simplicial sets. A stratification of a topological space X is some kind of map from X to $\mathbb{J}$. Analogously, we let:

Definition 4.31 *Let $\mathbb{J}$ be a poset and $N\mathbb{J}$ its nerve. An stratification of a simplicial set X by $\mathbb{J}$ is a map $s : X \to N\mathbb{J}$.*

Given $j \in \mathbb{J}$, we also introduce the notations

$$\begin{array}{lll} X_j := s^{-1}(j) & X_{\leq j} := s^{-1}(\mathbb{J}_{\leq j}) & X_{<j} := s^{-1}(\mathbb{J}_{<j}) \\ & X_{\geq j} := s^{-1}(\mathbb{J}_{\geq j}) & X_{>j} := s^{-1}(\mathbb{J}_{>j}) \end{array} \tag{4.54}$$

where $\mathbb{J}_{\leq j}$, $\mathbb{J}_{<j}$, $\mathbb{J}_{\geq j}$, and $\mathbb{J}_{>j}$ are the sub-posets of elements smaller or equal, smaller, greater or equal, and greater than j respectively.

As for topological spaces and simplicial sets, there is an adjunction between the category of $\mathbb{J}$-stratified spaces and $\mathbb{J}$-stratified simplicial sets. Given a stratified simplicial set X, its geometric realization $|X|$ is stratified via the composite

$$|X| \xrightarrow{|s|} |N\mathbb{J}| \to \mathbb{J}, \tag{4.55}$$

where the second map is given by $(j_0 \leq \ldots \leq j_k; t_0, \ldots, t_k) \mapsto \min_{r : t_r \neq 0} j_r$.

Geometric realization has a right adjoint $\mathrm{Sing}_{\mathbb{J}}$ given by

$$\mathrm{Sing}_{\mathbb{J}}(X)_k = \coprod_{s:\Delta[k]\to N\mathbb{J}} \mathrm{Map}_{\mathbb{J}}\Big(\big|(\Delta[k], s)\big|,\, X\Big).$$

The map $\mathrm{Sing}_{\mathbb{J}}(X) \to N\mathbb{J}$ is given by $(f : |(\Delta[k], s)| \to X) \mapsto s \in \mathrm{Hom}(\Delta[k], N\mathbb{J}) = (N\mathbb{J})_k$.

We now define our model structure on the category of $\mathbb{J}$-stratified simplicial sets. We use the criterion, originally due to Kan, for building (cofibrantly generated) model structures [18, section 11.3], [19, chapter 2]. Recall that the i-horn $\Lambda[n, i] \subset \Delta[n]$ is the sub-simplicial set of $\Delta[n]$ generated by all proper faces containing the ith vertex.

Definition 4.32 *Let $\mathbb{J}$ be a poset and $\mathsf{sSet} \downarrow N\mathbb{J}$ be the category of simplicial sets over $N\mathbb{J}$. The stratified model structure on $\mathsf{sSet} \downarrow N\mathbb{J}$ is given by:*

- *The weak equivalences are a map $X \to Y$ that induces weak equivalences of simplicial sets $X_{\leq j} \to Y_{\leq j}$ for all $j \in \mathbb{J}$.*

- *The generating cofibrations are the maps $\partial\Delta[n] \to \Delta[n]$, where the stratification $s : \Delta[n] \to N\mathbb{J}$ is arbitrary.*

- *The generating acyclic cofibrations are the maps $\Lambda[n, i] \subset \Delta[n]$, where $i < n$ <u>or</u> s assigns the same value to the nth and $n-1$st vertices of $\Delta[n]$.*

Conjecture 4.33 *Assuming that $\mathbb{J}$ doesn't have infinite descending chains, then definition 4.32 defines a model category structure on $\mathsf{sSet} \downarrow N\mathbb{J}$.*

The fibrations in this conjectured model category are what we call stratified fibrations of simplicial sets. Using this model category, there exists an analog of Theorem 4.29 for lists across stratified fibrations.

We expect that the geometric realization of a stratified fibration of simplicial sets is a stratified fibration of spaces.

4.8 Geometric realization

In [30], Quillen showed that the geometric realization of a Kan fibration is a Serre fibration. We beleive that the corresponding fact also holds for stratified fibrations (Theorem 4.38).

We have been able to prove this fact only assuming the following technical result:

Conjecture 4.34 *Let $p : E \to X$ be a map of stratified topological spaces and $j \in \mathbb{J}$ an element such that $E_k = X_k = \emptyset$ for all elements $k \in \mathbb{J}$ which are not comparable with j.*

Suppose that p can be written as

$$\begin{array}{c} E \\ \downarrow{\scriptstyle p} \\ X \end{array} = \textit{pushout} \left(\begin{tikzcd} E_{\geq j} \arrow[d, "\hat{p}"] & \widetilde{E} \arrow[l] \arrow[r, hook] \arrow[d, "\tilde{p}"] & \overline{E} \arrow[d, "\bar{p}"] \\ X_{\geq j} & \widetilde{X} \arrow[l] \arrow[r, hook] & \overline{X} \end{tikzcd} \right) \tag{4.56}$$

If the three maps $\hat{p}$, $\overline{p}$ and $\widetilde{E} \to E_{\geq j} \times_{X_{\geq j}} \widetilde{X}$ are stratified fibrations, then so is p.

The archetype for a Serre fibration is the projection map $F \times X \to X$. Any Serre fibration is locally a retract of one of that form. The following Lemma provides a similar archetype for stratified fibrations.

Lemma 4.35 *Let X be be a $\mathbb{J}$-stratified space and let $(\{F_j\}, \{r_{jj'} : F_j \to F_{j'}\}_{j<j'})$ be a diagram of spaces indexed by $\mathbb{J}$. Suppose moreover that all the $r_{jj'}$ are Serre fibrations. Let E be the space*

$$E = \coprod_j F_j \times X_{\geq j} \Big/ \sim, \tag{4.57}$$

where $(v, x) \sim (r_{jj'}(v), x)$ for all $x \in X_{\geq j'}$ and $v \in F_j$. Then the projection $p : E \to X$ is a stratified fibration.

Proof. Since stratified fibration is a local property, we may assume that $\mathbb{J}$ is finite. We prove the lemma by induction on the size of $\mathbb{J}$. We assume that it holds for $|\mathbb{J}| = j - 1$ and want to show that it holds for $|\mathbb{J}| = j$. Without loss of generality, we can assume that $\mathbb{J} = \{1, \dots, j\}$.

The map $p : E \to X$ is a pushout of

$$\begin{tikzcd} F_j \times X_j \arrow[d] & F_{j-1} \times X_j \arrow[l, two heads] \arrow[r, hook] \arrow[d] & E' \arrow[d] \\ X_j \arrow[r, equal] & X_j \arrow[r, hook] & X \end{tikzcd}$$

where

$$E' = \coprod_{j'<j} F_{j'} \times X_{\geq j'} \Big/ \sim \ . \tag{4.58}$$

The map $E' \to X$ is a stratified fibrations by induction. The conditions of Conjecture 4.34 are satisfied and so p is a stratified fibration. $\square$

In order to apply Lemma 4.35 to the question of geometric realization, we show the following technical lemma.

Lemma 4.36 *Let $p : E \to \Delta[n]$ be a stratified fibration. Given a face $\sigma : \Delta[k] \hookrightarrow \Delta[n]$ we let $\Gamma(\sigma; p)$ denote the simplicial set of sections of $\sigma^* p$. We then have a homeomorphism*

$$|E| = \coprod_{\sigma:\Delta[k]\hookrightarrow\Delta[n]} |\Gamma(\sigma; p)| \times \Delta^k \Big/ (s|_{d_i(\sigma)}, x) \sim (s, d^i(x)), \tag{4.59}$$

where $d_i : \Delta[n]_k \to \Delta[n]_{k-1}$ and $d^i : \Delta^{k-1} \to \Delta^k$ are the face and coface maps.

Proof. Given a face $\sigma : \Delta[k] \to \Delta[n]$, we build a map $f_\sigma : |\Gamma(\sigma; p)| \times \Delta^k \to |E|$. Let $((\tau, t), y) \in |\Gamma(\sigma; p)| \times \Delta^k$ be a point, where $\tau \in \Gamma(\sigma; p)_r$, $t \in \Delta^r$, and $y \in \Delta^k$. The adjoint of $\tau : \Delta[r] \to \Gamma(\sigma; p)$ is a map

$$\tilde{\tau} : \Delta[r] \times \Delta[k] \to \sigma^* E.$$

which commutes with the projections to $\Delta[k]$.

For suitable map $\kappa : \Delta[m] \to \Delta[k]$, there exists an embedding $\iota : \Delta[r] \times \Delta[k] \hookrightarrow \Delta[m]$ making the diagram

$$\begin{tikzcd} \Delta[r] \times \Delta[k] \arrow[r, "\tilde{\tau}"] \arrow[d, hook, "\iota"'] & \sigma^* E \arrow[d, "\sigma^* p"] \\ \Delta[m] \arrow[r, "\kappa"] \arrow[ur, dotted, "\nu"] & \Delta[k] \end{tikzcd} \tag{4.60}$$

commute (one can always pick $m = (r+1)(k+1) - 1$). The fibers of $|\kappa|$ and $|\kappa \circ \iota|$ are all contractible, so ι is a directed cofibration. Therefore (4.60) admits a lift ν. Let $\bar{\nu} : \Delta[m] \to E$ be the composite of ν with the inclusion $\sigma^* E \hookrightarrow E$. The adjoint of ι then fits into the following commutative triangle:

$$\begin{tikzcd} & \Gamma(\sigma;\kappa) \arrow[d, "\nu^*"] \\ \Delta[r] \arrow[ur, "\tilde{\iota}"] \arrow[r, "\tau"] & \Gamma(\sigma;p). \end{tikzcd} \tag{4.61}$$

Let $x := |\tilde{\iota}|(y)$. The preimage under κ of the ith vertex of $\Delta[k]$ is isomorphic to some sub-simplex $\Delta[\ell_i] \subset \Delta[m]$. The simplicial set of sections of κ is isomorphic to the product

$$\Gamma(\sigma;\kappa) = \Delta[\ell_0] \times \Delta[\ell_1] \times \ldots \times \Delta[\ell_k], \tag{4.62}$$

so we can view x as a point in $\prod \Delta^{\ell_i}$. The simplex Δ^m can be identified with the join $\Delta^{\ell_0} * \ldots * \Delta^{\ell_k}$ so we get a corresponding projection map

$$\varphi : \Big(\Delta^{\ell_0} \times \Delta^{\ell_1} \times \ldots \times \Delta^{\ell_k}\Big) \times \Delta^k \twoheadrightarrow \Delta^m. \tag{4.63}$$

At this point, we let $z := \varphi\big(x, y\big) \in \Delta^m$ and define $f_\sigma((\tau, t), y) := (\bar{\nu}, z)$.

We now show that f_σ is well defined. Let us replace ι by $\iota' := d^i \circ \iota : \Delta[r] \times \Delta[k] \to \Delta[m+1]$. Then we have $d_i(\nu') = \nu$ and therefore also $d_i(\bar{\nu}') = \bar{\nu}$. Composing with $d^i : \Delta[m] \to \Delta[m+1]$ induces a map $\Gamma(\sigma,\kappa) \to \Gamma(\sigma,\kappa')$ which, under the isomorphism (4.62) becomes

$$\begin{aligned} 1\times \ldots \times d^j \times \ldots \times 1 : \Delta[\ell_0] \times \ \ldots \ \times \Delta[\ell_a] \times \ \ldots \ \times \Delta[\ell_k] \\ \to \Delta[\ell_0] \times \ldots \times \Delta[\ell_a + 1] \times \ldots \times \Delta[\ell_k] \end{aligned}$$

for some appropriate j and a. We have a commutative diagram

$$\begin{tikzcd} \big(\Delta^{\ell_0} \times \ldots \times \Delta^{\ell_a} \times \ldots \times \Delta^{\ell_k}\big) \arrow[r, "\varphi"] \arrow[d, dashed, "1\times\ldots\times d^j\times\ldots\times 1"'] & \Delta^m \arrow[d, "d^i"] \\ \big(\Delta^{\ell_0} \times \ldots \times \Delta^{\ell_a+1} \times \ldots \times \Delta^{\ell_k}\big) \arrow[r, "\varphi'"] & \Delta^m \end{tikzcd}$$

and we can now compute

$$\begin{aligned} (\bar{\nu}', z') = \big(\bar{\nu}', \varphi'(x', y)\big) = \big(\bar{\nu}', \varphi'(|\tilde{\iota}'|(t), y)\big) = \Big(\bar{\nu}', \varphi'\big(\big|(1\times \ldots \times d^j \times \ldots \times 1) \circ \tilde{\iota}\big|(t), y\big)\Big) \\ = \big(\bar{\nu}', d^i\varphi\big(|\tilde{\iota}|(t), y\big)\big) = \big(\bar{\nu}', d^i\varphi\big(x, y\big)\big) = \big(\bar{\nu}', d^i(z)\big) = \big(d_i(\bar{\nu}'), z\big) = (\bar{\nu}, z). \end{aligned}$$

This shows us that $f_\sigma((\tau, t), y)$ doesn't depend on the choice of $\iota : \Delta[r] \times \Delta[k] \to \Delta[m]$.

We now assemble all the f_σ to get a map f from the RHS of (4.59) to $|E|$. Suppose that we have two equivalent points $((\tau, t), y) \in |\Gamma(\sigma;p)| \times \Delta^k$ and $((\tau', t'), y') \in |\Gamma(\sigma';p)| \times \Delta^{k-1}$ where $\sigma' = d_i(\sigma)$. We can assume that $t = t'$, $y = d^i(y')$ and $\tau' = \tau \circ res$ where $res : \Gamma(\sigma, p) \to \Gamma(\sigma';p)$ is the restriction map. The map $\tilde{\tau}' : \Delta[r] \times \Delta[k-1] \to \sigma'^* E$ is then the pullback of $\tilde{\tau}$ along the map $d^i : \Delta[k-1] \to \Delta[k]$. We may also take ι', κ', ν' to be the corresponding pullbacks of ι, κ, ν. We then have

$\bar{\nu}' = d_I \bar{\nu}$, where d_I is some appropriate composition of face maps and $\tilde{\iota}' = res \circ \tilde{\iota}$, where $res : \Gamma(\sigma, \kappa) \to \Gamma(d_i(\sigma), \kappa')$ is the restriction. Under the isomorphism (4.62), this restriction map become the projection

$$\mathrm{pr}_i : \Delta[\ell_0] \times \ldots \times \Delta[\ell_i] \times \ldots \times \Delta[\ell_k] \to \Delta[\ell_0] \times \ldots \times \widehat{\Delta[\ell_i]} \times \ldots \times \Delta[\ell_k].$$

It satisfies the equation $\varphi(t, d^i(x)) = d^I\big(\varphi'(\mathrm{pr}_i(t), x)\big)$. Putting all this together, we see that

$$\begin{aligned} f_{\sigma'}((\tau', t), y') = (\bar{\nu}', z') = \big(\bar{\nu}', \varphi'(x', y')\big) &= \big(\bar{\nu}', \varphi'(|\tilde{\iota}'|(t), y')\big) \\ = \big(d_I\bar{\nu}, \varphi'(|res \circ \tilde{\iota}|(t), y')\big) &= \big(\bar{\nu}, d^I\varphi'(|\mathrm{pr}_i \circ \tilde{\iota}|(t), y')\big) \\ = \big(\bar{\nu}, \varphi(|\tilde{\iota}|(t), d^i(y'))\big) = \big(\bar{\nu}, \varphi(|\tilde{\iota}|(t), y)\big) &= \big(\bar{\nu}, \varphi(x, y)\big) = (\bar{\nu}, z) = f_\sigma((\tau, t), y). \end{aligned}$$

This finishes the proof that f : [RHS of (4.59)] $\to |E|$ is well defined.

Finally, we show that f is an isomorphism. We do this by constructing an inverse f^{-1}. Let $(\bar{\nu}, z) \in |E|$ be a point given by $\bar{\nu} \in E_m$ and $z \in \Delta^m$. We assume furthermore that $\bar{\nu}$ is non-degenerate and that $z \in \mathring{\Delta}^m$. Let σ be the smallest sub-simplex of $\Delta[n]$ such that $p \circ \bar{\nu}$ factorizes as

$$\begin{tikzcd} \Delta[m] \arrow[r, "\bar{\nu}"] \arrow[d, "\kappa"'] & E \arrow[d, "p"] \\ \Delta[k] \arrow[r, "\sigma"] & \Delta[n]. \end{tikzcd} \tag{4.64}$$

The map (4.63) is an isomorphism over $\mathring{\Delta}^k$, so we can define $(x, y) := \varphi^{-1}(z)$.

