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Advances in Geometry

An analogue of convexity for complements of amoebas of varieties of higher codimension An answer to a question asked by B. Sturmfels

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1 Introduction and statement of result

Let $V \subset (\mathbb{C}^*)^n$ be a variety and let Log denote the logarithmic moment map $\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$, $(z_1, \dots, z_n) \mapsto (\log|z_1|, \dots, \log|z_n|)$. The amoeba $\mathscr{A}$ of the variety $V \subset (\mathbb{C}^*)^n$ is its image under that map. Amoebas were introduced by Gelfand, Kapranov and Zelevinsky in [6]. They show that for varieties of codimension 1, the complement ${}^c\mathscr{A}$ of the amoeba $\mathscr{A}$ in $\mathbb{R}^n$ is a disjoint union of convex regions.

If V is of codimension $k+1$ and ${}^c\mathscr{A}$ is its amoeba complement, we consider oriented $(k+1)$-planes π in $\mathbb{R}^n$. Let us call a class in the reduced homology group[1] $\tilde{H}_m(\pi \cap {}^c\mathscr{A})$ *non-negative* if its image in $\tilde{H}_k(\pi \backslash \{p\}) \simeq \mathbb{Z}$ is non-negative for all $p \in \pi \cap \mathscr{A}$. We show that such a class is never sent to zero in $\tilde{H}_k({}^c\mathscr{A})$, except if it is already zero. In other words, the maps $\tilde{H}_k(\pi \cap {}^c\mathscr{A}) \to \tilde{H}_k({}^c\mathscr{A})$ induced by the inclusions $\pi \cap {}^c\mathscr{A} \to {}^c\mathscr{A}$ never send non-zero non-negative classes to zero. The author expects these maps on homology to actually be injective, but this is only known for $k = 0$. In that case, the result specializes to the one mentioned above and proven in [6].

Indeed, when $k = 0$, we are looking at lines ℓ and at the maps on $\tilde{H}_0$ induced by $\ell \cap {}^c\mathscr{A} \to {}^c\mathscr{A}$. Assuming our result, we want to show that ${}^c\mathscr{A}$ is a disjoint union of convex sets. Suppose by contradiction that X is a component of ${}^c\mathscr{A}$ that is not convex. Choose $p, q \in X$ such that the interval joining them is not contained in X, and let ℓ be the oriented line $\overrightarrow{pq}$. It is then clear that $0 \neq [q] - [p] \in \tilde{H}_0(\ell \cap {}^c\mathscr{A})$ is a non-negative class that is sent to zero in $\tilde{H}_0({}^c\mathscr{A})$. On the other hand, if all components of ${}^c\mathscr{A}$ are convex, then the map $H_0(\ell \cap {}^c\mathscr{A}) \to H_0({}^c\mathscr{A})$ is always injective, and so is the map $\tilde{H}_0(\ell \cap {}^c\mathscr{A}) \to \tilde{H}_0({}^c\mathscr{A})$.

[1] See [1] chapter IV, pages 172 and 181 for a nice introduction to the homology groups and reduced homology groups of a space X. The reduced homology is denoted $\tilde{H}_*(X)$ and differs from $H_*(X)$ only in degree zero, where $\tilde{H}_0(X)$ is the kernel of a map $H_0(X) \to \mathbb{Z}$.

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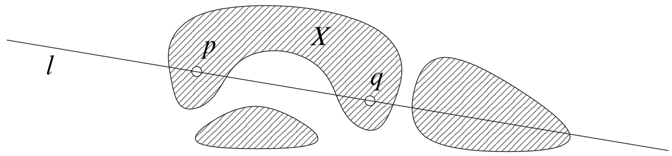

Amoebas have been studied by Passare, Rullgård, Forsberg and Tsikh, see [3], [13], [4] and [11], where they explain in what sense they are dual to Newton polytopes. See also [9] and [7] for an application to the topology of real algebraic curves. A good survey of the subject is provided by Mikhalkin [8]. In Chapter 9 of his book [15], Sturmfels explains the close relationship between amoebas and some piecewise linear objects called "tropical varieties". Some additional references are [12], [10], [14] and [16].

2 Various kinds of chains

For technical reasons it is convenient to define our homology groups using some kinds of chains other than the familiar singular chains ${}^{\mathrm{sing}}C_\bullet$. Our chains will be defined on triangulated spaces X (see [2] chapter II, Section 5), but we will only use them for open subsets of a real vector space π (complements of amoebas). We shall show that all these chains give rise to the same homology groups as singular chains with $\mathbb{Z}$ coefficients.

Let Δ_k, $k \geqslant 0$, be the standard k-simplex. Let ${}^{\mathrm{pl}}C_k(X)$ and ${}^{\mathrm{ps}}C_k(X)$ be the subgroups of singular k-chains generated by piecewise linear, respectively piecewise smooth maps $\Delta_k \to X$. Let $\Omega^k(X)$ be the space of piecewise smooth k-forms, smooth on the cells of some triangulation of X and compatible with restrictions to faces. We call two k-chains $\sigma = \sum \lambda_i \sigma_i$ and $\tau = \sum \mu_i \tau_i$ *geometrically equivalent*, and denote it by $\sigma \sim \tau$, if they are not distinguished by k-forms. More precisely $\sigma \sim \tau$ if

$$\int_\sigma \alpha = \sum \lambda_i \int_{\Delta_k} \sigma_i^*(\alpha) = \sum \mu_i \int_{\Delta_k} \tau_i^*(\alpha) = \int_\tau \alpha \quad \text{for all } \alpha \in \Omega^k(X),$$

where $\sigma_i^*(\alpha)$ denotes the pullback of α along $\sigma_i : \Delta_k \to X$. Note that a chain is always geometrically equivalent to one of its subdivisions.

Definition 2.1. Let

$${}^{\Delta}C_\bullet(X) := {}^{\mathrm{pl}}C_\bullet(X)/\text{geometric equivalence},$$
$${}^{\infty}C_\bullet(X) := {}^{\mathrm{ps}}C_\bullet(X)/\text{geometric equivalence}.$$

The chains ${}^{\Delta}C_\bullet(X)$ where introduced by Whitney in [17] under the name *polyhedral chains*, but for completely different purposes.

