbb{G}}$. The category $\mathrm{Rep}(\mathbb{G})$ would thus be an example of a concrete bicommutant category.
Objective 19: Prove that $\mathrm{Rep}(\mathbb{G})$ acting on Hilb is a bicommutant category.
We guess all concrete bicommutant categories on Hilb arise from the above construction:
Objective 20: Prove that concrete bicommutant categories on Hilb are the same thing as locally compact quantum groups.

The celebrated result of [ENO10] states that G-graded fusion categories are classified (up to equivalence) by homotopy classes of maps from BG to classifiying space of the Brauer-Picard 3-groupoid. We expect an analogous result to hold true in the context of bicommutant categories, where G is now allowed to be a locally compact group.
Objective 21: Define the Brauer-Picard 3-stack of bicommutant categories and invertible bimodule categories, and prove that stack maps from BG to the Brauer-Picard 3-stack correspond bijectively to G-graded bicommutant categories.

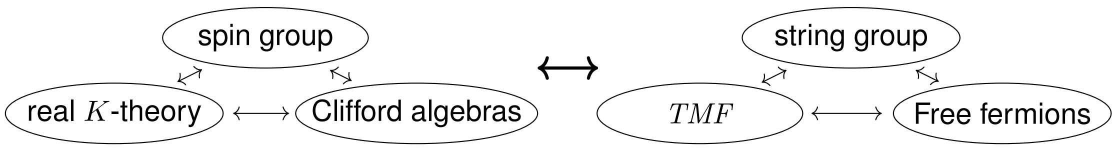
2-groups

Pointed unitary fusion categories are classified by a finite group G, together with a 3-cocycle in $H^3(G, U(1))$. If G is a locally compact group, then a class $[\omega] \in H^3(G, U(1))$ is represented geometrically by a multiplicative gerbe on G. For ω as above, we will construct a W^*-tensor category $\mathrm{Hilb}^\omega[G]$. Its underlying category is the category of representation of a continuous trace C^*-algebra with spectrum G.
Objective 22: Prove that $\mathrm{Hilb}^\omega[G]$ is a bicommutant category.
We expect all pointed bicommutant categories to be of the above form.
Objective 23: Define what it means for a bicommutant category to be 'pointed', and prove that pointed bicommutant categories are classified by a locally compact group G, together with a multiplicative gerbe on G.

The String group $\mathrm{String}(n)$ is the next group after $\mathrm{Spin}(n)$ in the Whitehead tower of $SO(n)$. It's a 2-group whose homotopy type is that of the 3-connected cover of $SO(n)$. If V is a finite dimensional real Hilbert space, one can form the free fermion conformal net based on V, which one then uses to construct a bicommutant category $\mathrm{Fer}(V)$. The construction $V \mapsto \mathrm{Fer}(V)$ is a higher categorical analog of the Clifford algebra construction. A folklore theorem says that a spin structure on an n-dimensional vector space V is equivalent to the data of a Morita equivalence between $\mathrm{Cliff}(V)$ and $\mathrm{Cliff}(\mathbb{R}^n)$.
Objective 24: Prove that a string structure on an n-dimensional vector space V is equivalent to a Morita equivalence between $\mathrm{Fer}(V)$ and $\mathrm{Fer}(\mathbb{R}^n)$.
Objective 25: Find a direct construction of the bicommutant category $\mathrm{Fer}(V)$ (ideally one which resembles the Clifford algebra construction), other that the category of solitonic representations of the free fermion conformal net associated to V.

Homotopy theory

Free fermions appear prominently in Stolz-Teichner's conjectural description of TMF, and play the same role as Clifford algebras do in the description of real K-theory. The cohomology theory TMF is string-oriented by the celebrated work of Ando-Hopkins-Rezk, while real K-theory is spin-oriented by Atiyah-Bott-Schapiro. Finally, we conjecture that free fermions can be used to describe string structures similarly to the way Clifford algebras can be used to describe spin structures. Two strongly analogous pictures emerge:



The 8-fold Morita periodicity of real Clifford algebras is directly related to the Bott periodicity of real K-theory. It is thus reasonable to ask whether the free fermions bicommutant categories exhibit a periodicity mirroring the 576-fold Bott periodicity[2] of TMF.
Question 26: Do the bicommutant categories $\mathrm{Fer}(\mathbb{R}^n)$ exhibit a Morita periodicity similar to the 8-fold Morita periodicity of real Clifford algebras?