Now we consider the standard triangulation of $\Delta^{\ell_0} \times \ldots \times \Delta^{\ell_k}$. Let $\alpha : \Delta^r \to \Delta^{\ell_0} \times \ldots \times \Delta^{\ell_k}$ be a simplex in the image of which x lies and $t := \alpha^{-1}(x) \in \Delta^r$. We let τ be the composite

$$\tau : \Delta[r] \xrightarrow{\alpha} \Delta[\ell_0] \times \ldots \times \Delta[\ell_k] = \Gamma(\sigma; \kappa) \hookrightarrow \Gamma(\sigma; p). \tag{4.65}$$

The image of $(\bar{\nu}, z)$ under the map f^{-1} is then declared to be $((\tau, t), y)$. This shows that f is an isomorphism and thus finishes the proof of the lemma. □

Lemma 4.36 and Lemma 4.35 together imply

Lemma 4.37 *Let $s : \Delta[n] \to N\mathbb{J}$ be a stratification and $|s| : \Delta^n \to \mathbb{J}$ the induced stratification on the geometric realization.*

Let $p : E \to \Delta[n]$ be a stratified fibration with respect to s. Then its geometric realization $|p| : |E| \to \Delta^n$ is a stratified fibration with respect to $|s|$.

Proof. Let $\tilde{s}$ be the stratification of Δ^n by it's poset of faces. We first show that $|p|$ is a stratified fibration with the respect $\tilde{s}$.

By Lemma 4.36, $|p|$ is of the form (4.57) where the F_j's are replaced by $|\Gamma(\sigma; E)|$. In order to apply Conjecture 4.34, we need to show that the restriction maps $|\Gamma(\sigma; E)| \to |\Gamma(\sigma'; E)|$ are Serre fibrations. We show that the corresponding maps of simplicial sets

$$r : \Gamma(\sigma; E) \to \Gamma(\sigma'; E) \tag{4.66}$$

are Kan fibrations. Indeed, a lift

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "\iota"'] & \Gamma(\sigma; E) \arrow[d, two heads] \\ B \arrow[r] \arrow[ur, dotted] & \Gamma(\sigma'; E) \end{tikzcd} \tag{4.67}$$

is equivalent to a lift

$$\begin{tikzcd} (A \times \sigma) \cup (B \times \sigma') \arrow[r] \arrow[d, hook] & E \arrow[d, two heads] \\ B \times \sigma \arrow[r] \arrow[ur, dotted] & \Delta[n]. \end{tikzcd} \tag{4.68}$$

If ι is an acyclic cofibration, then $(A\times\sigma)\cup(B\times\sigma') \hookrightarrow B\times\sigma$ is a directed cofibration. The diagram (4.68) has a lift, and so does (4.67). We conclude that (4.66) is a Kan fibration. So by Conjecture 4.34, $|p|$ is a stratified fibration with the respect to $\tilde{s}$.

In order to show that $|p|$ is a stratified fibration with respect to $|s|$, we appeal to Lemma 4.30. We need to show that the restriction (4.66) are homotopy equivalences whenever $|\sigma|$ and $|\sigma'|$ are in the same stratum.

So we go back to (4.67), but now $A \hookrightarrow B$ is an arbitrary cofibration. Again, $(A \times \sigma) \cup (B \times \sigma') \hookrightarrow B \times \sigma$ is a directed cofibration, so (4.68) and (4.67) have lifts. We conclude that (4.66) is an acyclic Kan fibration. $\square$

We can now prove the main theorem of this section (modulo Conjecture 4.34).

Theorem 4.38 *If $p : E \to X$ is a stratified fibration of simplicial sets, then its geometric realization $|p| : |E| \to |X|$ is a stratified fibration in the sense of definition 4.9.*

Proof. The space $|X|$ is the covered by its skeleta $|X^{(i)}|$. So by Lemma 4.19, it's enough to show the result when X is finite dimensional. Let us assume by induction that it holds when X has dimension $< n$.

Let X be a simplicial set of dimension n. The $(n-1)$-skeleton, along with the images of the n-simplices form a closed cover of $|X|$. So by Lemma 4.19, it's enough to show that the restrictions of $|p|$ to these various subspaces are all stratified fibrations.

The map $\left|E|_{X^{(n-1)}}\right| \to |X^{(n)}|$ is a stratified fibration by induction, so we concentrate on the other case. Let $\sigma : \Delta[n] \to X$ be a simplex and K its image in X. To show that $\left|E|_K\right| \to |K|$ is a stratified fibration, we write it as a pushout of

$$\begin{tikzcd} \left|E|_Z\right| \arrow[d, two heads] & \left|E|_{\partial\Delta[n]}\right| \arrow[l] \arrow[r, hook] \arrow[d, two heads] & \left|E|_{\Delta[n]}\right| \arrow[d, two heads] \\ |Z| & \partial\Delta^n \arrow[l] \arrow[r, hook] & \Delta^n, \end{tikzcd} \tag{4.69}$$

where Z is the image of $\partial\Delta[n]$ in X. The leftmost vertical map is a stratified fibration by the induction hypothesis. The rightmost map is a stratified fibration by Lemma 4.37. Therefore (4.69) satisfies the hypothesis of Conjecture 4.34 and $\left|E|_K\right| \to |K|$ is a stratified fibration. This finishes the inductive step. $\square$

4.9 Equivariant stratified fibrations

In the presence of groups actions, one can develop a similar theory of stratified fibrations. Let K be a topological group, and $\mathcal{F}$ a family of subgroups (i.e. a set of subgroups closed under subconjugacy). We work in the category $(K, \mathcal{F})$-spaces of K-spaces with stabilizers in $\mathcal{F}$. A stratification of an K-space X is a K-invariant upper semi-continuous function $s : X \to \mathbb{J}$.

Definition 4.39 *An equivariant generating directed cofibration is an inclusion of the form*

$$K/G \times \Lambda^n \hookrightarrow K/G \times \Delta^n, \tag{4.70}$$

where $\Lambda^n \hookrightarrow \Delta^n$ is a non-equivariant directed cofibration (see Definition 4.8), and $G \in \mathcal{F}$.

An equivariant stratified fibration is a map satisfying the right lifting property with respect to the set of equivariant generating directed cofibrations. An equivariant directed cofibration is a map satisfying the left lifting property with respect to the class of equivariant stratified fibrations.

The characterization of directed cofibrations given in Theorem 4.15 has an immediate generalization to the equivariant situation.

Theorem 4.40 *Let (B, s) be an equivariant $\mathbb{J}$-stratified space and $A \subset B$ a subspace. Suppose that the image of s has no infinite descending chains (for example B is compact). Then the following are equivalent:*

- *$A^G_{\leq j} \to B^G_{\leq j}$ is a homotopy equivalence for all $j \in \mathbb{J}$ and all $G \in \mathcal{F}$.*
- *$A \hookrightarrow B$ is an equivariant directed cofibration.*

Proof. The proof of Theorem 4.15 goes through word for word. $\square$

In other words, a map $A \to B$ in $(K, \mathcal{F})$-spaces is an equivariant directed cofibration if and only if the corresponding maps on fixed points $p : A^G \to B^G$ are directed cofibrations. Similarly, we get a characterization of equivariant stratified fibrations in terms of non-equivariant ones.

Theorem 4.41 *Let $p : E \to X$ be an equivariant stratified map. Then p is an equivariant stratified fibrations if and only if the corresponding maps on fixed points $p : E^G \to X^G$ are stratified fibrations.*

Proof. The lifting problems

$$\begin{tikzcd} K/G \times \Lambda^n \arrow[r] \arrow[d, hook] & E \arrow[d, two heads] \\ K/G \times \Delta^n \arrow[r] \arrow[ur, dotted] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Lambda^n \arrow[r] \arrow[d, hook] & E^G \arrow[d, two heads] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & X^G \end{tikzcd}$$

are equivalent. $\square$

Chapter 5

Stratified classifying spaces

Given a poset $\mathbb{J}$ and a functor $F : \mathbb{J} \to Ho(\mathsf{spaces})$, a natural problem is to classify stratified fibrations $E \to X$ such that ∇ factors as

$$\begin{tikzcd} \Pi_1(X) \arrow[rr, "\nabla"] \arrow[dr] & & Ho(\mathsf{spaces}) \\ & \mathbb{J} \arrow[ur, "F"'] & \end{tikzcd} \tag{5.1}$$

In other words, we ask for the classification question given specified homotopy types of fibers and homotopy classes of the gluing maps.

In analogy with classical topology, we want a classifying space B such that homotopy classes of stratified maps $X \to B$ are in bijection with fiber-wise homotopy equivalence classes of stratified fibrations $E \to X$ making (5.1) commute.

Instead of developing the general theory, we deal directly with our example of interest: orbispaces structures.

5.1 Orbispace structures

In this section, $\mathbb{J}$ denotes the poset of isomorphism classes of finite groups. The order relation is given by $[H] < [G]$ if there exists a monomorphism $H \to G$. Recall from Definition 3.4 and Theorem 3.5 that an orbispace structure on a $\mathbb{J}$-stratified space X is a stratified fibration $E \to X$. The fibers F_v over points v in the G-stratum are $K(G,1)$'s, and the maps $\nabla_\gamma : F_v \to F_w$ are injective on fundamental groups.

Definition 5.1 *Let $\mathcal{J}$ be a collection of representatives for each isomorphism class of finite groups. The* classifying space for orbispace structures $\underline{\mathsf{Orb}}$ *is the simplicial set whose n-simplices are given by:*

- *for each vertex of $\Delta[n]$, a group $G_i \in \mathcal{J}$,*
- *for each edge of $\Delta[n]$, a monomorphism $\phi_{ij} : G_i \to G_j$,*

- *for each 2-face of $\Delta[n]$, an element $g_{ijk} \in G_k$ satisfying*

$$\phi_{ik} = Ad(g_{ijk})\, \phi_{jk}\, \phi_{ij}, \tag{5.2}$$

- *for each 3-face of $\Delta[n]$ these group elements satisfy the cocycle condition*

$$g_{ij\ell}\; g_{jk\ell} = g_{ik\ell}\, \phi_{k\ell}(g_{ijk}). \tag{5.3}$$

We fix $\phi_{ii} = 1$ and $g_{iij} = g_{ijj} = 1$.

The stratification $\underline{\mathsf{Orb}} \to N\mathbb{J}$ is given by forgetting the ϕ_{ij} and the g_{ijk}. There is a universal orbispace structure $E_{\underline{\mathsf{Orb}}} \to \underline{\mathsf{Orb}}$. An n-simplex of $E_{\underline{\mathsf{Orb}}}$ is given by:

- *for each vertex of $\Delta[n]$, a group $G_i \in \mathcal{J}$,*

- *for each edge of $\Delta[n]$, a monomorphism $\phi_{ij} : G_i \to G_j$ and an element $x_{ij} \in G_j$,*

- *for each 2-face of $\Delta[n]$, an element $g_{ijk} \in G_k$ satisfying (5.2) and*

$$x_{ik} = g_{ijk}\, \phi_{jk}(x_{ij})\, x_{jk}, \tag{5.4}$$

- *for each 3-face of $\Delta[n]$, the g_{ijk} satisfy (5.3).*

We fix $x_{ii} = 1$. The map $E_{\underline{\mathsf{Orb}}} \to \underline{\mathsf{Orb}}$ is given by forgetting the x_{ij}.

The fact that $|E_{\underline{\mathsf{Orb}}}| \to |\underline{\mathsf{Orb}}|$ is an orbispace is not entirely obvious. We need to check that $E_{\underline{\mathsf{Orb}}} \to \underline{\mathsf{Orb}}$ is a stratified fibration and that its fibers have the correct homotopy type. This is done in the following lemma.

Lemma 5.2 *The map $|E_{\underline{\mathsf{Orb}}}| \to |\underline{\mathsf{Orb}}|$ is a stratified fibration. The fiber F_v over a point v in the G-stratum is a $K(G, 1)$. The maps $\nabla : F_v \to F_{v'}$ induce injective homomorphisms on π_1.*

Proof. It's enough by Theorem 4.38 to check that $E_{\underline{\mathsf{Orb}}} \to \underline{\mathsf{Orb}}$ is a stratified fibration of simplicial sets. Let $\Lambda[n, j] \hookrightarrow \Delta[n]$ be an acyclic cofibration and let

$$\begin{tikzcd} \Lambda[n,j] \arrow[r] \arrow[d, hook] & E_{\underline{\mathsf{Orb}}} \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & \underline{\mathsf{Orb}} \end{tikzcd} \tag{5.5}$$

be our usual lifting problem. So we are given groups G_i on the vertices of $\Delta[n]$, homomorphisms ϕ_{ij} on the edges, and elements g_{ijk} on the 2-faces, satisfying (5.2) and (5.3). We are also given elements x_{ij} on the edges of $\Lambda[n, j]$, and we want to extend them to the rest of the edges while respecting the relation (5.4). We do this by a case by case study.

If $n = 1$, then $x_{01} \in G_1$ can be taken arbitrarily.

If $n = 2$ and $j = 0$, we solve (5.4) for x_{12} in terms of x_{01} and x_{02}. If $n = 2$ and $j = 1$, we solve (5.4) for x_{02} is terms of x_{01} and x_{12}. If $n = 2$, $j = 2$ and $G_1 = G_2$ then ϕ_{12} is invertible and we can also solve (5.4) for x_{01} in terms of x_{02} and x_{12}.

If $n = 3$, then all the x_{ij} are already given to us, but we still need to check the condition (5.4) on the 2-face which is not in $\Lambda[n, j]$. If $n = 3$, $j = 0$ we compute

$$\begin{aligned} x_{13} = \phi_{13}(x_{01})^{-1} g_{013}^{-1}\, x_{03} &= g_{123}\, \phi_{23}(\phi_{12}(x_{01}))^{-1} g_{123}^{-1}\, g_{013}^{-1}\, g_{023}\, \phi_{23}(x_{02})\, x_{23} \\ &= g_{123}\, \phi_{23}(x_{01})^{-1} \phi_{23}(g_{012})^{-1} \phi_{23}(x_{02})\, x_{23} = g_{123}\, \phi_{23}(x_{12})\, x_{23}\,. \end{aligned}$$

If $n = 3$, and $j = 1$ or 2 we can write down

$$\begin{aligned} g_{013}\, \phi_{13}(x_{01})\, x_{13} &= g_{013}\, g_{123}\, \phi_{23}(\phi_{12}(x_{01})) g_{123}^{-1}\, g_{123}\, \phi_{23}(x_{12})\, x_{23} \\ & g_{013} g_{123} \phi_{23}(g_{012}^{-1}\, x_{02})\, x_{23} = g_{023}\, \phi_{23}(x_{02})\, x_{23}\,. \end{aligned}$$

If $j = 1$, we start from x_{03}=LHS and conclude that x_{03}=RHS. If $j = 2$, we start from x_{03}=RHS and conclude that x_{03}=LHS.
If $n = 3$, $j = 3$ and $G_2 = G_3$, then ϕ_{23} is invertible and we can compute

$$\begin{aligned} x_{02} = \phi_{23}^{-1}\big(g_{023}^{-1}\, x_{03}\, x_{23}^{-1}\big) &= \phi_{23}^{-1}\big(g_{023}^{-1}\, g_{013}\, \phi_{13}(x_{01})\, x_{13}\, x_{23}^{-1}\big) \\ &= \phi_{23}^{-1}\big(g_{023}^{-1}\, g_{013}\, g_{123}\, \phi_{23}(\phi_{12}(x_{01})) g_{123}^{-1}\, g_{123}\, \phi_{23}(x_{12})\big) = g_{012}\, \phi_{12}(x_{01})\, x_{12} \end{aligned}$$

If $n \geq 4$ all the data needed for a map $\Delta[n] \to E_{\underline{\mathsf{Orb}}}$ is already present in the map $\Lambda[n, j] \to E_{\underline{\mathsf{Orb}}}$. This finishes the verification that $E_{\underline{\mathsf{Orb}}} \to \underline{\mathsf{Orb}}$ is a stratified fibration.

We now check that the fibers F_v have the correct homotopy type. Since the homotopy type of fibers doesn't depend on the particular point in a stratum, we can assume that v is a vertex of $|\underline{\mathsf{Orb}}|$. So F_v is the geometric realization of the corresponding fiber of $E_{\underline{\mathsf{Orb}}} \to \underline{\mathsf{Orb}}$. An n-simplex in that fiber is an n-simplex in $E_{\underline{\mathsf{Orb}}}$ where $G_i = G$, $\phi_{ij} = 1$ and $g_{ijk} = 1$ for all i, j, k. The x_{ij}'s satisfy (5.4), which now reads $x_{ik} = x_{ij} x_{kj}$, so the fiber is isomorphic to the standard model of $K(G, 1)$.

Finally, given a path γ from v_0 to v_1, we check that $\nabla_\gamma : F_{v_0} \to F_{v_1}$ induces an injective homomorphism of fundamental groups. As before, we assume that v_0 and v_1 are vertices of $\underline{\mathsf{Orb}}$. Since ∇_γ only depends on the homotopy class γ, we may also assume that γ is an edge $\{G_0, G_1, \phi\}$ of $\underline{\mathsf{Orb}}$.