The boundary map $\partial_k : {}^{\mathrm{sing}}C_k(X) \to {}^{\mathrm{sing}}C_{k-1}(X)$ induces boundary maps on all

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the above chains, making them naturally into complexes. Indeed, if a chain σ is in ${}^{\mathrm{pl}}C_\bullet(X)$ or ${}^{\mathrm{ps}}C_\bullet(X)$, so will be $\partial\sigma$. Moreover, if $\sigma \sim \tau$ and $\alpha \in \Omega^\bullet(X)$, then

$$\int_{\partial\sigma} \alpha = \int_\sigma d\alpha = \int_\tau d\alpha = \int_{\partial\tau} \alpha,$$

therefore $\partial\sigma \sim \partial\tau$ and the boundary maps are well defined on ${}^{\Delta}C_\bullet(X)$ and ${}^{\infty}C_\bullet(X)$.

Define augmented versions ${}^{\alpha}\tilde{C}_\bullet(X)$ of these complexes[2], $\alpha \in \{\mathrm{sing}, \Delta, \infty\}$ by letting ${}^{\alpha}\tilde{C}_k(X) = {}^{\alpha}C_k(X)$ if $k \geqslant 0$, ${}^{\alpha}\tilde{C}_{-1}(X) = \mathbb{Z}$, and ${}^{\alpha}\tilde{C}_k(X) = 0$ if $k < -1$. The boundary map $\partial_0 : {}^{\alpha}\tilde{C}_0(X) \to \mathbb{Z}$ is induced by $\sum \lambda_i\sigma_i \mapsto \sum \lambda_i$.

Let ${}^{\mathrm{sing}}H_\bullet(X), {}^{\Delta}H_\bullet(X), {}^{\infty}H_\bullet(X)$ be the homology groups associated to the chain complexes ${}^{\mathrm{sing}}C_\bullet(X), {}^{\Delta}C_\bullet(X), {}^{\infty}C_\bullet(X)$ and let ${}^{\mathrm{sing}}\tilde{H}_\bullet(X), {}^{\Delta}\tilde{H}_\bullet(X), {}^{\infty}\tilde{H}_\bullet(X)$ be the reduced homology groups (see [1] page 181) associated to the corresponding augmented complexes.

Lemma 2.2. *For* $\alpha \in \{\mathrm{sing}, \Delta, \infty\}$ *we have* ${}^{\alpha}\tilde{H}_k(X) \simeq {}^{\alpha}H_k(X)$ *for* $k > 0$ *and* ${}^{\alpha}\tilde{H}_0(X) = \mathrm{Ker}(\omega)$, *where* $\omega : {}^{\alpha}H_0(X) \to {}^{\alpha}H_0(\mathrm{point}) \simeq \mathbb{Z}$ *is the map induced by the projection.*

Proof. This is a consequence of the long exact sequence in homology associated to the short exact sequence of complexes

$$0 \to \mathbb{Z}[-1] \to {}^{\alpha}\tilde{C}_\bullet(X) \to {}^{\alpha}C_\bullet(X) \to 0,$$

where $\mathbb{Z}[-1]$ is the complex $\ldots 0 \to \mathbb{Z} \to 0 \ldots$ concentrated in degree -1. □

Proposition 2.3. *If* X *is a triangulated space, then* ${}^{\mathrm{sing}}H_\bullet(X) \simeq {}^{\Delta}H_\bullet(X) \simeq {}^{\infty}H_\bullet(X)$ *and* ${}^{\mathrm{sing}}\tilde{H}_\bullet(X) \simeq {}^{\Delta}\tilde{H}_\bullet(X) \simeq {}^{\infty}\tilde{H}_\bullet(X)$.

Proof. By Lemma 2.2 it is enough to show that ${}^{\mathrm{sing}}H_\bullet(X) \simeq {}^{\Delta}H_\bullet(X) \simeq {}^{\infty}H_\bullet(X)$. We shall apply the uniqueness theorem which says that any ordinary homology theory defined on the category of pairs of triangulated spaces is naturally isomorphic to ${}^{\mathrm{sing}}H_\bullet$, see [2] chapter III Theorem 10.1 and chapter VII Section 10.

We first need to extend the definition of ${}^{\Delta}H_\bullet$ and ${}^{\infty}H_\bullet$ to pairs of triangulated spaces and then show that they satisfy the Eilenberg–Steenrod axioms for an ordinary homology theory, see [2] chapter I Section 3. Let ${}^{\alpha}C_\bullet(X, A) = {}^{\alpha}C_\bullet(X)/{}^{\alpha}C_\bullet(A)$, $\alpha \in \{\Delta, \infty\}$ and let ${}^{\alpha}H_\bullet(X, A)$ be the associated homology groups. We need to show functoriality, homotopy invariance, exactness of the long exact sequence of a pair, the excision axiom and the dimension axiom.

The dimension axiom is trivial. The long exact sequence comes from the short exact sequence of complexes $0 \to {}^{\alpha}C_\bullet(A) \to {}^{\alpha}C_\bullet(X) \to {}^{\alpha}C_\bullet(X, A) \to 0$, see [1] chapter IV Example 5.7. To prove excision one can apply verbatim the argument of [2]

[2] Everything also works for ${}^{\mathrm{pl}}C_\bullet$ and ${}^{\mathrm{ps}}C_\bullet$ but we will not need these notions.

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chapter VII Section 9, everything is just much easier since our chains are not only homologous but actually equal to their subdivisions. Functoriality is defined using simplicial approximations of maps of triangulated spaces, see [2] chapter II Section 7. For the sake of simplicity, we will only treat the case of spaces and leave the case of pairs to the reader. Given a continuous map of triangulated spaces $f : X \to Y$ and a class $[\sigma] = [\sum \lambda_i \sigma_i] \in {}^{\alpha}C_\bullet(X)$, we choose a simplicial approximation g and let $f_*[\sigma] = [\sum \lambda_i (g \circ \sigma_i)]$. To show this is well defined, assume $\sigma \sim \tau$ and let $\alpha \in \Omega^\bullet(Y)$. We have

$$\int_{g_*\sigma} \alpha = \int_\sigma g^*\alpha = \int_\tau g^*\alpha = \int_{g_*\tau} \alpha,$$

therefore $g_*\sigma \sim g_*\tau$. We still need to show that this does not depend on the choice of the simplicial approximation. This is achieved by showing homotopy invariance with respect to simplicial homotopies. The argument is the same as the one used in proving homotopy invariance of singular homology and can be found in [2] chapter VII Section 7. □

For $\sigma = \sum \lambda_i \sigma_i \in {}^{\mathrm{pl}}C_k(X)$, $\lambda_i \neq 0$, define its support by $\mathrm{supp}(\sigma) := \bigcup \mathrm{Im}(\sigma_i)$, for $c = [\sigma] \in {}^{\Delta}C_k(X)$, let $\mathrm{supp}(c) := \bigcap_{\tau \sim \sigma} \mathrm{supp}(\tau)$.