The Dixmier-Douady classification by $H^3(-, \mathbb{Z})$ of stable continuous trace C^*-algebras holds true because $\mathrm{Aut}(\mathcal{K}(H)) = PU(H)$ is a $K(\mathbb{Z}, 2)$. Taking bundles with fiber $B(H)$ as

[2] $576 = 24 \times 24$.

opposed to $\mathcal{K}(H)$ yields the same classification. The higher categorical analog of the infinite dimensional Hilbert space H is the W^*-category of modules over the hyperfinite III_1 factor R. The analog of $B(H)$ is $\mathrm{Bim}(R)$. And the analog of $\mathrm{Aut}(B(H))$ is $\mathrm{Aut}(\mathrm{Bim}(R))$.
Objective 27: Prove that $\mathrm{Aut}(\mathrm{Bim}(R))$ is a group (as opposed to a higher group), and that it is isomorphic to $\mathrm{Out}(R)$, the outer automorphism group of R.
The group $\mathrm{Out}(R)$ is not a topological group (viewed as a topological group, its topology would be coarse), and the most natural structure that it has is that of a condensed group.
Objective 28: Prove that the homotopy type of $\mathrm{Out}(R)$ is that of a $K(\mathbb{Z}, 3)$. (This should follow easily from a recent result of Ozawa [Oza25].)
Objective 29: Define what it means to be a bundle with fibers $\mathrm{Bim}(R)$ over some topological space X, and prove that those admit a Dixmier-Douady classification by $H^4(X, \mathbb{Z})$. This would align well with the fact that K-theory admits twists by $H^3(-, \mathbb{Z})$, while TMF admits twists by $H^4(-, \mathbb{Z})$.

In [HP23], we showed that for every Morita equivalence class $[\mathcal{C}]$ of unitary fusion categories, there exists a canonical bicommutant category $T_{[\mathcal{C}]}$ associated to it.
Objective 30: Prove that $\mathrm{Aut}(T_{[\mathcal{C}]})$ is a group (as opposed to a higher group).
As above, we treat $\mathrm{Aut}(T_{[\mathcal{C}]})$ is a condensed group.
Objective 31: Prove that the homotopy type of $B\mathrm{Aut}(T_{[\mathcal{C}]})$ agrees with that of the geometric realisation of the Brauer-Picard 3-groupoid of $[\mathcal{C}]$.

Algebra objects

For $\mathcal{C}$ a fusion category, Ostrik's theorem provides a useful correspondence between (certain) $\mathcal{C}$-module categories and (certain) algebra objects in $\mathcal{C}$. We expect this correspondence to survive for module categories over a bicommutant category T, when it comes from a rigid C^* tensor category $\mathcal{C}$.
Objective 32: For $T = \hat{\mathcal{C}}$ the unitary ind-completion of $\mathcal{C}$, prove an Ostrik's theorem relating T-module categories, and W^*-algebra objects in $\mathcal{C}$ as defined in [JP17].
Question 33: Investigate whether a version of Ostrik's theorem can be formulated for module categories over general bicommutant categories.

Let T be the unitary ind-completion of a rigid C^* tensor category $\mathcal{C}$. Given a W^*-algebra object A, we wish to understand the associated T-module category $\mathcal{M} := A\text{-}\mathrm{Mod}$. A W^*-category is called factorial if it is the category of modules over a factor.
Objective 34: Let A and $\mathcal{M}$ be as above, and let

$$B := \bigoplus_{c \in \mathcal{C},\, \text{simple}} \mathrm{Hom}(c, A) \,\bar{\otimes}\, \mathrm{Hom}(\bar{c}, A),$$

Prove that if $T \to \mathrm{End}(\mathcal{M})$ is fully faithful and (the underlying W^*-category of) $\mathcal{M}$ is factorial, then B is a factor. Prove that when $\mathcal{C}$ is fusion, then this is an if and only if.