Recall that ∇_γ is defined using a lift ℓ of the diagram (4.43). Such a lift

$$\ell : K(G_0, 1) \times \Delta[1] \to E_{\underline{\mathsf{Orb}}} \tag{5.6}$$

can be written explicitly. An n-simplex σ of $K(G_0, 1) \times \Delta[1]$ consists of elements $g_{ij} \in G_0$ and numbers $\varepsilon_i \in \{0, 1\}$ satisfying $g_{ik} = g_{ij}\, g_{jk}$ and $\varepsilon_i \leq \varepsilon_j$ for $i \leq j$, where $i \leq j$ range over the set of vertices of $\Delta[n]$. Its image $\ell(\sigma)$ assigns the group G_{ε_i} to the vertex i, the element $g_{ij} \in G_0$ to the edges ij with $\varepsilon_j = 0$ and $\phi(g_{ij}) \in G_1$ to the edges ij with $\varepsilon_j = 1$.

The map ∇_γ was defined by Precomposing ℓ with the inclusion of $K(G_0, 1) \times \{1\}$. In our case, we get the map $K(G_0, 1) \to K(G_1, 1)$ induced by ϕ. So ∇_γ induces ϕ on fundamental groups, which is injective by definition of $\underline{\mathsf{Orb}}$. $\square$

If a map $X \to |\underline{\mathsf{Orb}}|$ is simplicial with respect to some triangulation $\mathcal{T}$ of X, then we get a group for every vertex a monomorphism for every edge and a group element for each triangle. All these subject to the cocycle conditions (5.2) and (5.3). Inversely, such a collection of data determines a simplicial map $X \to |\underline{\mathsf{Orb}}|$.

Definition 5.3 *Let X be a $\mathbb{J}$-stratified space and $\mathcal{T}$ an oriented triangulation of X. Recall the set $\mathcal{J}$ from Definition 5.1.*

An $\underline{\mathsf{Orb}}$-valued cocycle of X consists of a group $G_v \in \mathcal{J}$ for every vertex v, a monomorphism $\phi_{vw} : G_v \to G_w$ for every edge vw, and an element $g_{uvw} \in G_w$ for each triangle uvw, all subject to the relations (5.2) and (5.3). Moreover, if v belongs to the G-stratum of X, we require that $G_v \simeq G$.

An $E_{\underline{\mathsf{Orb}}}$-valued cocycle is an $\underline{\mathsf{Orb}}$-valued cocycle along with elements $x_{vw} \in G_w$ for each edge vw, subject (5.4).

Two cocycles c, c' on X are equivalent if $c \sqcup c'$ extends to a cocycle on $X \times [0,1]$.

Given a stratified map $X \to |\underline{\mathsf{Orb}}|$, the pullback of $|E_{\underline{\mathsf{Orb}}}|$ to X induces an orbispace structure on X. This provides a bijection between homotopy classes of maps into $|\underline{\mathsf{Orb}}|$ and orbispace structures on X. In other words, $|\underline{\mathsf{Orb}}|$ is a classifying space for orbispace structures.

Theorem 5.4 *Let $\mathbb{J}$ be the poset of isomorphism classes of finite sets. Given a $\mathbb{J}$-stratified space X, the assignment $f \mapsto f^*|E_{\underline{\mathsf{Orb}}}|$ provides a bijection*

$$\left\{ \begin{array}{c} \textit{Homotopy classes of} \\ \mathbb{J}\textit{-stratified maps } X \to |\underline{\mathsf{Orb}}|. \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Isomorphism classes of} \\ \textit{orbispace structures on } X. \end{array} \right\} \tag{5.7}$$

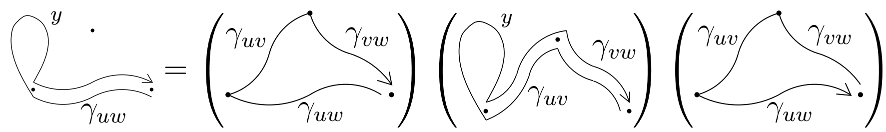
Proof. We first show that the orbifold structure $f^*|E_{\underline{\mathsf{Orb}}}|$ only depends on the homotopy class of f. Let f' be another map and $h : X \times [0,1] \to |\underline{\mathsf{Orb}}|$ a homotopy between f and f'. Composing the lift of

$$\begin{tikzcd} f^*|E_{\underline{\mathsf{Orb}}}| \arrow[r, hook] \arrow[d, hook, "i_0"'] & h^*|E_{\underline{\mathsf{Orb}}}| \arrow[d] \\ f^*|E_{\underline{\mathsf{Orb}}}| \times [0,1] \arrow[r] \arrow[ur, dotted] & X \times [0,1] \end{tikzcd} \tag{5.8}$$

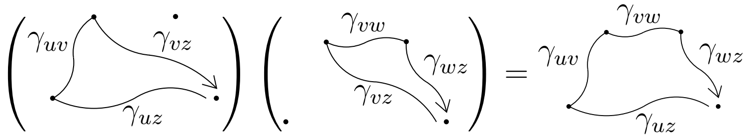
with the inclusion $i_1 : f^*|E_{\underline{\mathsf{Orb}}}| \hookrightarrow f^*|E_{\underline{\mathsf{Orb}}}| \times [0,1]$ produces a map $f^*|E_{\underline{\mathsf{Orb}}}| \to f'^*|E_{\underline{\mathsf{Orb}}}|$. Another application of the homotopy lifting property shows that $f^*|E_{\underline{\mathsf{Orb}}}|$ and $f'^*|E_{\underline{\mathsf{Orb}}}|$ are actually homotopy equivalent as spaces over X (one could also use Theorem 4.21).

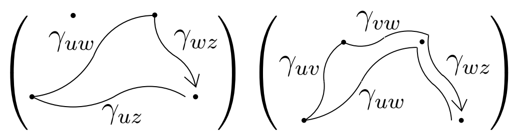
Given an orbispace structure $E \to X$, we now construct a map $f : X \to |\underline{\mathsf{Orb}}|$ such that $E \simeq f^*|E_{\underline{\mathsf{Orb}}}|$. Let $\mathcal{T}$ be an oriented triangulation of X. For every vertex v, we pick a point b_v in the fiber F_v and an isomorphism $\alpha_v : \pi_1(F_v, b_v) \xrightarrow{\sim} G_v$ for an appropriate $G_v \in \mathcal{J}$. For each edge vw, we pick a path γ_{uv} over that edge, linking b_v to b_w. Such an edge induces a group homomorphism $\tilde{\phi}_{vw} : \pi_1(F_v, b_v) \to \pi_1(F_w, b_w)$ by sending a loop $y \in \pi_1(F_v, b_v)$ to $\gamma_{vw}^{-1} \cdot y \cdot \gamma_{vw}$ and then pushing it into F_w. Let $\phi_{vw} := \alpha_w \circ \tilde{\phi}_{vw} \circ \alpha_v^{-1}$.For each triangle uvw of X, the three paths γ_{uw}, γ_{uv} and γ_{vw}

assemble to a loop $S^1 \to E$. Pushing it into F_w produces an element of $\pi_1(F_w, b_w)$. We let $g_{uvw} \in G_w$ be the image of that loop under α_w. One now checks that $c = \{G_v, \phi_{vw}, g_{uvw}\}$ is an $\underline{\mathsf{Orb}}$-valued cocycle on X i.e. a map $f : X \to |\underline{\mathsf{Orb}}|$:

$\phi_{uw}(y) =$ 

$$= g_{uvw}\, \phi_{vw}(\phi_{uv}(y)) g_{uvw}^{-1} = \big(Ad(g_{uvw})\phi_{vw}\phi_{uv}\big)(y).$$

$g_{uvw}\, g_{vwz} =$ 

$=$  $= g_{vwz}\, \phi_{wz}(g_{uvw}).$

Suppose that we make some other choices $\{\mathcal{T}', b'_{v'}, \alpha'_{v'}, \gamma'_{v'w'}\}$. These give a cocycle c' and a corresponding map f'. Let $\mathcal{T}''$ be a triangulation of $X \times [0,1]$ that restricts to $\mathcal{T}$ on $X \times \{0\}$ and to $\mathcal{T}'$ on $X \times \{1\}$. Extend $\{\gamma_{vw}, \gamma'_{v'w'}\}$ to all the edges of $\mathcal{T}''$. This produces a cocycle c'' on $X \times [0,1]$ extending $c \sqcup c'$, in other words a homotopy between f and f'.

We now show that f doesn't change when we replace E by another space E' which is homotopy equivalent over X. Let $e : E \to E'$ be such an equivalence. One then checks that $b'_v := e(b_v)$, $\alpha'_v := \alpha_v \circ \pi_1(e|_{F_v})^{-1}$, and $\gamma'_{vw} := e \circ \gamma_{vw}$ induce the same cocycle as $b_v, \alpha_v, \gamma_{vw}$. This finishes the proof that $f : X \to |\underline{\mathsf{Orb}}|$ is well defined up to homotopy.

We now show that $f^*|E_{\underline{\mathsf{Orb}}}| \to X$ is equivalent to the original orbispace $E \to X$. A simplex of $f^*|E_{\underline{\mathsf{Orb}}}|$ is the same thing as a simplex $\sigma \subset X$ along with data $\{x_{vw}\}$ extending the $\underline{\mathsf{Orb}}$-valued cocycle $c|_\sigma$ to an $E_{\underline{\mathsf{Orb}}}$-valued cocycle. We use the notation $[\sigma, \{x_{vw}\}]$ to denote such a simplex.

We now describe an equivalence $e : f^*|E_{\underline{\mathsf{Orb}}}| \to E$. It sends the 0-simplex of $K(G_v, 1)$ to x_v. Given an edge $[vw, x]$ of $f^*|E_{\underline{\mathsf{Orb}}}|$, consider the path $\gamma_{vw} \cdot \lambda_{vw}$, where $\lambda_{vw} = e(\alpha_w^{-1}(x))$. Using Lemma 4.17, that path can be rectified to a path ε_{vw} over the edge vw. We then let $e(vw, x) := \varepsilon_{vw}$.

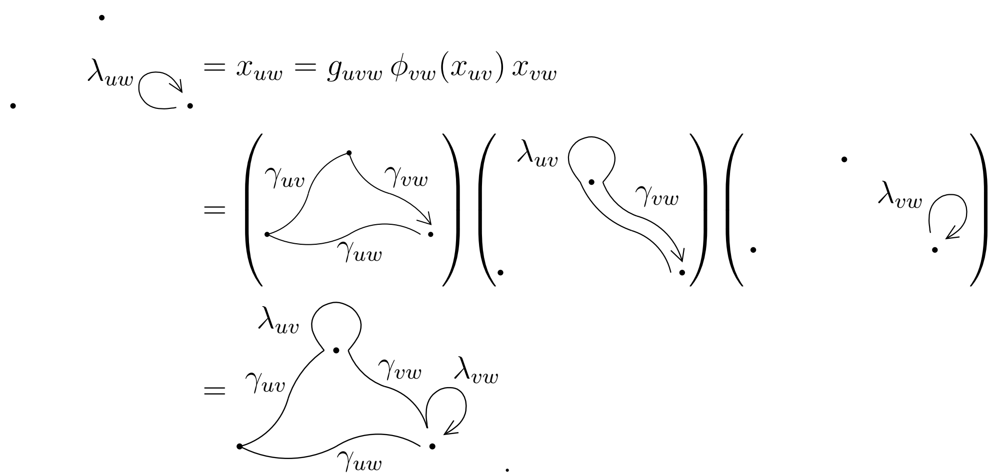
So far, we have constructed the map e on $\mathrm{sk}_1(f^*|E_{\underline{\mathsf{Orb}}}|)$. We now use obstruction theory to extend it to the whole $f^*|E_{\underline{\mathsf{Orb}}}|$. By Theorem 4.29, the obstructions to finding a lift

$$\begin{tikzcd}
\mathrm{sk}_1(f^*|E_{\underline{\mathsf{Orb}}}|) \arrow[r, "e"] \arrow[d, hook] & E \arrow[d] \\
f^*|E_{\underline{\mathsf{Orb}}}| \arrow[r] \arrow[ur, dotted] & X
\end{tikzcd} \tag{5.9}$$

lie in $H^{k+1}\big(f^*|E_{\underline{\mathsf{Orb}}}|, \mathrm{sk}_1(f^*|E_{\underline{\mathsf{Orb}}}|); \pi_k F\big)$. Since all the fibers are $K(\pi,1)$'s, we only need to check that the obstruction in

$$H^2\big(f^*|E_{\underline{\mathsf{Orb}}}|\,, \mathrm{sk}_1(f^*|E_{\underline{\mathsf{Orb}}}|); \pi_1 F\big) = \prod_{\substack{\text{2-simplices}\\ [uvw,\{x_{uv},x_{uw},x_{vw}\}]}} G_w \tag{5.10}$$

vanishes. So, for each 2-simplex $[uvw, \{x_{uv}, x_{uw}, x_{vw}\}]$ of $f^*|E_{\underline{\mathsf{Orb}}}|$, we need to check that the loop composed of ε_{uv}, ε_{uw}, and ε_{vw} represents the trivial element in $\pi_1(F_w)$. Equivalently, we show that the loops λ_{uw} and $\gamma_{uw}^{-1} \cdot \gamma_{uv} \cdot \lambda_{uv} \cdot \gamma_{vw} \cdot \lambda_{vw}$ are homotopic:



(5.11)

In order to finish the proof, we need to show that the map $f' : X \to |\underline{\mathsf{Orb}}|$ associated to $f^*|E_{\underline{\mathsf{Orb}}}|$ is homotopic to f. By naturality, it's enough to check this when $X = |\underline{\mathsf{Orb}}|$ and $f = \mathrm{Id}$. In that case, the triangulation of $|\underline{\mathsf{Orb}}|$ may be taken to be the standard one. The points x_v may be taken to be the 0-simplices of $|E_{\underline{\mathsf{Orb}}}|$, we can pick the α_v to be the identity, and we can choose the edges of $|E_{\underline{\mathsf{Orb}}}|$ with $x_{01} = 1$ for our paths γ_{vw}. It is then easy to check that the corresponding $\underline{\mathsf{Orb}}$-valued cocycle on $|\underline{\mathsf{Orb}}|$ gives the identity map $\mathrm{Id} : |\underline{\mathsf{Orb}}| \to |\underline{\mathsf{Orb}}|$. □

The orbispace $|E_{\underline{\mathsf{Orb}}}| \to |\underline{\mathsf{Orb}}|$ has the following other universal property. It is a terminal object up to homotopy in the category of orbispaces and representable maps.

Theorem 5.5 *Let $E \to X$ be an orbispace. Then the space of representable orbispace maps $(E, X) \to (|E_{\underline{\mathsf{Orb}}}|, |\underline{\mathsf{Orb}}|)$ is weakly contractible.*

Proof. To show that $M := \mathrm{Map}\big((E,X), (|E_{\underline{\mathsf{Orb}}}|, |\underline{\mathsf{Orb}}|)\big)$ is weakly contractible, it's enough that any map $S^n \to M$ is nullhomotopic. But a map $S^n \to M$ is the same thing as a map $S^n \times (E, X) \to (|E_{\underline{\mathsf{Orb}}}|, |\underline{\mathsf{Orb}}|)$. So, by letting $S^n \times (E,X)$ play the role of (E, X), it's enough to show that any two maps $(f, g), (f', g') : (E, X) \to (|E_{\underline{\mathsf{Orb}}}|, |\underline{\mathsf{Orb}}|)$ are homotopic. Without loss of generality, we may assume that g is a map classifying $E \to X$.

Let $\mathcal{T}$ be a triangulation of X such that the image $g'(\sigma)$ of every simplex $\sigma \in \mathcal{T}$ lands in a simplex of $|\underline{\mathsf{Orb}}|$. For each vertex $v \in X$, we consider the simplex $\tau_v \subset |\underline{\mathsf{Orb}}|$ in the interior of which $g'(v)$ lies. By sending v to the 0th vertex of τ, and extending linearly we obtain a new map $g'' : X \to |\underline{\mathsf{Orb}}|$. Moreover, the straight line homotopy between g' and g'' preserves the stratification of $|\underline{\mathsf{Orb}}|$. So, by the homotopy lifting property, it lifts to a homotopy from f' to some other map $E \to |E_{\underline{\mathsf{Orb}}}|$. This reduces us to the case when $g' : X \to |\underline{\mathsf{Orb}}|$ is compatible with the triangulations.

Now, fix $g : X \to |\underline{\mathsf{Orb}}|$ to be the particular map constructed in the proof of Theorem 5.4 using $\mathcal{T}$. At this point, we may also replace (E, X) by the equivalent orbispace $(f^*|E_{\underline{\mathsf{Orb}}}|, X)$. Finally, using a fiber-wise homotopy, the map $f^*|E_{\underline{\mathsf{Orb}}}| \to |E_{\underline{\mathsf{Orb}}}|$ can be replaced by a map respecting the two triangulations. We have equipped all our spaces with triangulations and reduced to the case when all the maps respect the triangulations. Therefore, we can assume that E and X were simplicial sets to begin with.