Lemma 2.4. *If π is a vector space, then any class $c = [\sigma] \in {}^{\Delta}C_k(\pi)$ has a representative in ${}^{\mathrm{pl}}C_k(\pi)$ that realizes* $\mathrm{supp}(c)$.

Proof. To construct such a representative, one first subdivides the pl-chain σ so that the simplices are all represented by linear maps. This is possible by definition of ${}^{\mathrm{pl}}C_k$. We then omit the degenerate simplices because they pair trivially with $\Omega^k(X)$. Our next goal is to make the images of the σ_i's only intersect in dimension less than k. We first treat the case when all the σ_i's span the same affine k-plane E. For every i, let $\mathscr{T}_i$ be a triangulation of E that contains $\mathrm{Im}(\sigma_i)$, and let $\mathscr{T}$ be a common refinement of the $\mathscr{T}_i$'s. For every k-simplex of $\mathscr{T}$, choose an affine map $\tau_j : \Delta_k \to E$ onto it. We can now replace every σ_i by the linear combination of τ_j's to which it is geometrically equivalent. For an arbitrary chain σ, we need to repeat the above construction for every k-plane E spanned by the σ_i's. It is then clear that the resulting chain τ will not have any $\mathrm{Im}(\tau_i) \cap \mathrm{Im}(\tau_j)$ of dimension k. We claim that the support of τ cannot be decreased further.

Suppose $\tau' \sim \tau$. Given $p \in \mathrm{Im}(\tau_i)$, we want to show $p \in \mathrm{supp}(\tau')$. Since $\mathrm{supp}(\tau')$ is closed, it is enough to show that any neighborhood $\mathscr{V}$ of p intersects it non-trivially. Let $p' \in \mathrm{Im}(\tau_i) \cap \mathscr{V}$ be such that $p' \notin \mathrm{Im}(\tau_j)$ if $j \neq i$. Such a point exists because the $\mathrm{Im}(\tau_j)$'s were assumed to intersect in dimension no more than $k-1$. Let $\mathscr{V}' \subset \mathscr{V}$ be a neighborhood of p' that does not meet $\mathrm{Im}(\tau_j)$ for $j \neq i$. Finally, let α be a form supported in $\mathscr{V}'$ with the property that $\int_{\tau_i} \alpha \neq 0$. We have $0 \neq \int_{\tau_i} \alpha = \int_\tau \alpha = \int_{\tau'} \alpha$, therefore $\mathrm{supp}(\tau') \cap \mathrm{supp}(\alpha) \neq \varnothing$, which implies $\mathrm{supp}(\tau') \cap \mathscr{V} \neq \varnothing$ and we are done. □

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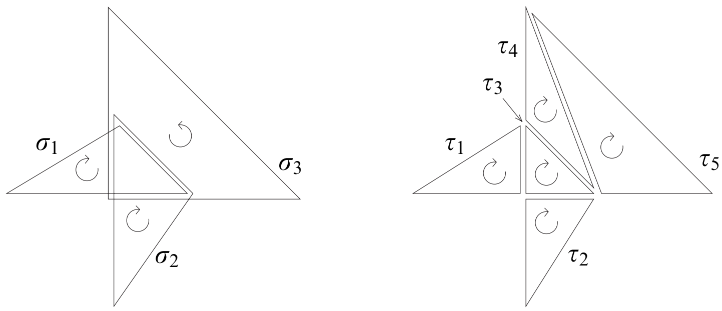
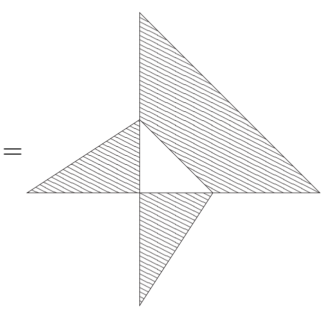


$$\begin{aligned}\sigma_1 + \sigma_2 + 2\sigma_3 &\sim (\tau_1 + \tau_3) + (\tau_2 + \tau_3) + 2(-\tau_3 - \tau_4 - \tau_5)\\ &= \tau_1 + \tau_2 - 2\tau_4 - 2\tau_5.\end{aligned}$$

$\operatorname{supp}([\sigma]) = \operatorname{supp}(\tau) =$



Remark 2.5. The statement of Lemma 2.4 is true when replacing π by any triangulated space, but false when replacing ${}^{\Delta}C_{\bullet}$ by ${}^{\infty}C_{\bullet}$.

From now on, let X be an open subset of a real vector space π.

Corollary 2.6. *For $X \subset \pi$, the group ${}^{\Delta}C_k(X)$ can be identified with the set of chains $c \in {}^{\Delta}C(\pi)$ satisfying* $\operatorname{supp}(c) \subset X$.

Proof. Let $A = \{c \in {}^{\Delta}C_k(\pi) \mid \operatorname{supp}(c) \subset X\}$. By Lemma 2.4 the map ${}^{\Delta}C_k(X) \to A$ is onto. To show it is injective suppose $[\sigma] \mapsto 0$, namely $\int_\sigma \alpha = 0$ for all $\alpha \in \Omega^k(\pi)$. We want to show that $\int_\sigma \beta = 0$ for all $\beta \in \Omega^k(X)$. Let β be such a form. Choose a smooth cut off function φ that takes the value 1 on $\operatorname{supp}(\sigma)$ and 0 outside of a compact set of X, and let $\gamma \in \Omega^k(\pi)$ be the extension of $\varphi\beta$ by zero outside of X. One has $\int_\sigma \beta = \int_\sigma \varphi\beta = \int_\sigma \gamma = 0$. □

Let ${}^{\Delta}\tilde{Z}_k(X)$ denote the kernel of $\partial_k : {}^{\Delta}\tilde{C}_k(X) \to {}^{\Delta}\tilde{C}_{k-1}(X)$.