Given a W^*-algebra object A, we expect the standard form L^2A and Connes fusion $\boxtimes_A$ to make sense, and $\mathrm{Bim}(A)$ to be a bicommutant category.
Objective 35: Prove that $\mathrm{Bim}(A) \cong \mathrm{End}_T(A\text{-}\mathrm{Mod})$, and that it is a bicommutant category.
Objective 36: Prove that ${}_AL^2A_A \in \mathrm{Bim}(A)$ is irreducible if and only if (the underlying W^*-category of) $A\text{-}\mathrm{Mod}$ is factorial.

Conformal field theory

The $c = 1$ Virasoro conformal net is the fixed points of the $SU(2)$ level 1 under the adjoint action of $SO(3)$. This suggests a group-theoretic description of $\mathrm{Rep}(Vir_{c=1})$.
Objective 37: Prove that $\mathrm{Rep}(Vir_{c=1})$ is equivalent to the category $(\mathrm{Hilb}^{\omega}[SU(2)])^{SO(3)}$ of $SO(3)$-equivariant objects of $\mathrm{Hilb}^{\omega}[SU(2)]$, for ω the generator of the relevant H^3 group.
Our expectation generalises to other principal W-algebras as follows. Let G be a compact, simple, simply connected, simply laced Lie group of rank r, and let $W(\mathfrak{g})$ denote the $c \to r$ limit of the principal W-algebra associated to G.
Objective 38: Prove that, for G as above, $\mathrm{Rep}(W(\mathfrak{g})) = (\mathrm{Hilb}^{\omega}[G])^{G_{ad}}$, where ω the generator of the relevant H^3 group, and $G_{ad} := G/Z(G)$.

If $\mathcal{A}$ is a conformal net, and if $Vir_c \subset \mathcal{A}$ is its Virasoro sub-net, then we expect the dualisable part of $\mathrm{Sol}_{Vir_c}(\mathcal{A})$ to be equivalent to the category of topological defect in the $2D$ chiral CFT associated to $\mathcal{A}$. Unfortunately, the notion of topological defect in a chiral CFT has not yet been defined mathematically.
Objective 39: Given a chiral CFT, define its tensor category of topological defect, and prove that it is equivalent to the dualisable part of $\mathrm{Sol}_{Vir_c}(\mathcal{A})$.
Question 40: What does the non-dualisable part of $\mathrm{Sol}_{Vir_c}(\mathcal{A})$ correspond to physically? (It should correspond to topological defects that are somehow not allowed to bend...)

Given a rational VOA V, [FRS02] provides a (still conjectural) classification of full CFTs with chiral algebra V in terms of module categories for $\mathrm{Rep}(V)$. Under that correspondence, the objects of a module category correspond bijectively to boundary conditions of the associated full CFT that respect the given chiral symmetry. Given an arbitrary conformal net $\mathcal{A}$, not necessarily rational, we expect this story to survive with only minor modifications.
Objective 41: For V a unitary VOA that integrates to a conformal net $\mathcal{A}$, and $T := \mathrm{Rep}(\mathcal{A})$, show that unitary full CFTs with chiral algebra V correspond to T-module categories.
We do not expect all module categories to correspond to full CFTs.
Objective 42: Show that a T-module category M corresponds to a full CFT if and only if for every T-linear functors $F_1, F_2 : T_{abs} \to M$, the map $\overline{\mathrm{End}(F_1)} \odot \mathrm{End}(F_2) \to \mathrm{End}(F_1^{\dagger} \circ F_2)$ extends to $\overline{\mathrm{End}(F_1)} \,\bar{\otimes}\, \mathrm{End}(F_2)$.
Objective 43: Given a T-module category M corresponding to a full CFT, show that the objects of M correspond bijectively to unitary boundary conditions that respect the given chiral symmetry.