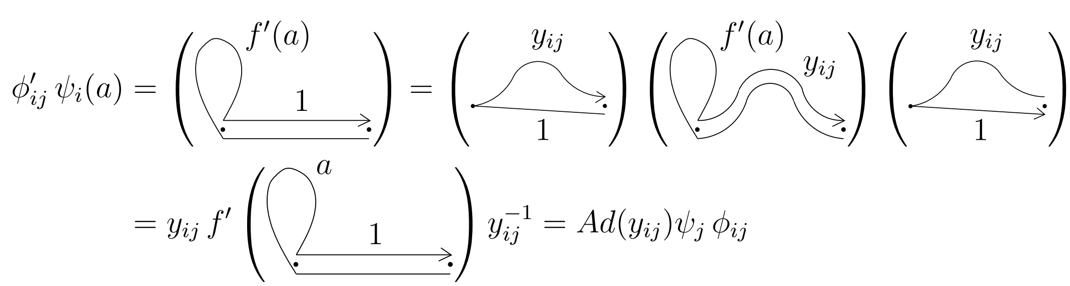
We are reduced to the following situation. We have a simplicial set X and a map $g : X \to \underline{\mathsf{Orb}}$. We let $f : g^*E_{\underline{\mathsf{Orb}}} \to E_{\underline{\mathsf{Orb}}}$ be the canonical map. We then have another pair of maps (f', g') commuting with the projections, and we want a diagram homotopy between the two diagrams

$$\begin{tikzcd} g^*E_{\underline{\mathsf{Orb}}} \arrow[r, "f'"] \arrow[d] & E_{\underline{\mathsf{Orb}}} \arrow[d] \\ X \arrow[r, "g'"] & \underline{\mathsf{Orb}} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} g^*E_{\underline{\mathsf{Orb}}} \arrow[r, "f"] \arrow[d] & E_{\underline{\mathsf{Orb}}} \arrow[d] \\ X \arrow[r, "g"] & \underline{\mathsf{Orb}}. \end{tikzcd} \tag{5.12}$$

We begin by writing down a homotopy $h : X \times \Delta[1] \to E_{\underline{\mathsf{Orb}}}$ between g and g'. For each n-simplex $\sigma \in X$, we write down the restriction of h to the corresponding sub-simplicial set $\Delta[n] \times \Delta[1] \subset X \times \Delta[1]$. Namely, we write down an $\underline{\mathsf{Orb}}$-valued cocycle on $\Delta[n] \times \Delta[1]$. Let $\{G_i, \phi_{ij}, g_{ijk}\}) := g(\sigma)$ and $\{G'_i, \phi'_{ij}, g'_{ijk}\}) := g'(\sigma)$. Let $F_i = K(G_i, 1) \subset E_{\underline{\mathsf{Orb}}}$ be the fiber over the ith vertex of $g(\sigma)$, and $F'_i = K(G'_i, 1)$ the fiber over the ith vertex of $g'(\sigma)$. Let $\psi_i : G_i \to G'_i$ be the homomorphism induced by $f' : F_i \to F'_i$. Note that ψ_i is injective because f' was assumed to be representable. Finally, let $y_{ij} \in G'_j$ be such that $\{G'_i, G'_j, \phi'_{ij}, y_{ij}\}$ is the image of $\{G_i, G_j, \phi_{ij}, 1\}$ under f'. The y_{ij} satisfy the conditions

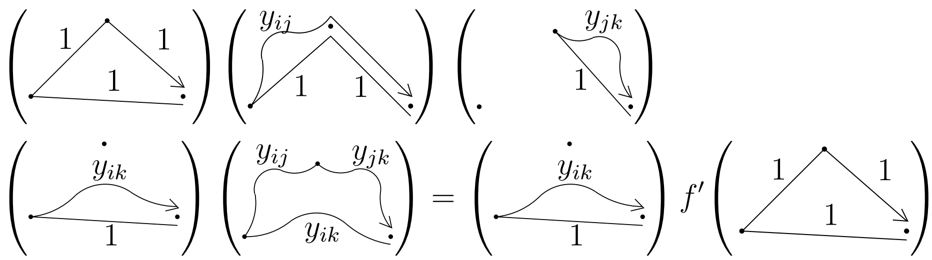
$$\phi'_{ij}\,\psi_i = Ad(y_{ij})\,\psi_j\,\phi_{ij} \qquad \text{and} \qquad g'_{ijk}\,\phi'_{jk}(y_{ij})\,y_{jk} = y_{ik}\,\psi_k(g_{ijk}), \tag{5.13}$$

which are best verified pictorially:



and

$$g'_{ijk}\,\phi'_{jk}(y_{ij})y_{jk} =$$



$$= y_{ik}\,\psi_k(g_{ijk}).$$

We now write the cocycle on $\Delta[n]\times\Delta[1]$ explicitly. We use lower indices for the $\Delta[n]$ coordinate and upper indices for the $\Delta[1]$ coordinate:

$$\begin{array}{lllll} G_i^0 = G_i & G_i^1 = G'_i & \phi_{ij}^{00} = \phi_{ij} & \phi_{ij}^{01} = \phi'_{ij}\,\psi_i & \phi_{ij}^{11} = \phi'_{ij} \\ g_{ijk}^{000} = g_{ijk} & g_{ijk}^{001} = y_{ik}\,\psi_k(g_{ijk})y_{jk}^{-1} & & g_{ijk}^{011} = g'_{ijk} & g_{ijk}^{111} = g'_{ijk}. \end{array} \tag{5.14}$$

We then check the cocycle conditions (5.2) and (5.3) using (5.13):

$$\begin{aligned} \phi_{ik}^{01} = \phi'_{ik}\,\psi_i &= Ad(y_{ik})\psi_k\,\phi_{ik} = Ad\big(g_{ijk}^{001}\,y_{jk}\,\psi_k(g_{ijk}^{-1})\big)\psi_k\,\phi_{ik} \\ &= Ad(g_{ijk}^{001})Ad(y_{jk})\psi_k Ad(g_{ijk}^{-1})\phi_{ik} \\ &= Ad(g_{ijk}^{001})\phi'_{jk}\,\psi_j\,\phi_{jk}^{-1}\,\phi_{jk}\,\phi_{ij} = Ad(g_{ijk}^{001})\phi'_{jk}\,\psi_j\,\phi_{ij} = Ad(g_{ijk}^{001})\phi_{jk}^{01}\,\phi_{ij}^{00} \end{aligned}$$

$$\begin{aligned} \phi_{ik}^{01} = \phi'_{ik}\,\psi_i &= Ad(y_{ik})\psi_k\,\phi_{ik} = Ad(g_{ijk}^{011})Ad({g'_{ijk}}^{-1})Ad(y_{ik})\psi_k\,\phi_{ik} \\ &= Ad(g_{ijk}^{011})Ad({g'_{ijk}}^{-1})\phi'_{ik}\,\psi_i = Ad(g_{ijk}^{011})\phi'_{jk}\,\phi'_{ij}\,\psi_i = Ad(g_{ijk}^{011})\phi_{jk}^{11}\,\phi_{ij}^{01} \\ g_{ijk}^{001}\,g_{jk\ell}^{001} &= y_{i\ell}\,\psi_\ell(g_{ij\ell})\,y_{j\ell}^{-1}\,y_{j\ell}\,\psi_\ell(g_{jk\ell})\,y_{k\ell}^{-1} = y_{i\ell}\,\psi_\ell(g_{ik\ell}\,\phi_{k\ell}(g_{ijk}))y_{k\ell}^{-1} \\ &= y_{i\ell}\,\psi_\ell(g_{ik\ell})y_{k\ell}^{-1}Ad(y_{k\ell})\big(\psi_\ell(\phi_{k\ell}(g_{ijk}))\big) \\ &= y_{i\ell}\,\psi_\ell(g_{ik\ell})y_{k\ell}^{-1}\,\phi'_{k\ell}(\psi_k(g_{ijk})) = g_{ik\ell}^{001}\,\phi_{k\ell}^{01}(g_{ijk}^{000}) \\ g_{ij\ell}^{001}\,g_{jk\ell}^{011} &= y_{i\ell}\,\psi_\ell(g_{ij\ell})y_{j\ell}^{-1}\,g'_{jk\ell} = y_{i\ell}\,\psi_\ell(g_{ij\ell})\psi_\ell(g_{jk\ell})y_{k\ell}^{-1}\,\phi'_{k\ell}(y_{jk})^{-1} \\ &= y_{i\ell}\,\psi_\ell(g_{ik\ell})\psi_\ell(\phi_{k\ell}(g_{ijk}))y_{k\ell}^{-1}\,\phi'_{k\ell}(y_{jk}^{-1}) \\ &= g'_{ik\ell}\,\phi'_{k\ell}(y_{ik})Ad(y_{k\ell})\big(\psi_\ell(\phi_{k\ell}(g_{ijk}))\big)\phi'_{k\ell}(y_{jk}^{-1}) \\ &= g'_{ik\ell}\,\phi'_{k\ell}\big(y_{ik}\,\psi_k(g_{ijk})y_{jk}^{-1}\big) = g_{ik\ell}^{011}\,\phi_{k\ell}^{11}(g_{ijk}^{001}) \\ g_{ij\ell}^{011}\,g_{jk\ell}^{111} &= g'_{ij\ell}\,g'_{jk\ell} = g'_{ik\ell}\,\phi'_{k\ell}(g'_{ijk}) = g_{ik\ell}^{001}\,\phi_{k\ell}^{11}(g_{ijk}^{001}). \end{aligned}$$

Assembling the cocycles (5.14) over the simplices of X produces an $\underline{\mathsf{Orb}}$-valued cocycle on $X\times\Delta[1]$. This is our simplicial homotopy between g and g'.

To build the corresponding homotopy between f and f', we proceed similarly, but with $(g^*E_{\underline{\mathsf{Orb}}})\times\Delta[1]$ instead of $X\times\Delta[1]$. For each simplex in $g^*E_{\underline{\mathsf{Orb}}}$, we write an $E_{\underline{\mathsf{Orb}}}$-valued cocycle on $\Delta[n]\times\Delta[1]$. It is given by (5.14) and

$$x_{ij}^{00} = x_{ij} \qquad x_{ij}^{01} = x'_{ij} \qquad x_{ij}^{11} = x'_{ij} \tag{5.15}$$

where x_{ij} and x'_{ij} are provided by the image of our simplex under f and f' respectively. They satisfy $x'_{ij} = y_{ij}\psi_j(x_{ij})$ which allows us to check that (5.15) satisfy the cocycle condition (5.4):

$$\begin{aligned} x^{01}_{ik} &= x'_{ik} = y_{ik}\psi_k(x_{ik}) = y_{ik}\psi_k\big(g_{ijk}\phi_{jk}(x_{ij})x_{jk}\big) \\ &= y_{ik}\psi_k(g_{ijk})y^{-1}_{jk}y_{jk}\psi_k((\phi_{jk}(x_{ij}))\psi_k(x_{jk}) \\ &= g^{001}_{ijk}y_{jk}\psi_k(\phi_{jk}(x_{ij}))y^{-1}_{jk}x'_{jk} = g^{001}_{ijk}\phi'_{jk}(\psi_j(x_{ij}))x'_{jk} = g^{001}_{ijk}\phi^{01}_{jk}(x^{00}_{ij})x^{01}_{jk} \end{aligned}$$

$$x^{01}_{ik} = x'_{ik} = g'_{ijk}\phi'_{jk}(x'_{ij})x'_{jk} = g^{011}_{ijk}\phi^{11}_{jk}(x^{01}_{ij})x^{11}_{jk}.$$

Assembling all the cocycles over the simplices of $g^*E_{\underline{\mathsf{Orb}}}$ produces an $E_{\underline{\mathsf{Orb}}}$-valued cocycle on $(g^*E_{\underline{\mathsf{Orb}}}) \times \Delta[1]$. This is our simplicial homotopy between f and f'. We have constructed a commutative diagram

$$\begin{tikzcd} (g^*E_{\underline{\mathsf{Orb}}}) \times \Delta[1] \arrow[r, "f'"] \arrow[d] & E_{\underline{\mathsf{Orb}}} \arrow[d] \\ X \times \Delta[1] \arrow[r, "g'"] & \underline{\mathsf{Orb}}, \end{tikzcd}$$

providing a homotopy between the the two diagrams (5.12). This finishes the proof that any two representable maps into $(|E_{\underline{\mathsf{Orb}}}|, |\underline{\mathsf{Orb}}|)$ are homotopic. □

5.2 Quotient structures

Given a topological group K and an action $K \circlearrowright Y$ with finite stabilizers, we can form the quotient orbispace $(EK \times Y)/K \to Y/K$. We want to know which orbispaces $E \to X$ are of the above form. As a first step, we classify the ways a stratified space X can be written as Y/K.

Definition 5.6 *Let $\mathbb{J}$ be the poset of isomorphism classes of finite sets, and X a $\mathbb{J}$-stratified space.*

A K-quotient structure, on X consists of a K-space P with finite stabilizers, and a stratified homeomorphism $P/K \xrightarrow{\sim} X$, where the stratification on P/K is by the stabilizer group. We also require that $P \to X$ be a K-equivariant stratified fibration, where X is given the trivial action.

Like for orbispace structures, K-quotient structures have a classifying space.

Definition 5.7 *Let K be a topological group, and let $\mathcal{J}_K$ be a set of representatives for the conjugacy classes of finite subgroups of K. The classifying space for quotient structures $\underline{B}K$ is the simplicial space whose n-simplices are given by:*

- *for each vertex of $\Delta[n]$, a group $G_i \in \mathcal{J}_K$,*
- *for each edge of $\Delta[n]$, an element $k_{ij} \in K$ satisfying*

$$k_{ij}\, G_i\, k^{-1}_{ij} \subseteq G_j, \tag{5.16}$$

- *for each 2-face of $\Delta[n]$, these group elements satisfy the cocycle condition*

$$k_{ik}\, k_{ij}^{-1} k_{jk}^{-1} \in G_k \tag{5.17}$$

We fix $k_{ii} = 1$. The stratification $\underline{B}K \to N\mathbb{J}$ is given by sending the groups G_i to their isomorphism class and forgetting the rest of the data.

There is a universal K-quotient structure $\underline{E}K \to \underline{B}K$. An n-simplex of $\underline{E}K$ is given by:

- *for each vertex of $\Delta[n]$, a group $G_i \in \mathcal{J}_K$ and an element $y_i G_i \in K/G_i$.*

- *for each edge of $\Delta[n]$, an element $k_{ij} \in K$ satisfying (5.16) and*

$$y_i G_i k_{ij}^{-1} \subseteq y_j G_j \tag{5.18}$$

- *for each 2-face of $\Delta[n]$, the k_{ij} satisfy (5.17)*

The action $K \circlearrowright \underline{E}K$ is induced by the action on the K/G_i's.

For convenience of notation, we write $(\underline{B}K)_n$ as a disjoint union of spaces $(\underline{B}K)_n^{\{G_i\}}$, according to the values of the G_i's. We now check that the quotient $|\underline{E}K|/K$ is homeomorphic to $|\underline{B}K|$ and the stratification by stabilizers agrees with the stratification coming from $\underline{B}K \to N\mathbb{J}$.

Lemma 5.8 *Let $p : \underline{E}K \to \underline{B}K$ be the canonical projection, and let $x \in |\underline{B}K|$ be a point in the G-stratum. Then the fiber $|p|^{-1}(x)$ is K-equivariantly isomorphic to K/G. Moreover, letting $f : (\underline{B}K)_n^{\{G_i\}} \to |\underline{B}K|$ be the natural map, we can be write the pullback of $|\underline{E}K|$ explicitly:*

$$f^*|\underline{E}K| = \coprod_{i=0}^{n} (K/G_i) \times (\underline{B}K)_n^{\{G_i\}} \times \Delta^i \Big/ \sim\,, \tag{5.19}$$

where

$$(y_i G_i; a, t) \sim (y_j k_{ij}(a)^{-1} G_j; a, t) \tag{5.20}$$

and Δ^i is identified with the last i-face of Δ^n.

Proof. Write $x = (\sigma, t)$ for some non-degenerate $\sigma \in |\underline{B}K|$ and some $t \in \mathring{\Delta}^n$. Let $\{G_i, k_{ij}, G_{ijk}\}$ be the data corresponding to σ. The fact that x lies in the G-stratum means that $G_0 = G$.

A simplex of $\underline{E}K$ whose image in $\underline{B}K$ is degenerate is necessarily itself degenerate. This means that $|p|^{-1}(x)$ is isomorphic to $(p_n)^{-1}(\sigma)$, where $p_n : (\underline{E}K)_n \to (\underline{B}K)_n$ is the map induced by p on the spaces on n-simplices.

So we need to show that $(p_n)^{-1}(\sigma) \simeq K/G$. A point in $(p_n)^{-1}(\sigma)$ is a collection $\{y_i G_i \in K/G_i\}$ satisfying (5.18). But (5.18) determines all the $y_i G_i$'s in terms of $y_0 G_0$. So the map $(p_n)^{-1}(\sigma) \to K/G = K/G_0$ given by $\{y_i G_i\} \mapsto y_0 G_0$ is injective.

