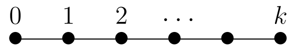
The modular tensor category of $SU(2)$ at level k

Simple objects: $V_0, V_1, V_2, \ldots, V_k$

Fusion rules: $V_i \boxtimes V_1 = \bigoplus_{\substack{j\,:\,\text{adjacent} \\ \text{to } i}} V_j$, where adjacency is with respect to this graph:



From the above rules, we get that $V_i \boxtimes V_j = V_{i-j} \oplus V_{i-j+2} \oplus V_{i-j+4} \oplus \ldots \oplus V_{i+j-2} \oplus V_{i+j}$ if $0 \leq i-j$ and $i+j \leq k$. And more generally:

$$V_i \boxtimes V_j = V_a \oplus V_{a+2} \oplus \ldots \oplus V_{k-b-2} \oplus V_{k-b} \qquad (*)$$

with $a = |i-j|$ and $b = |k-(i+j)|$.

The ***quantum dimensions*** $d_i = \dim(V_i)$ are the entries of the Perron-Frobenius eigenvector of the adjacency matrix of the above graph, normalized so that $d_0 = 1$:

$$d_i = [i+1]_q = \frac{q^{i+1} - q^{-(i+1)}}{q - q^{-1}} = q^{-i} + q^{-i+2} + q^{-i+4} + \ldots + q^{i-2} + q^i.$$

The ***balancing phases*** $\theta_i : V_i \to V_i$ are given by $\theta_i = q^{i+\frac{1}{2}i^2}$

[*Note:* If one tries to adopt $(*)$ as *definition* of the fusion product, then the braiding $\beta : V_i \boxtimes V_j \to V_j \boxtimes V_i$ and associator $\alpha : (V_i \boxtimes V_j) \boxtimes V_l \to V_i \boxtimes (V_j \boxtimes V_l)$ must be provided by explicit formulas. This can be done, but the formulas are messy.]

Realizations: • It is the semi-simple quotient of the category of (finite dimensional) tilting modules of the restricted quantum group $U_q^{\text{res}}\mathfrak{sl}(2)$ specialized at $q = e^{\frac{\pi i}{k+2}}$.

• It is also the category of projective positive energy representations of the loop group $LSU(2)$ with Lie algebra cocycle given by $c(f,g) = \frac{-k}{2\pi i}\int_{S^1} tr(f dg)$.

Low level examples:

level	$k=1$	$k=2$	$k=3$
$q = e^{\frac{\pi i}{k+2}}$	$q^6 = 1$	$q^8 = 1$	$q^{10} = 1$
d_i	$1 \quad 1$	$1 \quad \sqrt{2} \quad 1$	$1 \quad \phi \quad \phi \quad 1$
$D = \sqrt{\sum d_i^2}$	$\sqrt{2}$	2	$5+\sqrt{5}$
$\frac{1}{2\pi}\arg(\theta_i)$	$0 \quad 1/4$	$0 \quad 3/16 \quad 1/2$	$0 \quad 3/20 \quad 2/5 \quad 3/4$
$p_+ = \sum d_i^2 \theta_i = D \cdot e^{2\pi i \frac{3k}{8(k+2)}}$	$1+i$	$2 \cdot e^{2\pi i \frac{3}{16}}$	$(5+\sqrt{5}) \cdot e^{2\pi i \frac{9}{40}}$