Let $\mathcal{A}$ be a conformal net, let $T := \mathrm{Rep}(\mathcal{A})$, and let M be a T-module category. Since T is braided, there's a natural map $f : T \otimes T^{mop} \to \mathrm{End}_T(M)$.
Objective 44: Show that the module category M corresponds to a full CFT iff f restricts to a map

$$f_{abs} : (T \otimes T^{mop})_{abs} \to \mathrm{End}_T(M)_{abs},$$

and that one recovers the state space of the full CFT by the formula:

$$H_{full} = f_{abs}^{\dagger} \circ f_{abs} \in \mathrm{End}_{T \otimes T^{mop}}((T \otimes T^{mop})_{abs}) = T \otimes T^{mop}.$$

Objective 45: Check that for $\mathcal{A} = Vir_{c=1}$, the above recipe recovers the expected full CFT state spaces.

The central charge is an important invariant of a chiral CFT and it is natural to ask to what extent the associated bicommutant category remembers that invariant. If the chiral CFT is rational, the center of the bicommutant category is a modular tensor category that remembers the central charge mod 8. The framed TQFT associated to the bicommutant category should further remember the central charge mod 24.
Question 46: Is the central charge an invariant of bicommutant categories?
Question 47: Is the central charge modulo 24 an invariant of bicommutant categories? Is the central charge invariant under Morita equivalence of bicommutant categories?

Thompson group

Given a conformal net, we expect its category of solitons to always be a bicommutant category. Conversely, given a bicommutant category T, an absorbing object $\Omega \in T$, and a unitary $u : \Omega \otimes \Omega \to \Omega$, we can construct a net of von Neumann algebras on the circle. This net is not $\mathrm{Diff}(S^1)$-covariant, but instead covariant with respect to the Thompson group $\mathbb{T} \subset \mathrm{Homeo}(S^1)$. We call such a thing a Thompson net.
Objective 48: Show that applying the above construction to (the unitary ind-completion of) a unitary fusion category reproduces the Thompson net studied in [Jon17].

We expect bicommutant categories to be equivalent to Thompson nets in the following precise sense:
Objective 49: Show that the category of solitons associated to an arbitrary Thompson net is a bicommutant category. Prove that the composite

$$\{\text{bicommutant categories}\} \to \{\text{Thompson nets}\} \to \{\text{bicommutant categories}\}$$

is the identity (regardless of the choice of unitary $u : \Omega \otimes \Omega \to \Omega$ used in the construction). The bicommutant category associated to Thompson net comes equipped with a distinguished absorbing object Ω, and a distinguished isomorphism $u : \Omega \otimes \Omega \to \Omega$.
Objective 50: Prove that the composite

$$\{\text{Thompson nets}\} \to \{\text{bicommutant categories}\} \to \{\text{Thompson nets}\}$$

is the identity, provided one uses the u from above .

Ameanability and property (T)

The Fell topology on the category of representations of a locally compact (quantum) group $\mathbb{G}$ is a priori not an invariant of the W^*-category, but depends on knowing that it came from $\mathbb{G}$. Given a bicommutant category T and an absorbing object $\Omega \in T_{abs}$, let $\mathfrak{A} := End_{T_*}(\Omega \otimes \Omega)$. By writing $T = Func_{ln}(T_*, \mathrm{Hilb}) = \mathrm{Rep}_{ln}(\mathfrak{A})$, we get an intrinsic Fell topology on T, which depends only on the structure of T as a W^*-tensor category.
Objective 51: Prove that when T comes from a locally compact (quantum) group, then its intrinsic Fell topology agrees with the usual Fell topology on $\mathrm{Rep}(\mathbb{G})$.

The intrinsic Fell topology on a bicommutant category T allows one to talk about weak containment, hence about ameanability and property (T). Let $1_T \in T$ be the unit object, and Ω_T the absorbing object of T. A bicommutant category is ameanable if $1_T \prec \Omega_T$, and it has Khazdan's property (T) if $1_T \prec X$ implies $1_T \subset X$ for all $X \in T$.
Objective 52: Prove that if $T = \mathrm{Rep}(\mathbb{G})$ for some locally compact (quantum) group $\mathbb{G}$, then T is ameanable / has property (T) if and only if the corresponding property holds for $\mathbb{G}$.