Suppose now that we are given $y_0 G_0 \in K/G_0$. Then letting

$$y_i G_i := y_0 k_{0i}^{-1} G_i \tag{5.21}$$

produces a collection satisfying (5.18):

$$\begin{aligned} y_i G_i k_{ij}^{-1} &= y_0 k_{0i}^{-1} G_i k_{ij}^{-1} = y_0 k_{0i}^{-1} k_{ij}^{-1} k_{ij} G_i k_{ij}^{-1} \\ &\subseteq y_0 k_{0i}^{-1} k_{ij}^{-1} G_j = y_0 k_{0j}^{-1} k_{0j} k_{0i}^{-1} k_{ij}^{-1} G_j = y_0 k_{0j}^{-1} G_j = y_j G_j. \end{aligned} \tag{5.22}$$

This shows that $\{y_i G_i\} \mapsto y_0 G_0$ is a bijection. We have identified the fiber of p_n, hence the fiber $|p|$.

If $t \in \Delta^n$ lies in some face Δ^I, $I = \{i_0 \dots i_s\} \subset \{0 \dots n\}$, then its fiber $\{y_i G_i\}_{i \in \{0 \dots n\}} \simeq K/G_0$ gets identified with $\{y_i G_i\}_{i \in I} \simeq K/G_{i_0}$. As we saw in (5.21), this is exactly relation (5.20). $\square$

We now finish checking that $|\underline{E}K| \to |\underline{B}K|$ is a K-quotient structure.

Lemma 5.9 *Let $|\underline{B}K|$ be given the trivial K-action. Then the map $|\underline{E}K| \to |\underline{B}K|$ is a K-equivariant stratified fibration.*

Proof. By Theorem , we need to check that the map of H-fixed points $|\underline{E}K|^H \to |\underline{B}K|^H = |\underline{B}K|$ is a stratified fibration for all subgroups $H < K$. The space $|\underline{B}K|$ is covered by its skeleta, so by Lemma 4.19, it's enough to show that $|\underline{E}K|_n^H \to |\underline{B}K|_n$ are stratified fibrations.

We assume by induction that $|\underline{E}K|_{n-1}^H \to |\underline{B}K|_{n-1}$ is a stratified fibration. The map $|\underline{E}K|_n^H \to |\underline{B}K|_n$ can be written as the pushout of

$$\begin{tikzcd} {|\underline{E}K|_{n-1}^H} \arrow[d] & \alpha^*|\underline{E}K|^H \arrow[l] \arrow[r, hook] \arrow[d] & \beta^*|\underline{E}K|^H \arrow[d] \\ {|\underline{B}K|_{n-1}} & (\underline{B}K)_n \times \partial\Delta^n \arrow[l] \arrow[r, hook, "\iota"] & (\underline{B}K)_n \times \Delta^n, \end{tikzcd} \tag{5.23}$$

where $\beta : (\underline{B}K)_n \times \Delta^n \to |\underline{B}K|$ is the canonical map and $\alpha = \beta \circ \iota$. So by Conjecture 4.34, it's enough to show that $\beta^*|\underline{E}K|^H \to (\underline{B}K)_n \times \Delta^n$ is a stratified fibration.

But $(\underline{B}K)_n$ is the disjoint union of the $(\underline{B}K)_n^{\{G_i\}}$. Write $f : (\underline{B}K)_n^{\{G_i\}} \times \Delta^n$ for the canonical map and recall the expression (5.19) for $f^*|\underline{E}K|$. Taking H-fixed points, we get

$$f^*|\underline{E}K|^H = \coprod_{i=0}^{n} (K/G_i)^H \times (\underline{B}K)_n^{\{G_i\}} \times \Delta^i \Big/ \sim, \tag{5.24}$$

where

$$(y_i G_i; a, t) \sim (y_i k_{ij}(a)^{-1} G_j; a, t). \tag{5.25}$$

and Δ^i is identified with the last i-face of Δ^n. Let $G_i' := k_{0i}^{-1} G_i k_{0i}$. If we apply the change of coordinates $(y_i G_i; a, t) \mapsto (y_i k_{0i}(a) G_i'; a, t)$ to (5.24), the relation (5.25) becomes $(y_i G_i'; a, t) \sim (y_i G_j'; a, t)$. So we can apply Lemma 4.35 to our situation. The projections $(K/G_i)^H \to (K/G_j)^H$ are covering spaces, in particular fibrations, so $f^*|\underline{E}K|^H \to (\underline{B}K)_n^{\{G_i\}} \times \Delta^n$ is a stratified fibration. $\square$

As was the case with $\underline{\mathsf{Orb}}$, it's useful to introduce a notion of cocycle corresponding $\underline{B}K$.

Definition 5.10 *Let X be a $\mathbb{J}$-stratified space and $\mathcal{T}$ an oriented triangulation. Recall the set $\mathcal{J}_K$ from definition 5.7.*

A $\underline{B}K$-valued cocycle on X consists of the following data. To each vertex v, we associate a group in $G_v \in \mathcal{J}_K$. For each simplex $\sigma : \Delta^n \to X$ of $\mathcal{T}$, and for each $0 \leq i, j \leq n$, we have a function $k_{ij} = k_{ij}^{\sigma} : \Delta^n \to K$. These functions satisfy the conditions (5.16) and (5.17) where the groups G_i are the ones associated to the vertices of σ.

Moreover, if $\iota : \Delta^k \to \Delta^n$ is the inclusion of a face, and if $\tau = \sigma \circ \iota$, we require that

$$k_{ij}^{\tau} = k_{\iota(i),\iota(j)}^{\sigma} \circ \iota. \tag{5.26}$$

Two cocycles c, c' are equivalent if $c \sqcup c'$ extends to a cocycle on $X \times [0,1]$.

A $\underline{B}K$-valued cocycle c on X produces a map $f : X \to |\underline{B}K|$ as follows. Let $\sigma : \Delta^n \to X$ be a simplex of $\mathcal{T}$, and let $\{G_i\}$ be the groups associated to the vertices of σ. The function f is then given by

$$f : \sigma(t) \mapsto (\{G_i, k_{ij}^{\sigma}(t)\}, t) \in |\underline{B}K|. \tag{5.27}$$

It is well defined because (5.26).

Given X an a $\mathbb{J}$-stratified map $X \to |\underline{B}K|$, we can pull back $|\underline{E}K|$ to get a quotient structure on X. This provides a bijection between homotopy classes of maps $X \to |\underline{B}K|$ and quotient structures on X.

Theorem 5.11 *Let $\mathbb{J}$ be the poset of isomorphism classes of finite groups, and K a topological group. Then, for any $\mathbb{J}$-stratified space X, the assignment $f \mapsto f^*|\underline{E}K|$ provides a bijection*

$$\left\{ \begin{array}{c} \textit{Homotopy classes of} \\ \mathbb{J}\textit{-stratified maps } X \to |\underline{B}K|. \end{array} \right\} \quad \leftrightarrow \quad \left\{ \begin{array}{c} \textit{Isomorphism classes of} \\ K\textit{-quotient structures on } X. \end{array} \right\} \tag{5.28}$$

Proof. We first note that $f^*|\underline{E}K|$ only depends on the homotopy class of f. This is done as in (5.8), using Lemma 5.9. We also note that a K-equivariant homotopy equivalence $P \xrightarrow{\sim} P'$ between two K-quotient structures is necessarily a homeomorphism.

Given a K-quotient structure $p : P/K \xrightarrow{\sim} X$, we build a map $f : X \to |\underline{B}K|$ such that $P \simeq f^*|\underline{E}K|$. We do this by constructing a $\underline{B}K$-valued cocycle on X with respect to some oriented triangulation $\mathcal{T}$.

First, for every vertex v, we pick a point $b_v \in P_v := p^{-1}(v)$ whose stabilizer is in $\mathcal{J}_K$. The groups for our $\underline{B}K$-valued cocycle are then given by

$$G_v = \mathrm{Stab}_K(b_v). \tag{5.29}$$

Given a simplex $\sigma : \Delta^n \to X$, we let G_i be the group associated to its ith vertex. We use G_i^σ when we want to stress the dependence on σ. We also refer to the subset

$$\{(t_0, \dots, t_n) \in \Delta^n \,|\, t_i = 0 \text{ for } i < j, t_j \neq 0\} \subset \Delta^n \tag{5.30}$$

as the j-stratum of Δ^n, and use the notation $\Delta^n_{\leq j}$, $\Delta^n_{\geq j}$ as in Definition 4.4.

We now produce a sequence of K-equivariant quotients of $\sigma^* P$

$$\sigma^* P = Q_0 \overset{q_1}{\twoheadrightarrow} Q_1 \overset{q_2}{\twoheadrightarrow} Q_2 \cdots \overset{q_n}{\twoheadrightarrow} Q_n. \tag{5.31}$$

They come with maps to $p_i : Q_i \to \Delta^n$ satisfying $p_{i-1} = p_i \circ q_i$. The q_i are isomorphisms over $\Delta^n_{\geq i}$, and the isotropy groups of Q_i are conjugate to G_i over $\Delta^n_{\leq i}$. We use Q_i^σ when we want to stress the dependence on σ. If $\iota : \Delta^k \to \Delta^n$ is the inclusion of a face, and $\tau = \sigma \circ \iota$, we also want inclusions $Q_i^\tau \to Q^\sigma_{\iota(i)}$ commuting with the q_i, and such that

$$\begin{tikzcd} Q_i^\tau \arrow[r] \arrow[d] & Q^\sigma_{\iota(i)} \arrow[d] \\ \Delta^k \arrow[r, "\iota"] & \Delta^n \end{tikzcd} \tag{5.32}$$

is a pullback diagram.

We construct the Q_i^σ by induction on $\dim(\sigma)$ and on i. If $\dim(\sigma) = 0$ or $i = 0$ we don't have to do anything. So let's assume that we have Q_j^τ for all j if $\dim(\tau) < n$ and for $j < i$ if $\dim(\tau) = n$. Let $\sigma : \Delta^n \to X$ be an n-simplex and $i > 0$. By (5.32), Q_i is determined on the ith horn $\Lambda^{n,i}$. Let $r : \Delta^n \to \Lambda^{n,i}$ be a deformation retraction sending $\Delta^n \setminus \Lambda^{n,i}$ into $\Delta^n_{\leq i}$. By the homotopy lifting property, r is covered by a K-equivariant deformation retraction $\tilde{r} : Q_{i-1} \to Q_{i-1}|_{\Lambda^{n,i}}$. We define Q_i by identifying $x, y \in Q_{i-1}$ if $\tilde{r}(x)$ and $\tilde{r}(y)$ have the same image in $Q_i|_{\Lambda^{n,i}}$.

We now construct sections $s_i = s_i^\sigma$ of Q_i^σ, compatible with (5.32), and such that

$$\mathrm{Stab}_K\big(s_i(t)\big) = G_i \qquad \text{for} \quad t \in \Delta^n_{\leq i}. \tag{5.33}$$

On 0-simplices v, the value of s_0 is given by b_v. Assume now by induction that we have s_i^τ for all simplices τ of dimension smaller that n. Let σ be an n-simplex. The compatibility with (5.32) forces the value of s_i on the i-horn. We then extend it to Δ^n by the homotopy extension property.

Finally, we construct the functions $k_{ij} = k_{ij}^\sigma : \Delta^n \to K$ needed for our cocycle. We make sure that they satisfy

$$k_{ij}(t)\, q_{ij}(s_i(t)) = s_j(t), \tag{5.34}$$

where q_{ij} stands for for $q_j q_{j-1} \dots q_{i+1}$. Note that (5.34) specifies $k_{ij}(t)$ up to left multiplication by an element of $\mathrm{Stab}_K(s_j(t))$. That stabilizer group is generically equal to G_j, so we only have a global ambiguity by G_j. If $\dim(\sigma) = 1$, we pick a value for k_{01} at the 0th vertex, subject to (5.34). This resolves the ambiguity, thus defining uniquely k_{01}. For arbitrary σ, the value of k_{ij} is given by (5.26) on some contractible subset of Δ^n. Again, the ambiguity is resolved, and so we have defined k_{ij}.

We now check that $\{G_i, k_{ij}\}$ satisfy the cocycle conditions (5.16) and (5.17). For the first one, we recall that G_i stabilizes s_i. It therefore also stabilizes $q_{ij}(s_i)$. Since G_j is the generic stabilizer of s_j, (5.16) holds generically, and by continuity it holds everywhere. The second condition holds because

$$\begin{aligned} s_k = k_{ik}\, q_{ik}\, s_i = k_{ik}\, q_{jk}\, q_{ij}\, s_i &= k_{ik}\, k_{ij}^{-1} k_{jk}^{-1} k_{jk}\, q_{jk}\, k_{ij}\, q_{ij}\, s_i \\ &= k_{ik}\, k_{ij}^{-1} k_{jk}^{-1} k_{jk}\, q_{jk}\, s_j = k_{ik}\, k_{ij}^{-1} k_{jk}^{-1} s_k \end{aligned}$$

and $\mathrm{Stab}_K(s_k) = G_k$ on a dense piece of Δ^n. This finishes the verification that $c := \{G_i, k_{ij}\}$ is a $\underline{B}K$-valued cocycle on X. Let $f : X \to |\underline{B}K|$ be the corresponding map.

Suppose now that we make some other choices $\{\mathcal{T}', b'_v, Q'_i, s'_i, k'_{ij}\}$. These give a cocycle c' and a corresponding map f'. We want to show that f and f' are homotopic. Let $\mathcal{T}''$ be a triangulation of $X \times [0,1]$ that restricts to $\mathcal{T}$ on $X \times \{0\}$ and to $\mathcal{T}'$ on $X \times \{1\}$. We can extend $\{Q_i, Q'_i, s_i, s'_i, k_{ij}, k'_{ij}\}$ to data $\{Q''_i, s''_i, k''_{ij}\}$ on the simplices of $\mathcal{T}''$ satisfying (5.31), (5.32), (5.34). This produces a cocycle c'' on $X \times [0,1]$ extending $c \sqcup c'$ i.e. a homotopy between f and f'. This finishes the proof that f is well defined up to homotopy.

We now show that $f^*|\underline{E}K|$ is isomorphic to the original K-quotient structure P. Let $x = \sigma(t)$ be a point of X, where $\sigma : \Delta^n \to X$ is a simplex of our triangulation $\mathcal{T}$, and $t \in \mathring{\Delta}^n$. We need to produce a K-equivariant map between its fiber P_x in P and the corresponding fiber $(f^*|\underline{E}K|)_x = |\underline{E}K|_{f(x)}$. Recall from (5.27) that $f(x)$ is given by $(\{G_i, k_{ij}(t)\}, t)$. Its fiber in $|\underline{E}K|$ is then given by

$$|\underline{E}K|_{f(x)} = \Big\{ (y_i G_i) \in \prod^n K/G_i \,\Big|\, y_i G_i k_{ij}^{-1}(t) \subseteq y_j G_j \Big\}, \tag{5.35}$$

and is isomorphic to K/G_0 by Lemma 5.8. Now consider the fibers $Q_i(t) := p_i^{-1}(t)$, where p_i are as in (5.31). Let $\psi_i : Q_i(t) \to K/G_i$ be the unique K-map sending $s_i(t)$ to the coset G_i. We can now write the equivalence on each fiber as

$$y \in P_x \;\mapsto\; \big(\psi_i\, q_{0i}(y)\big) \in |\underline{E}K|_{f(x)}\,, \tag{5.36}$$

where $q_{0i} = q_i \dots q_1 : P_x = Q_0(t) \to Q_i(t)$ is the projection. To check that the conditions in (5.35) are satisfied by the $\psi_i\big(q_{0i}(y)\big)$, we write $y = a\, s_0(t)$ for some $a \in K$ and compute

$$\psi_i\big(q_{0i}(y)\big) = \psi_i\big(a\, q_{0i}(s_0)\big) = \psi_i\big(a\, k_{0i}^{-1} k_{0i}\, q_{0i}(s_0)\big) = \psi_i(a\, k_{0i}^{-1} s_i) = a\, k_{0i}^{-1} G_i\,,$$

so $$\psi_i\big(q_{0i}(y)\big) k_{ij}^{-1} = a\, k_{0i}^{-1} G_i k_{ij}^{-1} \subseteq a\, k_{0i}^{-1} k_{ij}^{-1} G_j = a\, k_{0j}^{-1} G_j = \psi_j\big(q_{0j}(y)\big).$$

This finishes the proof that $f^*|\underline{E}K|$ is isomorphic to P.