Lemma 2.7. *Let $n \geqslant 1$, π be an n-dimensional real vector space and let $X \subset \pi$ be an open subset of it. Let $c \in {}^{\Delta}\tilde{Z}_{n-1}(X)$. There is a unique $C \in {}^{\Delta}C_n(\pi)$ such that $\partial C = c$.*

Moreover $[c]$ *is non-trivial in* $\tilde{H}_{n-1}(X)$ *if and only if there is a point* $p \in \pi \backslash X$ *in the support of* C.

Proof. The chain groups ${}^{\Delta}C_{\bullet}(\pi)$ vanish[3] above the top dimension n, and hence the boundary map $\partial_n : {}^{\Delta}C_n(\pi) \to {}^{\Delta}C_{n-1}(\pi)$ is injective. Indeed $\tilde{H}_n(\pi) = 0$, hence $\mathrm{Ker}(\partial_n) = \mathrm{Im}(\partial_{n+1}) = 0$. We also have[4] that $\tilde{H}_{n-1}(\pi) = 0$, therefore any $(n-1)$-cycle $c \in {}^{\Delta}\tilde{Z}_{n-1}(\pi)$ bounds exactly one n-chain C in ${}^{\Delta}C_n(\pi)$. Our class $[c]$ is trivial in $\tilde{H}_{n-1}(X)$ if and only if $C \in {}^{\Delta}C_n(X)$, that is, if and only if $\mathrm{supp}(C) \subset X$. Conversely $[c]$ is non-trivial if and only if $\mathrm{supp}(C) \backslash X \neq \varnothing$. □

3 Analogue of convexity

For $k \geqslant 0$, and π an oriented $(k+1)$-dimensional vector space with volume form dv, let us introduce the following notation:

Definition 3.1. We say that $C \in {}^{\Delta}C_{k+1}(\pi)$ is non-negative if $\int_C f\, dv \geqslant 0$ for all $f \geqslant 0$, and that $\partial C \in {}^{\Delta}\tilde{Z}_k(\pi)$ is non-negative if C is. The set of all such chains (respectively cycles) will be denoted ${}^{\Delta}C^{+}_{k+1}(\pi)$ (respectively ${}^{\Delta}\tilde{Z}^{+}_{k}(\pi)$).

Note that, by Lemma 2.7, the cycle ∂C determines C, and therefore ${}^{\Delta}\tilde{Z}^{+}_{k}(\pi)$ is well defined. For $X \subset \pi$, let ${}^{\Delta}\tilde{Z}^{+}_{k}(X)$ be ${}^{\mathrm{pl}}C_k(X) \cap {}^{\Delta}\tilde{Z}^{+}_{k}(\pi)$.

Lemma 3.2. *Let* $k \geqslant 0$, *and* π *be an oriented vector space of dimension* $k+1$. *A cycle* $c \in {}^{\Delta}\tilde{Z}_k(\pi)$ *is non-negative if and only if, for all* $p \notin \mathrm{supp}(c)$, *the class* $[c]$ *is non-negative in* $\tilde{H}_k(\pi \backslash \{p\}) \simeq \mathbb{Z}$.

Proof. Without loss of generality, we may assume that $\pi = \mathbb{R}^{k+1}$. Let c be an element of ${}^{\Delta}\tilde{Z}_k(\mathbb{R}^{k+1})$. We know from [5] p. 327 that for $k \geqslant 1$, the form

$$\frac{1}{\omega_k}\sum_{i=0}^{k}(-1)^i \frac{x_i}{\|x\|^{k+1}}\, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \qquad (1)$$

represents the standard generator of the De Rahm cohomology group $H^k_{\mathrm{DR}}(\mathbb{R}^{k+1} \backslash \{0\}) \simeq \mathbb{R}$, where $\omega_k = \mathrm{vol}(\mathbb{S}^k)$, $\|\cdot\|$ denotes the euclidian norm and $\hat{}$ means that the term is omitted. If $k = 0$, then (1) is the function that takes the value $-1/2$ on the negatives and $1/2$ on the positives. It represents the generator of the reduced[5] De Rahm cohomology group $\tilde{H}^k_{\mathrm{DR}}(\mathbb{R} \backslash \{0\}) \simeq \mathbb{R}$.

Putting all the cases together and replacing 0 by a point $p \notin \mathrm{supp}(c)$, the generator of $\tilde{H}_{\mathrm{DR}}(\mathbb{R}^{k+1} \backslash \{p\})$ is therefore represented by the form

[3] This is the crucial property that motivates the introduction of ${}^{\Delta}C_{\bullet}$.

[4] This would fail if we used $H_{\bullet}$ instead of $\tilde{H}_{\bullet}$.

[5] The reduced De Rahm cohomology of X is given by $\tilde{H}^k_{\mathrm{DR}}(X) = H^k_{\mathrm{DR}}(X)$ if $k > 0$ and $\tilde{H}^0_{\mathrm{DR}}(X) = H^0_{\mathrm{DR}}(X)/\text{constant functions}$.