Let $\mathcal{C}$ be a rigid C^*-tensor category, and let $\hat{\mathcal{C}}$ be its unitary ind-completion. $\hat{\mathcal{C}}$ is always ameanable and always has property (T) in the sense of bicommutant categories.
Objective 53: Prove that $\mathcal{C}$ is ameanable / has property (T) in the sense of [PV15] iff $Z(\hat{\mathcal{C}})$ has the corresponding property in the sense of bicommutant categories.

We expect that every unitary chiral CFT has an associated bicommutant category. If true, ameanability and property (T) would be notions that make sense for arbitrary unitary chiral CFTs. Interpreting 'unitary chiral CFT' to mean 'conformal net', ameanability and property (T) are certainly notions that make sense for conformal nets.
Objective 54: Prove that if a conformal net $\mathcal{A}$ is such that $\mathrm{Rep}(\mathcal{A})$ is both ameanable and has property (T), then the conformal net $\mathcal{A}$ is rational.
Objective 55: Prove that ameanaility and property (T) are preserved under finite index extensions of the conformal net $\mathcal{A}$.

To understand ameanaility and property (T) in conformal field theory, it is imporant to have examples of non-rational conformal nets that are ameanable, as well as examples of non-rational conformal nets that have property (T).
Objective 56: Prove that the Virasoro conformal net $Vir_{c=1}$ is a non-rational conformal net which is ameanable.
Objective 57: Prove that Vir_c for $c > 1$ is not ameanable, and does not have property (T).
Objective 58: Prove that the conformal nets associated to the principal W-algebras of higher rank Lie algebras are ameanable when the central charge c is equal to the rank r of the Lie algebra, and have property (T) when $c > r$.

3-category

von Neumann algebras and their bimodules form the objects and 1-morphisms of a 2-category. We expect bicommutant categories to similarly assemble into a 3-category. In Bartels-Douglas-Henriques, it was proven that conformal nets form a 3-category. But that 3-category is not very well behaved as it is not (visibly) idempotent complete. Bicommutant categories should fix that problem.
Objective 59: Prove that bicommutant categories and their bimodule categories form the objects and 1-morphisms of a symmetric monoidal 3-category, and prove that this 3-category is idempotent complete.
Objective 60: Prove that the 3-category of conformal nets embeds fully faithfully inside the 3-category of bicommutant categories.

If a W^*-algebra A acts faithfully on a Hilbert space, then A and $(A')^{op}$ are called Morita equivalent. By analogy, if a bicommutant category T acts faithfully on a module category, then we call T and $(T')^{mop}$ Morita equivalent. This generalises the existing notion of Morita equivalence for unitary fusion categories.
Objective 61: Prove that the above notion of Morita equivalence is an equivalence relation on bicommutant categories.
Objective 62: Prove that two bicommutant categories are Morita equivalent if and only if they are isomorphic in the 3-category of bicommutant categories.

It is unclear whether one should expect to be able to classify bicommutant categories that are invertible up to Morita equivalence.

Question 63: Can the group of invertible bicommutant categories be computed? Is it generated by the bicommutant category associated to the E_8 level 1 chiral CFT?

TQFT

Given a 3-category, it is important to identify those objects which are 1-dualisable, those which are 2-dualisable (i.e. fully dualisable), and those which are invertible.

Objective 64: Prove that a bicommutant category T is 1-dualisable iff the corresponding Thompson net satisfies the split property, iff $\mathrm{End}(\Omega_1) \odot \mathrm{End}(\Omega_3) \to \mathrm{End}(\Omega_1 \otimes \Omega_2 \otimes \Omega_3)$ extends to $\mathrm{End}(\Omega_1) \bar{\otimes} \mathrm{End}(\Omega_3)$, for any absorbing objects $\Omega_1, \Omega_2, \Omega_3 \in T$.

Objective 65: Prove that a bicommutant category is 2-dualisable iff the corresponding Thompson net has the property that the four-interval bimodule ${}_{A_1 \bar{\otimes} A_3}(H_0)_{A_2 \bar{\otimes} A_4}$ is dualisable.