Finally, suppose we start with a map $f : X \to |\underline{B}K|$. We need to show that the map f' associated to $P = f^*|\underline{E}K|$ is homotopic to f. By naturality, we may assume that $X = |\underline{B}K|$ and $f = \mathrm{Id}$. Triangulate each of the spaces $(\underline{B}K)_n \times \Delta^n$. Make sure that the projections $(\underline{B}K)_n \times \Delta^n \to \Delta^n$ and the attaching maps $(\underline{B}K)_n \times \partial\Delta^n \to |\underline{B}K|_{n-1}$ respect the triangulations. This induces a triangulation $\mathcal{T}$ of $|\underline{B}K|$, which refines the skeleton filtration. Note that the set of vertices necessarily agrees with $|\underline{B}K|_0$. Given $v = G \in \mathcal{J}_K$, we let b_v be the identity coset in $P_v = |\underline{E}K|_v = K/G$. Now, we need to choose quotients Q_i of $Q_0 = \sigma^*|\underline{E}K|$ as in (5.31). By Lemma 5.8, Q_0 can be written as

$$Q_0 = \coprod_{j=0}^{n} K/G_j \times \Delta^n_{\geq j} \Big/ \sim, \tag{5.37}$$

where $(y_i G_i, t) \sim (y_i k_{ij}^{-1}(t) G_j, t)$ and G_j are the groups associated to the vertices of σ, and k_{ij} are inherited from $\underline{B}K$. We then let

$$Q_i := \coprod_{j=i}^{n} K/G_j \times \Delta^n_{\geq i} \Big/ \sim \ . \tag{5.38}$$

Pick $s_i(t)$ to be the identity coset in K/G_i. Finally, the functions k_{ij} can be taken equal the ones which were already given to us from the beginning. They satisfy (5.34) since

$$k_{ij} q_{ij} s_i = k_{ij} q_{ij} G_i = k_{ij} k_{ij}^{-1} G_j = G_j = s_j. \tag{5.39}$$

One then checks that the corresponding map $f : |\underline{B}K| \to |\underline{B}K|$ is the identity. $\square$

The K-space $|\underline{E}K|$ actually has an other name. It is the classifying space for the family of finite subgroups of K. (See [Lück] for a survey of the subject.) This is the content of the following proposition:

Proposition 5.12 *Let H be a finite subgroup of K. Then the fixed point set $|\underline{E}K|^H$ is contractible.*

Proof. The simplicial space $\underline{E}K$ is isomorphic to the nerve of a category $\mathcal{C}$. The objects of $\mathcal{C}$ are the orbits K/G for $G \in \mathcal{J}_K$, equipped with a choice of base point. The morphisms are the K-equivariant, base points preserving maps between them. Taking H-fixed points means that we restrict the set of objects to those where the base point is H-fixed. So $(\underline{E}K)^H$ is the nerve of a full subcategory $\mathcal{C}' \subset \mathcal{C}$. An object of $\mathcal{C}'$ can also be viewed as an object of $\mathcal{C}$ equipped with a map from $(K/H, H) \in \mathcal{C}$. But $\mathcal{C}'$ has an initial object, so its nerve is contractible:

$$|\underline{E}K|^H = |(\underline{E}K)^H| = |\mathcal{C}'| \simeq *.$$

$\square$

Given a K-quotient structure $P \to X$, we can form the corresponding orbispace $(P \times EK)/K \to X$. This operation is represented by a stratified map $|\underline{B}K| \to |\underline{\mathsf{Orb}}|$ which we can write explicitly:

Proposition 5.13 *Let K be a topological group, and recall the sets $\mathcal{J}$ and $\mathcal{J}_K$ from Definitions 5.1 and 5.7. Let $\kappa : \mathcal{J}_K \to \mathcal{J}$ be the map sending a group $G \in \mathcal{J}_K$ to the unique group in $\mathcal{J}$ to which it is isomorphic. Let $\beta_G : G \to \kappa(G)$ be an isomorphism.*

The orbispace $(|\underline{E}K| \times EK)/K \to |\underline{B}K|$ is then represented by (the realization *of) the map*

$$\begin{array}{rccc} \Psi: & \underline{B}K & \to & \underline{\mathsf{Orb}} \\ & \{G_i, k_{ij}\} & \mapsto & \{\kappa(G_i),\ \beta_j Ad(k_{ij})\beta_i^{-1},\ \beta_k(k_{ik}k_{ij}^{-1}k_{jk}^{-1})\}, \end{array} \tag{5.40}$$

where β_i is a shorthand notation for β_{G_i}.

Proof. Triangulate the spaces $(\underline{B}K)_n \times \Delta^n$ and make sure that the projections $(\underline{B}K)_n \times \Delta^n \to \Delta^n$ and the attaching maps $(\underline{B}K)_n \times \partial\Delta^n \to |\underline{B}K|_{n-1}$ respect the triangulations. This induces a triangulation $\mathcal{T}$ of $|\underline{B}K|$ refining the skeleton filtration. The vertices of $\mathcal{T}$ agree with $|\underline{B}K|_0 = \mathcal{J}_k$.

Let $\sigma : \Delta^n \to |\underline{B}K|$ be a simplex of that triangulation. By Lemma 5.8, we can write $\sigma^*|\underline{E}K|$ as

$$\sigma^*|\underline{E}K| = \coprod_{i=0}^{n} K/G_i \times \Delta^i \Big/ \sim,$$

where $(y_i k_{ij}(a)G_i; a, t) \sim (y_j G_j; a, t)$ and Δ^i is identified with the last i-face of Δ^n. Therefore, we can also write

$$(\sigma^*|\underline{E}K| \times EK)/K = \coprod_{i=0}^{n} EK/G_i \times \Delta^i \Big/ \sim, \tag{5.41}$$

where $(z_i k_{ij}(a)G_i; a, t) \sim (z_j G_j; a, t)$.

We now make the choices $\{b_i, \alpha_i, \gamma_{ij}\}$ used in the proof of Theorem 5.4. Pick some base point $e \in EK$. Over the ith vertex of Δ^n, the fiber of (5.41) is EK/G_i. We choose our base point $b_i := eG_i$. We have $\pi_i(EK/G_i) = G_i$ and we pick the isomorphism $\alpha_i := \beta_i$. For each i, j we pick some path $\delta_{ij} : [0,1] \to EK$ from e to ek_{ij}. The paths γ_{ij} are given by

$$\gamma(x) = \begin{cases} \delta_{ij}(x)G_i & \text{if } i < 1, \\ ek_{ij}G_i = eG_j & \text{if } i = 1, \end{cases}$$

where we have omitted the Δ^n coordinate.

We now check that

$$\phi_{ij} = \beta_j\, Ad(k_{ij})\beta_i^{-1} \quad \text{and} \quad g_{ijk} = \beta_k(k_{ik}k_{ij}^{-1}k_{jk}^{-1}). \tag{5.42}$$

The isomorphisms β_i don't play any role in the argument, so we omit them from the notation and identify G_i with $\kappa(G_i)$. Let $y \in G_i$ be an element represented by a path $z : [0,1] \to EK/G_i$. It admits a lift $\tilde{z} : [0,1] \to EK$ going from e to ey^{-1}. The element $\phi_{ij}(y)$ is represented by the path $\varepsilon := \bar{\delta}_{ij}^{-1} \cdot zk_{ij}^{-1}G_j \cdot \bar{\delta}_{ij}$, where $\bar{\delta}_{ij}$ is the projection of δ_{ij} in EK/G_j. The lift of ϵ in EK is given by $\tilde{\varepsilon} = \delta_{ij}^{-1}k_{ij}^{-1} \cdot \tilde{z}k_{ij}^{-1} \cdot \bar{\delta}_{ij}yk_{ij}^{-1}$. We check that $\tilde{\varepsilon}(0) = e$ and $\tilde{\varepsilon}(1) = ek_{ij}y^{-1}k_{ij}^{-1}$ which gives us (5.42.a). Now, the element g_{ijk} is represented by the path $\eta := \bar{\delta}_{ik}^{-1} \cdot \bar{\delta}_{ij} \cdot \bar{\delta}_{jk}$. Its lift in EK is given by $\tilde{\eta} = \delta_{ik}^{-1}k_{ik}^{-1} \cdot \delta_{ij}k_{ik}^{-1} \cdot \delta_{jk}k_{ij}k_{ik}^{-1}$. We check that $\tilde{\eta}(0) = e$ and $\tilde{\eta}(1) = ek_{jk}k_{ij}k_{ik}^{-1}$ which gives us (5.42.b).

We have checked the formulas (5.40) on every individual simplex of $\mathcal{T}$. The choices can made in a compatible way over all the simplices of $\mathcal{T}$, so we have proven the result. □

If we are given a collection of subgroups of K, we can do the following useful variation on Definition 5.7.

Definition 5.14 *Let K be a topological group and $\mathcal{F}$ a collection of subgroups (i.e. a set of subgroups, closed under conjugation). Let $\mathcal{J}_{\mathcal{F}}$ be a set of representatives of the conjugacy classes in $\mathcal{F}$.*

Then one defines $B_{\mathcal{F}}K$ by replacing $\mathcal{J}_K$ by $\mathcal{J}_{\mathcal{F}}$ everywhere in Definition 5.7. Similarly, one defines $E_{\mathcal{F}}K \to B_{\mathcal{F}}K$.

One then has the following straightforward generalizations of Theorems 5.11.

Theorem 5.15 *Let $\mathbb{J}$ be the poset of isomorphism classes of finite groups, K a topological group and $\mathcal{F}$ a collection of subgroups. Then, for any $\mathbb{J}$-stratified space X, the assignment $f \mapsto f^*|E_{\mathcal{F}}K|$ provides a bijection*

$$\left\{ \begin{array}{c} \textit{Homotopy classes of} \\ \mathbb{J}\textit{-stratified maps} \\ X \to |B_{\mathcal{F}}K|. \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Isomorphism classes of} \\ K\textit{-quotient structures on } X \\ \textit{with stabilizers in } \mathcal{F}. \end{array} \right\} \tag{5.43}$$

□

Assume now that $\mathcal{F}$ is a family of subgroup (i.e. if it's closed under conjugation and under taking subgroups). Then then $|E_{\mathcal{F}}K|$ is a classifying space for $\mathcal{F}$:

Proposition 5.16 *The space $|E_{\mathcal{F}}K|^H$ is contractible for $H \in \mathcal{F}$ and empty otherwise.*

The proof is identical to that of Proposition 5.12. □

Chapter 6

Global quotients

The purpose of this chapter is to show that every orbispace (E, X) where X is compact is a global quotient by a compact Lie group K. Namely, there exists a K-space Y such that (E, X) is isomorphic to $[Y/K] := ((Y \times EK)/K, Y/K)$. It turns out that it's easier to answer the question if we relax the condition that K be a Lie group. In that case, we can prove the result even when X is non-compact.

6.1 A convenient group

Let U be the inductive limit of unitary groups

$$U := \varinjlim U(n!) \tag{6.1}$$

where the inclusions $U(n!) \hookrightarrow U((n+1)!)$ are given by $A \mapsto A \otimes \mathrm{Id}_{n+1}$ (See [12], [1], [15] for other occurrences of this group.) By Bott periodicity $\pi_i(U(n)) = 0$ for even i and $\mathbb{Z}$ for odd i, as long as $i < 2n$. The inclusion $U(n!) \hookrightarrow U((n+1)!)$ induces multiplication by $n+1$ on π_i so we conclude that

$$\pi_i(U) = \varinjlim \pi_i\big(U(n!)\big) = \begin{cases} 0 & \text{if } i \text{ is even} \\ \mathbb{Q} & \text{if } i \text{ is odd.} \end{cases} \tag{6.2}$$

The group U can also be defined as a colimit of all the $U(n)$'s where the colimit is instead indexed by the lattice $\mathbb{N}$, ordered by divisibility.

Let (E, X) be an orbispace represented by a map $f : X \to \underline{\mathsf{Orb}}$. It is of the form $[Y/U]$ if and only if there is a lift up to homotopy

$$\begin{tikzcd} & {|\underline{B}U|} \arrow[d, "|\Psi|"] \\ X \arrow[ur, dotted] \arrow[r, "f"] & {|\underline{\mathsf{Orb}}|} \end{tikzcd} , \tag{6.3}$$

where Ψ is the map described in Proposition 5.13.

For technical reasons, it will be useful to replace $|\underline{B}U|$ by another space mapping to it. Let $\mathcal{F}$ be the family of finite subgroups of U which are embedded by their regular representation $\lambda : G \to U(n) \hookrightarrow U$. We have

Lemma 6.1 *For any group $G \in \mathcal{F}$, the map $N_U(G) \to \mathrm{Aut}(G)$ is surjective.*

Proof. Let $\lambda : G \to U(n) \hookrightarrow U$ denote the regular representation. The action of $\mathrm{Aut}(G)$ on G induces a permutation representation $\rho : \mathrm{Aut}(G) \to U(n) \hookrightarrow U$ which normalizes $\lambda(G)$. Clearly, ρ is a section of the map $N_U(G) \to \mathrm{Aut}(G)$ so that map is surjective. $\square$

Let $B_{\mathcal{F}}U$ be as in Definition 5.14 and let $\mathrm{Sing}(B_{\mathcal{F}}U)$ be the simplicial set obtained by applying level-wise the singular functor, and then taking the diagonal (i.e. the "geometric realization" functor from bisimplicial sets to simplicial sets). Note that a map into $\mathrm{Sing}(B_{\mathcal{F}}U)$ is the same thing as a $B_{\mathcal{F}}U$-valued cocycle as described in Definition 5.10.

Lemma 6.2 *Let $\tilde{\Psi}$ be the composite*

$$\tilde{\Psi} : \mathrm{Sing}(B_{\mathcal{F}}U) \hookrightarrow \mathrm{Sing}(\underline{B}U) \xrightarrow{\mathrm{Sing}\Psi} \mathrm{Sing}(\underline{\mathsf{Orb}}) = \underline{\mathsf{Orb}}, \tag{6.4}$$

and let $\tilde{\Psi}^{op} : \mathrm{Sing}(B_{\mathcal{F}}U)^{op} \to \underline{\mathsf{Orb}}^{op}$ be the map between opposite simplicial sets. Then $\tilde{\Psi}^{op}$ is a stratified fibration.

Proof. To show that $\tilde{\Psi}^{op}$ is a stratified fibration, we replace the usual lifting diagram by its opposite

$$\begin{tikzcd} \Lambda[n,j]^{op} \arrow[r] \arrow[d, hook] & \mathrm{Sing}(B_{\mathcal{F}}U) \arrow[d, "\tilde{\Psi}"] \\ \Delta[n]^{op} \arrow[r] \arrow[ur, dotted] & \underline{\mathsf{Orb}}. \end{tikzcd} \tag{6.5}$$

Since $\Delta[n]^{op} = \Delta[n]$ and $\Lambda[n,j]^{op} = \Lambda[n, n-j]$, we can replace (6.5) by

$$\begin{tikzcd} \Lambda[n,j] \arrow[r] \arrow[d, hook] & \mathrm{Sing}(B_{\mathcal{F}}U) \arrow[d, "\tilde{\Psi}"] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & \underline{\mathsf{Orb}}\,, \end{tikzcd} \tag{6.6}$$

where $\Lambda[n,j] \hookrightarrow \Delta[n]$ now satisfies the opposite condition to that in Definition 4.32. Namely, $j > 0$ <u>or</u> the 0th and 1st vertices of $\Delta[n]$ are in the same stratum.

To check (6.6), we do a case by case analysis, similarly to the proof of Lemma 5.2. We have an $\underline{\mathsf{Orb}}$-valued cocycle $c = \{G_i, \phi_{ij}, g_{ijk}\}$ on $\Delta[n]$, a $B_{\mathcal{F}}U$-valued cocycle $c' = \{H_i, k_{ij}\}$ on $\Lambda[n,j]$, and they satisfy $\tilde{\Psi}(c') = c|_{\Lambda[n,j]}$. Namely

$$G_i = \kappa(H_i) \qquad \phi_{ij} = \beta_j Ad(k_{ij})\beta_i^{-1} \qquad g_{ijk} = \beta_k(k_{ik}k_{ij}^{-1}k_{jk}^{-1}), \tag{6.7}$$

where κ and β_i are as in (5.40). We want extend c' to the whole $\Delta[n]$ while preserving (6.7).

If $n = 1$ and $j = 1$ we let $H_0 \in \mathcal{J}_\mathcal{F}$ be the unique group isomorphic to G_0. The group $H_0' := \beta_1^{-1}\phi_{ij}\beta_0(H_0)$ belongs to $\mathcal{F}$ because it's a subgroup of H_1. By Lemma 6.1, any isomorphism between groups of $\mathcal{F}$ can be obtained by conjugating by an appropriate element of U. So we may pick $k_{01} \in U$ such that $Ad(k_{01})|_{H_0} = \beta_1^{-1}\phi_{ij}\beta_0$. This defines a $B_\mathcal{F}U$-valued cocycle by viewing k_{01} as a constant function on Δ^1.

If $n = 1$, $j = 0$ and ϕ_{01} is invertible, we can run the same argument as above with ϕ_{01}^{-1} playing the role of ϕ_{01}.