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$$\alpha_p = \frac{1}{\omega_k}\sum_{i=0}^{k}(-1)^i \frac{x_i - p_i}{\|x-p\|^{k+1}}\, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k$$

for any $k \geqslant 0$. The functional $[c] \mapsto \int_c \alpha_p$ gives the standard isomorphism $\tilde{H}_k(\mathbb{R}^{k+1}\setminus\{p\}) \to \mathbb{Z}$. To check that $[c]$ is non-negative in $\tilde{H}_k(\mathbb{R}^{k+1}\setminus\{p\})$, it is thus enough to check that $\int_c \alpha_p \geqslant 0$. Now let C be the unique chain such that $\partial C = c$. If we call δ the euclidian distance between p and $\mathrm{supp}(c)$ and let $\mu : \mathbb{R}_+ \to [0,1]$ be a non-decreasing function, vanishing on a neighborhood of the origin and one on $[\delta, \infty)$, we can compute

$$\begin{aligned}
\int_c \alpha_p &= \int_{\partial C} \mu(\|x-p\|)\alpha_p = \int_C d(\mu(\|x-p\|)\alpha_p) \\
&= \int_C \Big[d(\mu(\|x-p\|)) \wedge \alpha_p + \mu(\|x-p\|) \underbrace{d\alpha_p}_{=0 \quad \text{since } \alpha_p \text{ is closed}} \Big] \\
&= \int_C \mu'(\|x-p\|)\, d(\|x-p\|) \wedge \alpha_p \\
&= \int_C \mu'(\|x-p\|) \sum_i \frac{(x_i - p_i)\, dx_i}{\|x-p\|} \\
&\quad \wedge \frac{1}{\omega_k}\sum_i (-1)^i \frac{x_i - p_i}{\|x-p\|^{k+1}}\, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \\
&= \frac{1}{\omega_k}\int_C \mu'(\|x-p\|) \sum_i \frac{(x_i - p_i)^2}{\|x-p\|^{k+2}}\, dx_0 \wedge \cdots \wedge dx_k \\
&= \int_C \underbrace{\frac{\mu'(\|x-p\|)}{\omega_k \|x-p\|^k}}_{\geqslant 0}\, dx_0 \wedge \cdots \wedge dx_k.
\end{aligned}$$

The right hand side is non-negative for all p and μ if and only if $C \in {}^{\Delta}C^+_{k+1}$, i.e. $c \in {}^{\Delta}\tilde{Z}^+_k$. On the other hand, $\int_c \alpha_p$ is non-negative for all p if and only if $[c]$ is in $\tilde{H}^+_k(\mathbb{R}^{k+1}\setminus\{p\})$ for all p. This proves our lemma. □

Definition 3.3. Let π be an oriented vector space of dimension $k+1$ and $X \subset \pi$ an open subset. A class $[c] \in \tilde{H}_k(X)$ is called non-negative if its image in $\tilde{H}_k(\pi\setminus\{p\}) \simeq \mathbb{Z}$ is non-negative for all $p \notin X$. The set of such classes will be denoted $\tilde{H}^+_k(X)$.

It follows from Lemma 3.2 that the class of a cycle $c \in {}^{\Delta}\tilde{Z}^+_k(X)$ is always in $\tilde{H}^+_k(X)$. The converse also turns out to be true.

Lemma 3.4. *Let π be an oriented vector space of dimension $k+1$ and let X be an open subset. Then an element of $\tilde{H}^+_k(X)$ can always be represented by a cycle in ${}^{\Delta}\tilde{Z}^+_k(X)$.*

Proof. Let $[c] \in \tilde{H}_k^+(X)$. By Lemma 2.7 there is a $C \in {}^{\Delta}C_{k+1}(\pi)$ such that $\partial C = c$. One decomposes $C = C_+ - C_-$, with both C_+ and C_- non-negative and having disjoint supports (up to a set of smaller dimension). If there were a point $p \in \operatorname{supp}(C_-) \backslash X$, then the same argument as in Lemma 3.2 would show that c represents a negative element in $\tilde{H}_k(\pi \backslash \{p\})$, which contradicts our hypothesis. So $\operatorname{supp}(C_-) \subset X$ and $[c]$ can also be represented by $c + \partial C_- = \partial C_+ \in {}^{\Delta}\tilde{Z}_k^+(X)$. □

Definition 3.5 (analogue of convexity). A subset X of a vector space is *k-convex* if, for all oriented affine $(k+1)$-planes π, the map $\tilde{H}_k(\pi \cap X) \to \tilde{H}_k(X)$ does not send non-zero elements of $\tilde{H}_k^+(\pi \cap X)$ to zero.

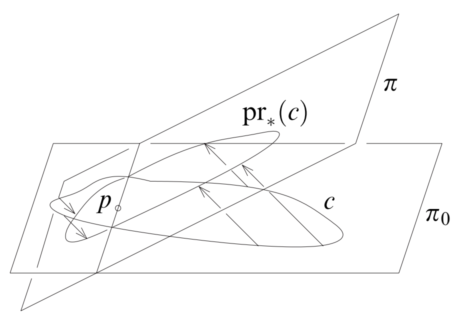
As explained in the introduction, a set is 0-convex if and only if it is a disjoint union of convex sets.

When X is open, one can check k-convexity on a dense subset of plane directions.

Lemma 3.6. *Let X be open in $\mathbb{R}^n$. Then if X is not k-convex, there is a $(k+1)$-plane π with rational slope and a non-zero class in $\tilde{H}_k^+(\pi \cap X)$ that is sent to zero under $\tilde{H}_k(\pi \cap X) \to \tilde{H}_k(X)$.*

Proof. If X is not k-convex, there is an oriented $(k+1)$-plane π_0 and a class $0 \neq a \in \tilde{H}_k^+(\pi_0 \cap X)$ that is sent to zero in $\tilde{H}_k(X)$. By Lemma 3.4, we may choose a representative $c \in {}^{\Delta}\tilde{Z}_k^+(\pi_0 \cap X)$ for the class a.

By Lemma 2.7 there is a chain $C \in {}^{\Delta}C_{k+1}^+(\pi_0)$ such that $\partial C = c$, and there is a point $p \in \operatorname{supp}(C) \backslash X$. For a $(k+1)$-plane π let $\mathrm{pr} : \pi_0 \to \pi$ be the orthogonal projection. Because X is open and c compactly supported, it is possible to choose π containing p with rational slope, and close enough to π_0 so that $(t + (1-t)\,\mathrm{pr})_*(c)$ stays supported in X for all $t \in [0,1]$.