Objective 66: Prove that a bicommutant category is invertible in the 3-category if and only if the four-interval bimodule is invertible.

Given a connected surface Σ, possibly with boundary, let $\Gamma(\Sigma)$ denote the quotient of $\mathrm{Diff}(\Sigma)$ by the relation of isotopy fixing the boundary. This group is an extension

$$1 \to MCG(\Sigma) \to \Gamma(\Sigma) \to \mathrm{Diff}(\partial\Sigma) \to 1 \tag{1}$$

were $MCG(\Sigma)$ denotes the mapping class group of Σ rel boundary. In [BDH17], we showed that for every rational conformal net $\mathcal{A}$, we get projective representations of the groups $\Gamma(\Sigma)$. If $\mathcal{A}$ is not rational, we expect the result to survive provided $\partial\Sigma \neq \emptyset$ or $\chi(\Sigma) < 0$.

Objective 67: Given a conformal net $\mathcal{A}$, and a surface $\Sigma \neq S^2$ or $S^1 \times S^1$, construct projective representations of $\Gamma(\Sigma)$ generalising the construction in [BDH17].

Given a 1-dualizable bicommutant category, an absorbing object Ω, and $u : \Omega \otimes \Omega \cong \Omega$, we expect a closely analogous picture to emerge. Let us write $\tilde{\Gamma}(\Sigma)$ for the version of the above group where $\mathrm{Diff}(\partial\Sigma)$ in (1) is replaced by the appropriate Thompson group.

Objective 68: Given a 1-dualizable bicommutant category T, an isomorphism $u : \Omega \otimes \Omega \cong \Omega$, and a surface $\Sigma \neq S^2$ or $S^1 \times S^1$, construct projective representations of $\Gamma(\Sigma)$.

Objective 69: Find an expression for the projective cocycle of these representations in terms of the bicommutant category T.

By a result of Kawahigashi-Longo-Mueger, if a conformal net $\mathcal{A}$ has finite index, its category of representation $\mathrm{Rep}(\mathcal{A}) = Z(\mathrm{Sol}(\mathcal{A}))$ is (the unitary ind-completion) of a unitary modular tensor category.

Objective 70: Prove that a bicommutant category T is 2-dualisable iff its Drinfel'd center is the unitary ind-completion of a finite direct sum of unitary modular tensor categories.

Objective 71: Prove that a bicommutant category is invertible iff its Drinfel'd center is Hilb.

Unitary modular tensor categories are unitary braided fusion categories with trivial Mueger center, and examples come from Drinfel'd centers of unitary fusion categories.

Objective 72: Prove that the Drinfel'd center $Z_1(T)$ of a bicommutant category T is a braided bicommutant category, and that the Mueger center $Z_2(T)$ of a braided bicommutant category is a symmetric bicommutant category.

Given a bicommutant category T, let $\underline{Z}(T) := \{z \in \mathrm{End}(1_T) \,|\, z \otimes 1_X = 1_X \otimes z, \forall X \in T\}$.

Objective 73: Prove that the braiding on the Drinfel'd center $Z_1(T)$ of a bicommutant category is non-degenerate in the sense that the Mueger center of $Z_1(T)$ is $\underline{Z}(T)\text{-Mod}$.

Fix a compact connected Lie group G and a level $k \in H^4_+(BG, \mathbb{Z})$. In [Hen17], we introduced the category $\mathrm{Rep}^k(\Omega G)$ of representations of the based loop group at level k, and argued that it is what Chern–Simons theory assigns to a point.

Objective 74: Prove that the bicommutant category $\mathrm{Rep}^k(\Omega G)$ is fully dualisable, and thus admits an associated TQFT by the cobordism hypothesis.

The main piece of evidence for the claim in [Hen17] is that the Drinfel'd center of $\mathrm{Rep}^k(\Omega G)$ is equivalent to (the unitary ind-completion of) the modular tensor category $\mathrm{Rep}^k(LG)$ of representations of the free loop group. To complete the argument, we would need:

Objective 75: Prove that for every fully dualisable bicommutant category T, the value of the associated TQFT on S^1 agrees with the Drinfel'd center of T, as braided tensor category.

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