If $n = 2$, $j = 2$, we extend k_{02} and k_{12} to the whole Δ^2 while preserving (6.7.b). We then solve (6.7.c) for k_{01}. We now check (6.7.b) for k_{01}:

$$\begin{aligned}\beta_1 Ad(k_{01})\beta_0^{-1} &= \beta_1 Ad\big(k_{12}^{-1}\,\beta_2^{-1}(g_{012}^{-1})\,k_{02}\big)\beta_0^{-1} = \beta_1 Ad(k_{12}^{-1})\,\beta_2^{-1}Ad(g_{012}^{-1})\,\beta_2\,Ad(k_{02})\beta_0^{-1}\\ &= \beta_1 Ad(k_{12}^{-1})\,\beta_2^{-1}Ad(g_{012}^{-1})\,\phi_{02} = \beta_1 Ad(k_{12}^{-1})\,\beta_2^{-1}\phi_{12}\,\phi_{01}\\ &= \beta_1 Ad(k_{12}^{-1})\,\beta_2^{-1}\phi_{12}\,\beta_1\beta_1^{-1}\phi_{01} = \beta_1 Ad(k_{12}^{-1})Ad(k_{12})\,\beta_1^{-1}\phi_{01} = \phi_{01}.\end{aligned} \tag{6.8}$$

If $n = 2$, $j = 1$, we extend k_{01} and k_{12} to Δ^2, preserving (6.7.b). We solve (6.7.c) for k_{02} and check (6.7.b):

$$\begin{aligned}\beta_2\,Ad(k_{02})\beta_0^{-1} &= \beta_2 Ad\big(\beta_2^{-1}(g_{012})k_{12}\,k_{01}\big)\beta_0^{-1} = \beta_2\,\beta_2^{-1}Ad(g_{012})\,\beta_2\,Ad(k_{12})Ad(k_{01})\beta_0^{-1}\\ &= Ad(g_{012})\,\phi_{12}\,\beta_1 Ad(k_{01})\,\beta_0^{-1} = Ad(g_{012})\,\phi_{12}\,\phi_{01} = \phi_{02}.\end{aligned}$$

If $n = 2$, $j = 0$, and ϕ_{01} is invertible, we extend k_{01}, k_{02} to Δ^2, preserving (6.7.b), solve (6.7.c) for k_{12} and check (6.7.b):

$$\begin{aligned}\beta_2\,Ad(k_{12})\beta_1^{-1} &= \beta_2 Ad\big(\beta_2^{-1}(g_{012}^{-1})k_{02}k_{01}^{-1}\big)\beta_1^{-1} = \beta_2\,\beta_2^{-1}Ad(g_{012}^{-1})\beta_2\,Ad(k_{02})Ad(k_{01}^{-1})\beta_1^{-1}\\ &= Ad(g_{012}^{-1})\,\phi_{02}\,\beta_0\,Ad(k_{01}^{-1})\beta_1^{-1} = Ad(g_{012}^{-1})\,\phi_{02}\,\phi_{01}^{-1} = \phi_{12}.\end{aligned}$$

If $n \geq 3$, we extend k_{in} to the whole Δ^n while preserving (6.7.b) and let $k_{ij} := k_{jn}^{-1}\beta_n^{-1}(g_{ijn}^{-1})k_{in}$. They satisfy (6.7.b) by the same computation as (6.8), so we just check (6.7.c):

$$\begin{aligned}\beta_k\big(k_{ik}\,k_{ij}^{-1}k_{jk}^{-1}\big) &= \beta_k\big(k_{kn}^{-1}\beta_n^{-1}(g_{ikn}^{-1})k_{in}\,k_{in}^{-1}\beta_n^{-1}(g_{ijn})k_{jn}\,k_{jn}^{-1}\beta_n^{-1}(g_{jkn})k_{kn}\big)\\ &= \beta_k\big(k_{kn}^{-1}\,\beta_n^{-1}(g_{ikn}^{-1}\,g_{ijn}\,g_{jkn})\,k_{kn}\big) = \beta_k\big(k_{kn}^{-1}\,\beta_n^{-1}(\phi_{kn}(g_{ijk}))\,k_{kn}\big)\\ &= Ad(k_{kn}^{-1})\,\beta_n^{-1}\,\phi_{kn}(g_{ijk}) = g_{ijk}.\end{aligned}$$

This finishes the proof that $\tilde{\Psi}^{op}$ is a stratified fibration. $\square$

6.2 Orbispaces are global quotients

We begin by a couple of technical lemmas. Let $\mathcal{C}$ be the category whose objects are $\mathcal{J}$ (see Definition 5.1) and whose morphisms are injective homomorphisms modulo conjugacy in the target $\mathrm{Hom}_\mathcal{C}(G, H) := \mathrm{Mono}(G, H)/H$.

Lemma 6.3 *Let ν be the projection*

$$\nu : \underline{\mathsf{Orb}} \to N\mathcal{C} \tag{6.9}$$

given by $\{G_i, \phi_{ij}, g_{ijk}\} \mapsto \{G_i, [\phi_{ij}]\}$ and let $x \in |N\mathcal{C}|$ be a point. Then the fiber $|\nu|^{-1}(x)$ is a $K(A,2)$'s for some finite group A.

Proof. We first show that ν is a stratified fibration. Once more, we write down the lifting problem

$$\begin{tikzcd} \Lambda[n,j] \arrow[r] \arrow[d, hook] & \underline{\mathsf{Orb}} \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & N\mathcal{C}. \end{tikzcd} \tag{6.10}$$

So we are given groups G_i and morphisms $\psi_{ij} \in \mathrm{Hom}_{\mathcal{C}}(G_i, G_j)$, satisfying $\psi_{02} = \psi_{12}\,\psi_{01}$. We also have a compatible $\underline{\mathsf{Orb}}$-valued cocycle $\{G_i, \phi_{ij}, g_{ijk}\}$ on $\Lambda[n,j]$. We want to extend it to the whole $\Delta[n]$ preserving the relation $\psi_{ij} = [\phi_{ij}]$. We do it case by case:

If $n = 1$, we can take ϕ_{01} to be any representative of ψ_{01}.

If $n = 2$, we pick a representative $\phi_{k\ell}$ of $\psi_{k\ell}$, where $k\ell$ is the edge not in $\Lambda[n,j]$. Since $[\phi_{02}] = [\phi_{12}][\phi_{01}]$, we can pick an element g_{012} such that $\phi_{02} = Ad(g_{012})\phi_{12}\,\phi_{01}$.

If $n = 3$, $j \neq 3$ we can solve (5.3) for the missing g_{ijk}.

If $n = j$, $j = 3$ and ϕ_{23} is invertible, we can solve (5.3) for g_{012}.

This finishes the proof that ν is a stratified fibration. By Theorem 4.38, $|\nu|$ is a stratified fibration of spaces so by Corollary 4.24, it then enough to identify the homotopy type of the fibers of $|\nu|$ over the vertices of $|\underline{\mathsf{Orb}}|$.

Let $G \in \mathcal{J}$ be a group and $x \in N\mathcal{C}$ be the corresponding vertex. The fiber $\nu^{-1}(x)$ is the the simplicial set given by

$$\big(\nu^{-1}(x)\big)_n = \big\{\{G_i, \phi_{ij}, g_{ijk}\}_{i,j,k=0..n} \,\big|\, G_i = G,\ [\phi_{ij}] = 1\big\}. \tag{6.11}$$

It can easily be checked that (6.11) is a fibrant simplicial set and that it's only non-trivial homotopy group is $\pi_2 = Z(G)$. $\square$

Lemma 6.4 *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $f : \mathcal{C} \to \mathcal{D}$ be a functor such that for every $X \in Ob(\mathcal{D})$, the category $f^{-1}(X)$ is a groupoid. Let $x = (\sigma, y) \in |N\mathcal{D}|$ be a point, where $\sigma \in (N\mathcal{D})_n$ is a non-degenerate and $y \in \mathring{\Delta}^n$ and let $\sigma^* f$ denote the pullback*

$$\begin{tikzcd} \sigma^* N\mathcal{C} \arrow[r] \arrow[d, "\sigma^* f"'] & N\mathcal{C} \arrow[d, "f"] \\ \Delta[n] \arrow[r, "\sigma"] & N\mathcal{D}. \end{tikzcd}$$

Then $|f|^{-1}(x)$ is homeomorphic to the realization of the simplicial set of section of $\sigma^* f$.

Proof. Even if $f : N\mathcal{C} \to N\mathcal{D}$ is not a stratified fibration, the argument of Lemma 4.36 goes through. Indeed, the only place where that assumption is used, is in order to find a lift of (4.60). In our case, we have to solve

$$
\begin{tikzcd}
\Delta[r] \times \Delta[n] \arrow[r] \arrow[d, hook] & \sigma^* N\mathcal{C} \arrow[d, "\sigma^* f"] \\
\Delta[(r+1)(k+1)-1] \arrow[r] \arrow[ur, dotted] & \Delta[n].
\end{tikzcd}
\tag{6.12}
$$

The lift $\Delta[(r+1)(k+1)-1] \to \sigma^* N\mathcal{C}$ exists (and is unique) because of our assumption on the fibers of f. $\square$

Lemma 6.5 *Let Gr be the category of finite groups and monomorphisms and $\mathcal{C}$ be the category of finite groups and monomorphisms modulo conjugacy in the target* $\mathrm{Hom}_{\mathcal{C}}(G,H) := \mathrm{Hom}_{Gr}(G,H)/H$*. Let $p : Gr \to \mathcal{C}$ be the projection functor. Given a diagram $\alpha = (G_0 \xrightarrow{j_1} G_1 \xrightarrow{j_2} \dots \xrightarrow{j_n} G_n)$ in Gr and its image $\bar{\alpha} := p(\alpha)$ in $\mathcal{C}$, we let q and $\bar{q}$ be the maps*

$$
q : \mathrm{Aut}(\alpha) \to \mathrm{Aut}_{Gr}(G_0) \qquad \textit{and} \qquad \bar{q} : \mathrm{Aut}(\bar{\alpha}) \to \mathrm{Aut}_{\mathcal{C}}(G_0). \tag{6.13}
$$

For $\phi \in \mathrm{Aut}_{Gr}(G_0)$ we then have

$$
\phi \in Im(q) \;\Leftrightarrow\; p(\phi) \in Im(\bar{q}). \tag{6.14}
$$

Proof. Clearly $\phi \in Im(q) \Rightarrow p(\phi) \in Im(\bar{q})$, so we assume that $p(\phi) \in Im(\bar{q})$. Let $\{\psi_i \in \mathrm{Aut}_{\mathcal{C}}(G_i)\}$ an automorphism of $\bar{\alpha}$ such that $\psi_0 = p(\phi)$ and let $\tilde{\psi}_i$ be lifts of ψ_i in Gr. Make sure that $\tilde{\psi}_0 = \phi$. Since $\psi_i\, p(j_i) = p(j_i)\, \psi_{i-1}$, there exist elements $g_i \in G_i$ such that $\tilde{\psi}_i\, j_i = Ad(g_i) j_i\, \tilde{\psi}_{i-1}$. This is best described by the diagram

$$
\begin{tikzcd}
G_0 \arrow[r, "j_1"] \arrow[d, "\tilde{\psi}_0"'] \arrow[dr, phantom, "g_1"] & G_1 \arrow[r, "j_2"] \arrow[d, "\tilde{\psi}_1"'] \arrow[dr, phantom, "g_2"] & G_2 \arrow[r, dashed] \arrow[d, "\tilde{\psi}_2"'] & \cdots \arrow[r] & G_n \arrow[d, "\tilde{\psi}_n"'] \\
G_0 \arrow[r, "j_1"'] & G_1 \arrow[r, "j_2"'] & G_2 \arrow[r, dashed] & \cdots \arrow[r] & G_n.
\end{tikzcd}
$$

Let $h_i \in G_i$ be the elements inductively defined by $h_0 = 1$ and $h_i = j_i(h_{i-1})\, g_i^{-1}$. The automorphisms $\phi_i := Ad(h_i)\, \tilde{\psi}_i \in \mathrm{Aut}(G_i)$ satisfy

$$
\begin{aligned}
\phi_i\, j_i = Ad(h_i)\, \tilde{\psi}_i\, j_i &= Ad\big(j_i(h_{i-1}) g_i^{-1}\big)\, Ad(g_i)\, j_i\, \tilde{\psi}_{i-1} \\
&= Ad(j_i(h_{i-1}))\, j_i\, \tilde{\psi}_{i-1} = j_i\, Ad(h_{i-1})\, \tilde{\psi}_{i-1} = j_i\, \phi_{i-1},
\end{aligned}
$$

which shows that $\{\phi_i\}$ is an element of $\mathrm{Aut}(\bar{\alpha})$. Since $\phi_0 = \phi$, this also shows that $\phi \in Im(q)$. $\square$

We can now state the main theorem of this section.

Theorem 6.6 *Let $U = \varinjlim U(n!)$ be the group given in (6.1) and let (E,X) be an orbispace. Then there exists a U-space Y such that (E,X) is equivalent to $[Y/U]$.*

Proof. By Theorem 5.4 it is enough to solve the problem for $(E, X) = (|E_{\underline{\mathsf{Orb}}}|, |\underline{\mathsf{Orb}}|)$. Let

$$\Psi : \underline{B}U \to \underline{\mathsf{Orb}} \tag{6.15}$$

be the map (5.40) representing the functor $Y \mapsto [Y/U]$. By Theorem 5.11, finding a U-space Y such that $(E, X) \simeq [Y/U]$ is the same thing as finding a homotopy section of $|\Psi|$.

Let $\mathcal{F}$ be the family of finite subgroups which are embedded via a multiple of their regular representation, and let $\mathrm{Sing}(B_{\mathcal{F}}U)$ be as in Lemma 6.2. Since $|\mathrm{Sing}(B_{\mathcal{F}}U)|$ maps to $|\underline{B}U|$, it's enough to find a section of

$$\tilde{\Psi} : \mathrm{Sing}(B_{\mathcal{F}}U) \to \underline{\mathsf{Orb}}. \tag{6.16}$$

Having a section of $\tilde{\Psi}$ is the same thing as having a section of $\tilde{\Psi}^{op}$. Lemma 6.2 checks that this map is a stratified fibration of simplicial sets and by a simplicial set analog of Theorem 4.29, the obstructions to the existence of a section of (6.16) lie in the cohomology groups $H^{k+1}(\underline{\mathsf{Orb}}, \pi_k(F))$.

So we need to understand the homotopy type of the fibers of $\tilde{\Psi}$. Taking opposites, and applying the singular functor doesn't change the homotopy type, so these fibers are homotopy equivalent to the fibers of

$$\Psi_{\mathcal{F}} : B_{\mathcal{F}}U \to \underline{\mathsf{Orb}}. \tag{6.17}$$

Given a group $G \in \mathcal{J}$, let F denote the fiber of $\Psi_{\mathcal{F}}$ over the corresponding vertex of $|\underline{\mathsf{Orb}}|$. It's the simplicial set given by

$$F_n = \left\{ \{H_i, k_{ij}\}_{i,j=0..n} \middle| \, \kappa(H_i) = G,\ \beta_j Ad(k_{ij})\beta_i^{-1} = \mathrm{Id}_G,\ \beta_k(k_{ik}k_{ij}^{-1}k_{jk}^{-1}) = 1 \right\}. \tag{6.18}$$

There is the unique $H \in \mathcal{F}$ isomorphic to G, so all the H_i are equal to H. The k_{ij} centralize H and satisfy $k_{ik}k_{ij}^{-1}k_{jk}^{-1} = 1$, so F isomorphic to $B(Z_U(H))$.

Now, given a representation $\rho : H \to U(n)$, is well known that $Z_{U(n)}(H)$ is isomorphic to a product $U(n_1) \times \cdots \times U(n_r)$, where the n_i are the multiplicities of the irreducible representations ρ_i occurring in ρ. Taking the limit to infinity, we get that $Z_U(H) \simeq U^r$. We have therefore identified the fibers of $\Psi_{\mathcal{F}}$ as

$$F = B\big(Z_U(H)\big) \simeq B(U^r). \tag{6.19}$$

Combining (6.19) with (6.2), we also get that $\pi_k(F) = \mathbb{Q}^r$ for even k and 0 otherwise. In our case, we also know that ρ is the regular representation of H so we see that $\pi_{\text{even}}(F)$ is the free $\mathbb{Q}$-vector space on the set of irreps of H. Using character theory, this can also be rewritten as

$$\pi_{\text{odd}}(F) = 0, \qquad\qquad \pi_{\text{even}}(F) = \mathbb{Q}[H]^H, \tag{6.20}$$

where H acts on $\mathbb{Q}[H]$ by conjugation. Given a monomorphism $H \to H'$ the corresponding map on fibers is induced by the inclusion $Z_U(H') \to Z_U(H)$. The homomorphisms $\pi_{\mathrm{even}}(F') \to \pi_{\mathrm{even}}(F)$ is then the pullback $\mathbb{Q}[H']^{H'} \to \mathbb{Q}[H]^{H}$.