Assuming π and π_0 are not orthogonal, give π the orientation induced from π_0 by pr. The cycles c and $\mathrm{pr}_*(c)$ are homotopic, therefore $[\mathrm{pr}_*(c)] = [c] = 0$ in $\tilde{H}_k(X)$. It only remains to show that $\mathrm{pr}_*(c)$ represents a non-zero element of $\tilde{H}_k^+(\pi \cap X)$. The map $\mathrm{pr}_* : {}^{\Delta}C_k(\pi_0) \to {}^{\Delta}C_k(\pi)$ sends isomorphically ${}^{\Delta}\tilde{Z}_k^+(\pi_0)$ onto ${}^{\Delta}\tilde{Z}_k^+(\pi)$, therefore $\mathrm{pr}_*(c) \in {}^{\Delta}\tilde{Z}_k^+(\pi \cap X)$ and by Lemma 3.2 $[\mathrm{pr}_*(c)] \in \tilde{H}_k^+(\pi \cap X)$. Now $p = \mathrm{pr}(p) \in \operatorname{supp}(\mathrm{pr}_*(C)) \backslash X$, so by Lemma 2.7, $[\mathrm{pr}_*(c)]$ is non-zero in $\tilde{H}_k^+(\pi \cap X)$. □

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4 Amoeba complements

We can now state our main result.

Theorem 4.1. *Let* $V \subset (\mathbb{C}^*)^n$ *be a variety of codimension* $k+1$, *let* $\mathscr{A} = \mathrm{Log}(V)$ *be the amoeba of* V *and* ${}^c\mathscr{A} = \mathbb{R}^n \backslash \mathscr{A}$ *be the amoeba complement, where* $\mathrm{Log}(z_1, \dots, z_n) = (\log|z_1|, \dots, \log|z_n|)$. *Then* ${}^c\mathscr{A}$ *is k-convex.*

Proof. First note that, since Log is proper, ${}^c\mathscr{A}$ is open and we can compute homology groups using polyhedral chains ${}^\Delta C_\bullet$. Let π be an oriented $(k+1)$-plane and $a \in \tilde{H}_k^+(\pi \cap {}^c\mathscr{A})$ be a non-zero class. We want to show that the image of a in $\tilde{H}_k({}^c\mathscr{A})$ is non-zero. By Lemma 3.6 it is enough to show it when π has rational slope. We may by Lemma 3.4 represent a by a non-negative cycle $c \in {}^\Delta\tilde{Z}_k^+(\pi \cap {}^c\mathscr{A})$. Finally, by Lemma 2.7 there is a chain $C \in {}^\Delta C_{k+1}^+(\pi)$ bounded by c and a point $p \in \mathrm{supp}(C) \cap \mathscr{A}$.

Before going on, it is convenient to identify $(\mathbb{C}^*)^n$ with $\mathbb{R}^n \times \mathbb{T}^n$ under the map

$$\begin{aligned} \mathrm{Log}_{\mathbb{C}} : (\mathbb{C}^*)^n &\xrightarrow{\simeq} \mathbb{R}^n \times \mathbb{T}^n \\ z &\mapsto (\mathrm{Log}(z), \mathrm{Arg}(z)), \end{aligned}$$

where $\mathrm{Arg}(z_1, \dots, z_n) = \left(\frac{z_1}{|z_1|}, \dots, \frac{z_n}{|z_n|}\right)$. Put on $\mathbb{R}^n \times \mathbb{T}^n$ the complex structure induced from $(\mathbb{C}^*)^n$ by the isomorphism $\mathrm{Log}_{\mathbb{C}}$. For $(x, y) \in \mathbb{R}^n \times \mathbb{T}^n$, the tangent space $T_{(x,y)} \cdot (\mathbb{R}^n \times \mathbb{T}^n)$ splits as a direct sum $T_x\mathbb{R}^n \oplus T_y\mathbb{T}^n$ of two totally real subspaces that are exchanged under multiplication by i. Let pr_1 (resp. pr_2) be the projection to $\mathbb{R}^n$ (resp. $\mathbb{T}^n$). The point p being in $\mathscr{A}$, one can choose a point $v \in \mathrm{Log}_{\mathbb{C}}(V)$ such that $\mathrm{pr}_1(v) = p$. Let $q = \mathrm{pr}_2(v)$. Now, let us consider the sequence of maps

$$\begin{array}{ccccccccc} T_p\pi & \hookrightarrow & T_p\mathbb{R}^n & \hookrightarrow & T_p\mathbb{R}^n \oplus T_q\mathbb{T}^n & & T_p\mathbb{R}^n \oplus T_q\mathbb{T}^n & \twoheadrightarrow & T_q\mathbb{T}^n \xrightarrow{\exp} \mathbb{T}^n, \\ & & & & \wr\big| & & \wr\big| & & \\ & & & & T_{(p,q)}(\mathbb{R}^n \times \mathbb{T}^n) & \xrightarrow{\cdot i} & T_{(p,q)}(\mathbb{R}^n \times \mathbb{T}^n) & & \end{array}$$

where $\exp : T_q\mathbb{T}^n \twoheadrightarrow \mathbb{T}^n$ is the (riemannian) exponential map that presents the (flat) torus $\mathbb{T}^n$ as a quotient of the vector space $T_q\mathbb{T}^n$ by the lattice $\Lambda = \exp^{-1}(q)$. Let $\mathbb{T}_\pi$ be the image of $T_p\pi$ under the above sequence of maps. Since π was assumed to have rational slope in $\mathbb{R}^n$, the image of $T_p\pi$ in $T_q\mathbb{T}^n$ has rational slope with respect to Λ, and $\mathbb{T}_\pi$ is therefore a closed subtorus of $\mathbb{T}^n$ (as opposed to a dense leaf of some foliation).

Let $\pi_{\mathbb{C}} = \pi \times \mathbb{T}_\pi \subseteq \mathbb{R}^n \times \mathbb{T}^n$. By definition of $\mathbb{T}_\pi$, the tangent space $T_{(p,q)}\pi_{\mathbb{C}} \simeq T_p\pi \oplus T_q\mathbb{T}_\pi$ is invariant under i. This implies that all tangent spaces $T_{(x,y)}\pi_{\mathbb{C}}$ are invariant under i, and therefore $\pi_{\mathbb{C}}$ is a complex submanifold of $\mathbb{R}^n \times \mathbb{T}^n$. Indeed $\pi_{\mathbb{C}}$ can be identified with a coset of a subgroup of the complex Lie group $\mathbb{R}^n \times \mathbb{T}^n$, and translation in the group gives a complex isomorphism of $T_{(x,y)}(\mathbb{R}^n \times \mathbb{T}^n)$ with $T_{(p,q)} \cdot (\mathbb{R}^n \times \mathbb{T}^n)$ sending $T_{(x,y)}\pi_{\mathbb{C}}$ to $T_{(p,q)}\pi_{\mathbb{C}}$.

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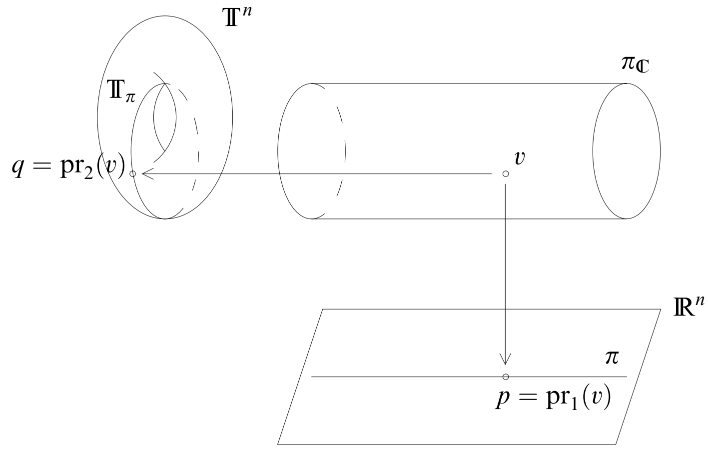
For the rest of the proof, the important properties of $\pi_{\mathbb{C}}$ are that it is complex and that $v \in \mathrm{Log}_{\mathbb{C}}(V) \cap \pi_{\mathbb{C}}$.



We orient $\mathbb{T}_\pi$ in the way that makes the product orientation on $\pi_{\mathbb{C}} = \pi \times \mathbb{T}_\pi$ agree with the orientation inherited from its complex structure. With that orientation, the torus $\mathbb{T}_\pi$ defines an element $[\mathbb{T}_\pi] \in {}^\infty C_{k+1}(\mathbb{T}^n)$. Let φ be the composed homomorphism

$$\begin{aligned}\varphi : {}^\Delta C_\bullet({}^c\mathscr{A}) &\xrightarrow{\times[\mathbb{T}_\pi]} {}^\infty C_{\bullet+k+1}({}^c\mathscr{A} \times \mathbb{T}^n)\\ &\longrightarrow {}^\infty C_{\bullet+k+1}((\mathbb{R}^n \times \mathbb{T}^n)\backslash \mathrm{Log}_{\mathbb{C}}(V))\\ &\xrightarrow{(\mathrm{Log}_{\mathbb{C}}^{-1})_*} {}^\infty C_{\bullet+k+1}((\mathbb{C}^*)^n \backslash V)\\ &\longrightarrow {}^\infty C_{\bullet+k+1}(\mathbb{C}^n \backslash \overline{V}),\end{aligned}$$

where the second and fourth maps are induced by the inclusions and $\overline{V}$ is the closure of V in $\mathbb{C}^n$. The map φ is compatible with ∂, and therefore induces a map $\bar{\varphi}$ on homology groups

$$\bar{\varphi} : H_\bullet({}^c\mathscr{A}) \to H_{\bullet+k+1}(\mathbb{C}^n \backslash \overline{V}).$$

Recall that we are trying to show that $a = [c]$ is not sent to zero in $\tilde{H}_k({}^c\mathscr{A}) \subseteq H_k({}^c\mathscr{A})$. It suffices to show that a is not sent to zero under the maps

$$\tilde{H}_k(\pi \cap {}^c\mathscr{A}) \xrightarrow{\iota_*} \tilde{H}_k({}^c\mathscr{A}) \subseteq H_k({}^c\mathscr{A}) \xrightarrow{\bar{\varphi}} H_{2k+1}(\mathbb{C}^n \backslash \overline{V}),$$

where ι denotes the inclusion.

We shall give a heuristic argument why $\bar{\varphi}\iota_*(a) = [\varphi(c)]$ should be non-zero in $H_{2k+1}(\mathbb{C}^n \backslash \overline{V})$ by claiming that the linking number $\mathrm{lk}(\varphi(c), \overline{V})$ is strictly positive.

Since $\partial(\varphi(C)) = \varphi(c)$, that linking number is by definition (see [1] pages 117–118) the number of intersections of $\varphi(C)$ and $\overline{V}$, counted with multiplicities. We need to know that all intersections give positive contributions, and that there is at least one. The first claim is true because $\mathrm{supp}(\varphi(C))$ and $\overline{V}$ are complex. Indeed $\varphi(C)$ was constructed by taking the chain $C \times [\mathbb{T}_\pi]$, which was supported on $\pi_{\mathbb{C}}$, and mapping it to $\mathbb{C}^n \backslash \overline{V}$ via the holomorphic map $\mathrm{Log}_{\mathbb{C}}^{-1}$. Also, because of the non-negativity of C and of the good choice of orientation of $\mathbb{T}_\pi$, we know that the "orientation" of $\varphi(C)$ coincides with the orientation coming from the complex structure of $\mathrm{supp}(\varphi(C))$. One then uses the fact that the intersection of two complex submanifolds always comes with positive sign. The last piece of information needed is that $\mathrm{Log}_{\mathbb{C}}^{-1}(v) \in \mathrm{supp}(\varphi(C)) \cap \overline{V}$. As there is at least one intersection, $\mathrm{lk}(\varphi(c), \overline{V}) > 0$.

Unfortunately, the above argument about linking numbers is not entirely rigorous (the main problem being that $\overline{V}$ is not compact), so let us start again. First note that when $k+1 = n$, k-convexity is the empty condition, so we may assume $k+1 < n$. Consider the sphere $\mathbb{C}^n \cup \{\infty\}$ and let $\overline{\overline{V}}$ be the closure of $\overline{V}$ in it. Since $\overline{\overline{V}}$ is compact, oriented and smooth in real codimension one, it has a fundamental class $[\overline{\overline{V}}] \in H_{2(n-k-1)}(\overline{\overline{V}}) \simeq \mathbb{Z}$ ([1] page 338). By Poincaré–Alexander–Lefschetz duality ([1] section VI Theorem 8.3)

$$\begin{aligned} H_{2(n-k-1)}(\overline{\overline{V}}) &\simeq H^{2(k+1)}(\mathbb{C}^n \cup \{\infty\}, (\mathbb{C}^n \cup \{\infty\}) \backslash \overline{\overline{V}}) \\ &\simeq H^{2(k+1)}(\mathbb{C}^n \cup \{\infty\}, \mathbb{C}^n \backslash \overline{V}). \end{aligned}$$

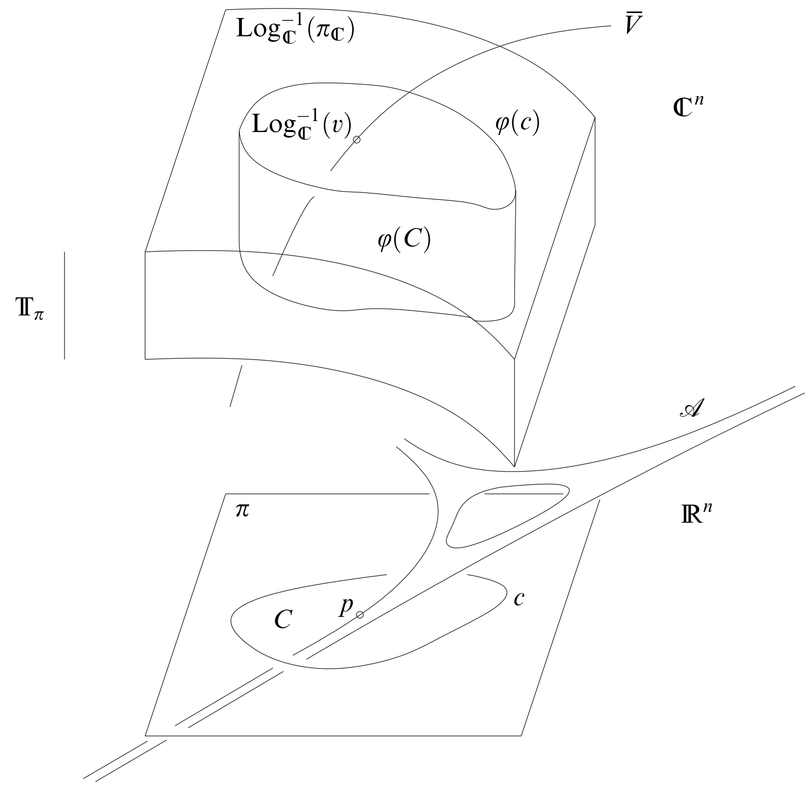
Now, by the long exact sequence in homology ([1] page 180)

$$\begin{aligned} 0 = H_{2(k+1)}(\mathbb{C}^n \cup \{\infty\}) &\to H_{2(k+1)}(\mathbb{C}^n \cup \{\infty\}, \mathbb{C}^n \backslash \overline{V}) \\ &\xrightarrow{\partial} H_{2k+1}(\mathbb{C}^n \backslash \overline{V}) \to H_{2k+1}(\mathbb{C}^n \cup \{\infty\}) = 0 \end{aligned}$$

we know that $H_{2(k+1)}(\mathbb{C}^n \cup \{\infty\}, \mathbb{C}^n \backslash \overline{V}) \simeq H_{2k+1}(\mathbb{C}^n \backslash \overline{V})$. Since $\partial(\varphi(C)) = \varphi(c)$ we have that $[\varphi(C)] \mapsto [\varphi(c)]$ under that isomorphism. We want to evaluate $[\varphi(C)] \in H_{2(k+1)}(\mathbb{C}^n \cup \{\infty\}, \mathbb{C}^n \backslash \overline{V})$ on the dual[6] of $[\overline{\overline{V}}]$ in $H^{2(k+1)}(\mathbb{C}^n \cup \{\infty\}, \mathbb{C}^n \backslash \overline{V})$. This can be computed by counting with multiplicities the intersections of $\mathrm{supp}(\varphi(C))$ and $\overline{\overline{V}}$ (see [1] section VI Theorem 11.9), but first we need to make $\varphi(C)$ and $\overline{\overline{V}}$ transverse to each other. To do this, we apply to $\varphi(C)$ a small affine unitary transformation u, centered at $\mathrm{Log}_{\mathbb{C}}^{-1}(v)$. The chain $u\varphi(C)$ is still complex in $\mathbb{C}^n$, therefore all the points of $\mathrm{supp}(u\varphi(C)) \cap \overline{\overline{V}}$ will give a positive contribution to the intersection number $[u\varphi(C)] \cdot [\overline{\overline{V}}]$. On the other hand, $\mathrm{Log}_{\mathbb{C}}^{-1}(v)$ is both in $\mathrm{supp}(u\varphi(C))$ and in $\overline{\overline{V}}$, so $[u\varphi(C)] \cdot [\overline{\overline{V}}] > 0$.

We have shown that the class $[u\varphi(C)] = [\varphi(C)]$ is non-zero in $H_{2(k+1)}(\mathbb{C}^n \cup \{\infty\}, \mathbb{C}^n \backslash \overline{V})$, which implies that $[\varphi(c)] = \bar{\varphi}\imath_*(a)$ is non-zero in $H_{2k+1}(\mathbb{C}^n \backslash \overline{V})$, which in turn implies that $\imath_*(a)$ is non-zero in $\tilde{H}_k({}^c\mathscr{A})$. □

[6] This is essentially the same as computing $\mathrm{lk}(\varphi(c), \overline{V})$.



The lower half of the above picture represents the amoeba $\mathscr{A}$ in $\mathbb{R}^n$ and the cycle $0 \neq [c] \in \tilde{H}_k(\pi \cap {}^c\mathscr{A})$. The chain C is bounded by c and intersects $\mathscr{A}$ at the point p.

The upper half represents the variety $\overline{V}$ in $\mathbb{C}^n$ and the cycle $\varphi(c)$ linked around $\overline{V}$. The chain $\varphi(C)$ is bounded by $\varphi(c)$ and lies on the complex manifold $\mathrm{Log}_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}})$ (which is actually a variety). It intersects $\overline{V}$ in $\mathrm{Log}_{\mathbb{C}}^{-1}(v)$, and because both $\varphi(C)$ and $\overline{V}$ are complex, this gives a positive contribution to the linking number $\mathrm{lk}(\varphi(c), \overline{V})$.

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