In order to finish the proof, we now need to compute the obstructions groups $H^*(\underline{\mathsf{Orb}}; \pi_{\mathrm{even}}(F))$, and show that they all vanish. Let $\mathcal{A}$ be the sheaf $\pi_{\mathrm{even}}(F)^{op}$, used to define the above cohomology group (see Definition 4.28 and Example 4.27). More concretely, if $x = (\sigma, t) \in |\underline{\mathsf{Orb}}|$ is a point, where $\sigma = \{G_i, \phi_{ij}, g_{ijk}\}$ is an n-simplex and $t \in \mathring{\Delta}^n$, we can write down the stalk of $\mathcal{A}$ at x as

$$\mathcal{A}_x = \mathbb{Q}[G_0]^{G_0}.$$

Let $\mathcal{C}$ be the category whose objects are $\mathcal{J}$ (see Definition 5.1) and whose morphisms are injective homomorphisms modulo conjugacy in the target. We compute $H^{k+1}(\underline{\mathsf{Orb}}, \mathcal{A})$ using the Leray Serre spectral sequence

$$H^p\big(|N\mathcal{C}|;\, H^q(\textit{fibers of } |\nu|; \mathcal{A})\big) \quad \Rightarrow \quad H^{p+q}(|\underline{\mathsf{Orb}}|; \mathcal{A}) \tag{6.21}$$

associated to the map (6.9). The sheaf $\mathcal{A}$ is constant along the fibers of $|\nu|$ and it's stalks are rational. So by Lemma 6.3, the spectral sequence (6.21) collapses and we get

$$H^*(|N\mathcal{C}|; \mathcal{A}) \;=\; H^*(|\underline{\mathsf{Orb}}|; \mathcal{A}). \tag{6.22}$$

Now we compute (6.22) using the Leray Serre spectral sequence

$$H^p\big(|N\mathbb{J}|;\, H^q(\textit{fibers of } |\mu|; \mathcal{A})\big) \quad \Rightarrow \quad H^{p+q}(|N\mathcal{C}|; \mathcal{A}) \tag{6.23}$$

associated to the map $\mu : N\mathcal{C} \to N\mathbb{J}$ (as usual, $\mathbb{J}$ is the poset of isomorphism classes of finite groups).

Given a point $x \in |N\mathbb{J}|$, in the interior of some simplex $G_0 < \ldots < G_n$ we let $\mathcal{G}_x$ be the groupoid whose objects are the diagrams $G_0 \to \ldots \to G_n$ in $\mathcal{C}$. By Lemma 6.4, we know that $|\mu|^{-1}(x) = |N\mathcal{G}_x|$. In particular, the connected components of $|\mu|^{-1}(x)$ are in bijection with the isomorphism classes of diagrams $G_0 \to \ldots \to G_n$. Since the objects of $\mathcal{G}_x$ have finite isomorphism groups, and since finite groups have trivial rational cohomology, we get that

$$H^*(\mu^{-1}(x), \mathcal{A}) = H^0(\mu^{-1}(x), \mathcal{A}) = \bigoplus_{\substack{\text{isomorphism} \\ \text{classes of diagrams in } \mathcal{C} \\ \alpha = (G_0 \to \ldots \to G_n)}} \big(\mathbb{Q}[G_0]^{G_0}\big)^{\mathrm{Aut}(\alpha)}. \tag{6.24}$$

So the spectral sequence (6.23) collapses, and we get

$$H^*(|\underline{\mathsf{Orb}}|; \mathcal{A}) \;=\; H^*(|N\mathcal{C}|; \mathcal{A}) \;=\; H^*(|N\mathbb{J}|, \mathcal{B}), \tag{6.25}$$

where $\mathcal{B}$ is the sheaf whose stalks are given by (6.24). By Lemma 6.5, we can rewrite these stalks as

$$\mathcal{B}_x = \bigoplus_{\substack{\text{isomorphism} \\ \text{classes of diagrams in } \mathsf{Gr} \\ \alpha = (G_0 \to \ldots \to G_n)}} \mathbb{Q}[G_0]^{\mathrm{Aut}(\alpha)}, \tag{6.26}$$

where Gr is the category of finite groups and monomorphisms.

Let Z be simplicial set given by

$$Z_k = \Big\{\text{iso. classes of } \big(g \in G_0 \hookrightarrow G_1 \hookrightarrow \ldots \hookrightarrow G_k\big)\Big\}, \tag{6.27}$$

and let $Z(n)$ be the connected component of Z where g that order n. To each k-simplex $\sigma = [g \in G_0 \to \ldots \to G_k] \in Z_k(n)$, we can associate a $(k+1)$-simplex $\sigma_+ := [g \in \langle g \rangle \to G_0 \to \ldots \to G_k] \in Z_{k+1}(n)$ satisfying $d_0(\sigma_+) = \sigma$ and $d_{i+1}(\sigma_+) = d_i(\sigma)$. These simplices assemble to a map

$$Cone\big(Z(n)\big) \to Z(n), \tag{6.28}$$

thus providing a homotopy between $\mathrm{Id}_{Z(n)}$ and the constant map at $[1 \in \mathbb{Z}/n\mathbb{Z}] \in Z(n)$. In particular, this shows that $Z(n)$ is contractible. We have shown that Z is a disjoint union of contractible connected components, and in particular that

$$H^*(Z;\mathbb{Q}) = 0 \qquad \text{for} \quad * > 0. \tag{6.29}$$

Now let's try to compute $H^*(Z;\mathbb{Q})$ using the Leray Serre spectral sequence

$$H^p\big(|N\mathbb{J}|;\, H^q(\mathit{fibers\ of}\ |\xi|;\mathbb{Q})\big) \quad \Rightarrow \quad H^{p+q}(|Z|;\mathbb{Q}). \tag{6.30}$$

associated to the projection $\xi : Z \to N\mathbb{J}$. Once again, we let $x \in |N\mathbb{J}|$ be a point in the interior of some simplex $G_0 < \ldots < G_n$, and we try to understand the fiber $|\xi|^{-1}(x)$. Clearly this fibers is discrete, so we get

$$H^*(|\xi|^{-1}(x),\mathbb{Q}) = H^0(|\xi|^{-1}(x),\mathbb{Q}) = \bigoplus_{\substack{\text{iso. classes of}\\ g\in G_0\hookrightarrow\ldots\hookrightarrow G_n}} \mathbb{Q}. \tag{6.31}$$

We now observe that the right hand sides of (6.26) and (6.31) are equal to each other. So we can assemble (6.25) and the collapsing spectral sequence (6.30) to get

$$H^*(|\underline{\mathsf{Orb}}|;\mathcal{A}) \;=\; H^*(|N\mathbb{J}|,\mathcal{B}) \;=\; H^*(Z,\mathbb{Q}) = 0 \qquad \text{for} \quad * > 0. \tag{6.32}$$

We have shown that are obstruction groups are all zero, which finishes the proof. □

As a corollary to our theorem, we get that every compact orbispace is a global quotient by the action of a compact Lie group.

Corollary 6.7 *Let (E,X) be an orbispace such that X is compact. Then there exists a compact Lie group K and a K-space Y such that $(E,X) \simeq [Y/K]$.*

Proof. Let $f : X \to |\mathsf{Orb}|$ be the map classifying (E, X). By Theorem 6.6, there exists a lift (up to homotopy)

$$\begin{tikzcd} & |\underline{B}U| \arrow[d] \\ X \arrow[r, "f"] \arrow[ur, dotted, "\tilde{f}"] & |\mathsf{Orb}| \end{tikzcd} .$$

Since X is compact and $|\underline{B}U| = \varinjlim |\underline{B}U(n!)|$, the map $\tilde{f}$ factors at some finite stage. The resulting map $X \to |\underline{B}U(n!)|$ classifies a $U(n!)$-space Y such that $(E, X) \simeq [Y/U(n!)]$. $\square$

If (E, X) is a compact orbifold, then the $U(n!)$-space Y such that $(E, X) \simeq [Y/U(n!)]$ is automatically a manifold. So we have also proven that every compact orbifold is the quotient of a compact manifold by a compact Lie group.

6.3 Enough vector bundles

There is an interesting connection between global quotients by compact Lie groups and (finite dimensional) vector bundles on orbispaces.

Definition 6.8 *An orbispace $p : E \to X$ has enough vector bundles if for every subspace $X' \subset X$ and every vector bundle V on $(E', X') := (p^{-1}(X), X)$, there exists a vector bundle W on (E, X) and an embedding $V \hookrightarrow W|_{(E',X')}$.*

Theorem 6.9 *Let $p : E \to X$ be a compact orbispace (i.e. X is compact). Then the following are equivalent:*

1. *(E, X) is a global quotient by a compact Lie group.*
2. *(E, X) has enough vector bundles.*
3. *There exists a vector bundle W on (E, X) such that for every point $y \in E$, the action of $\pi_1(F)$ on W_y is faithful. Here $F = p^{-1}(p(y))$ stands for the fiber of y.*

Proof. We first show 1. $\Rightarrow$ 2. Let $Y \circlearrowleft K$ be such that $(E, X) \simeq [Y/K]$, and let $Y' \subset Y$ be the K-invariant subspace corresponding to $X' \subset X$. Let V be a vector bundle on (E, X) and let $\tilde{V}$ be the corresponding K-equivariant vector bundles on Y. It is well known that any equivariant vector bundle $\tilde{V}$ on a compact space Y' embeds in one of the form $Y' \times M$, where M is a representation of K. Let W be the vector bundle on (E, X) corresponding to $Y \times M \to Y$. Since $\tilde{V}$ embeds in $Y \times M|_{Y'}$, the bundle V embeds in $W_{(E',X')}$, as desired.

We show 2. $\Rightarrow$ 3. Suppose that (E, X) has enough vector bundles, and let $\{U_i\}$ be a finite cover of X such that $(p^{-1}(U_i), U_i) \simeq [Y_i/G_i]$. Let M_i be faithful representations of G_i, and let V_i be the vector bundles on $(p^{-1}(U_i), U_i)$ corresponding to $Y_i \times M_i \to Y_i$. Since M_i is faithful, the isotropy groups of $(p^{-1}(U_i), U_i)$ act faithfully on the fibers of V_i. Let W_i be vector bundles on (E, X) such that $V_i \hookrightarrow W_i|_{(E',X')}$, and let $W := \bigoplus W_i$. Clearly, the isotropy groups of (E, X) act faithfully on the fibers of W.

We now show 3. $\Rightarrow$ 1. Let $(P, P/\!\sim)$ be the total space of the frame bundle of W. It has no isotropy groups, and so it's equivalent to the space $Y := P/\!\sim$. The fibers of $P \to Y$ are contractible and P has a free action of the orthogonal group $O(n)$. So $E = P/O(n)$ is a Borel construction for $Y \circlearrowleft O(n)$. Similarly $X = Y/O(n)$. We have identified (E, X) with the global quotient $[Y/O(n)]$, which finishes the proof. $\square$

Corollary 6.10 *Compact orbispaces have enough vector bundles.*

Proof. By corollary 6.7, all orbispaces are global quotients by compact Lie groups. $\square$

As a consequence of Corollary 6.10, we can prove excision for K-theory.

Proposition 6.11 *Let $p : E \to X$ be a compact orbispace and let $(F, A) := (p^{-1}(A), A)$ and $(E', X') := (p^{-1}(X'), X')$ be two sub-orbispaces and let (F', A') be the intersection of (F, A) and (E', X'). Let $V' = V_0' \ominus V_1'$ be a $\mathbb{Z}/2$-graded vector bundle on (E', X') and $f' : V_0'|_{(F',A')} \to V_1'|_{(F',A')}$ be an isomorphism.*

Then there exists a $\mathbb{Z}/2$-graded vector bundle V on (E, X) and an isomorphism $f : V_0|_{(F,A)} \to V_1|_{(F,A)}$ such that $(V, f)|_{(E',X')}$ represents the same class as (V', f') is the relative K-theory group $K^0((E', X'), (F', A'))$. More precisely $(V, f)|_{(E',X')}$ is the sum of (V', f') and $(Z \ominus Z, \mathrm{Id}_Z|_{(F',A')})$, for some vector bundle Z on (E', X').

Proof. Let $V' = V_0' \ominus V_1'$ and f be as above. By Theorem 6.9, we can find a vector bundle V_0 on (E, X) such that $V_0' \hookrightarrow V_0|_{(E',X')}$. Let Z be the orthogonal complement of V_0' in $V_0|_{(E',X')}$. We build V_1 by gluing $V_0|_{(F,A)}$ and $V_1' \oplus Z$ along the map

$$V_0|_{(F',A')} \simeq (V_0' \oplus Z)|_{(F',A')} \xrightarrow{f \oplus 1} .(V_1' \oplus Z)|_{(F',A')}.$$

We then let f be the natural map between $V_0|_{(F,A)}$ and $V_1|_{(F,A)}$. It is clear that (V, f) has the required properties. $\square$

Remark 6.12 Theorem 6.9 still holds for Lie orbispaces (with identical proof), but Corollary 6.10 is not true any more. So we cannot use finite dimensional vector bundles in order to define K-theory of Lie orbispaces. One should instead use bundles of $\mathbb{Z}/2$-graded Hilbert spaces equipped with odd self-adjoint Fredholm operators.

6.4 Contractible groups

Recall the group $U = \varinjlim U(n!)$ from (6.1) and the family $\mathcal{F}$ of subgroups embedded via their regular representation. There are two main properties of U and $\mathcal{F}$ used in the proof of Theorem 6.6. The first one is that for any $G, H \in \mathcal{F}$ and any monomorphism $G \to H$ there exists an element $k \in U$ such that $Ad(k)|_G = f$ (see Lemma 6.1). The second one is that the centralizers $Z_U(H)$ are rational spaces for all $H \in \mathcal{F}$ (see equations (6.19) and (6.20)).

The idea is that, from the point of finite groups, these centralizers behave as if they were contractible. This motivates the following definition.

Definition 6.13 *A topological group K is contractible with respect to a family $\mathcal{F}$ if for every groups $G, H \in \mathcal{F}$ and every monomorphism $f : G \to H$, the space*

$$\big\{k \in K \,\big|\, Ad(k)|_G = f\big\} \tag{6.33}$$

is contractible.

As one might expect, it is easy to build contractible groups:

Proposition 6.14 *Let K be a topological group and $\mathcal{F}$ a family of subgroups. Then there exists a group K' containing K, which is contractible with respect to the family generated by $\mathcal{F}$.*

Proof. Each time we find a non-trivial map from a sphere into one of the spaces (6.33), we add a cell to kill it. We then freely generate a groups, modulo the relation that $Ad(k)|_G = f$ for all points k in that cell. This process terminates by the small object argument. □

Another example of contractible group is the unitary group of an infinite dimensional Hilbert space:

Example 6.15 Let $\mathcal{H}$ be a countably infinite dimensional Hilbert space. Let $U(\mathcal{H})$ be it group of unitary automorphisms, and let $\mathcal{F}$ be the family of finite (or compact Lie) subgroup G such that each irrep of G appears infinitely many times in $\mathcal{H}$. Then $U(\mathcal{H})$ is contractible with respect to $\mathcal{F}$.

Indeed, let $f : G \to H$ be a monomorphism between elements of $\mathcal{F}$, and consider the space (6.33). Since f is injective, the inclusion $G \hookrightarrow U(\mathcal{H})$ and the map f are equivalent representations of G, hence (6.33) in non-empty. The space (6.33) carries a simply transitive action of the centralizer $Z_{U(\mathcal{H})}(G) \simeq (U(\mathcal{H}))^r$. By Kuiper's theorem [21], the group $U(\mathcal{H})$ is contractible. Therefore $Z_{U(\mathcal{H})}(G)$ is contractible, and so is the space (6.33).

If we replace U by a group which contains all finite groups, and which is contractible with respect to them, them the proof of Theorem 6.6 goes through. As in Lemma 6.2, the map

$$|\mathrm{Sing}\underline{B}K| \to |\underline{\mathsf{Orb}}| \tag{6.34}$$

is a stratified fibration, and its fibers are $B(Z_K(H))$ as in (6.19). By (6.33), these fibers are contractible, and so the map (6.34) is a stratified homotopy equivalence. Since $|\mathrm{Sing}\underline{B}K|$ maps to $|\underline{B}K|$ and $|\underline{B}K|$ maps to $|\underline{\mathsf{Orb}}|$, we also get that

$$|\underline{B}K| \simeq |\underline{\mathsf{Orb}}| \tag{6.35}$$

as stratified spaces. This suggests the following improvement of Theorem 6.6.

Theorem 6.16 *Let K be a group and let $\mathcal{F}$ be the family of its finite subgroups. Suppose that every finite group is isomorphic to an element of $\mathcal{F}$, and that K is contractible with respect to $\mathcal{F}$.*

Then the natural functor from $(K, \mathcal{F})$-spaces to the category of orbispaces and representable maps (i.e. injective on stabilizer groups) is an equivalence of topologically enriched categories.

Proof. Let $\mathcal{O}_K$ denote the category of orbits K/G for $G \in \mathcal{F}$ and K-equivariant maps between them. Let $\mathcal{O}_K$-spaces be the category of continuous contravariant functors $\mathcal{O}_K \to \mathsf{spaces}$. By Elmendorf's theorem [8], the categories of $(K, \mathcal{F})$-spaces and $\mathcal{O}_K$-spaces are topologically equivalent. The functor K-$\mathsf{spaces} \to \mathcal{O}_K$-$\mathsf{spaces}$ is given by

$$Y \mapsto \big(K/G \mapsto \mathrm{Map}_K(K/G, Y)\big) \tag{6.36}$$

and its homotopy inverse is the bar construction

$$F \mapsto B(F, \mathcal{O}_K, \iota) \tag{6.37}$$

where ι is the inclusion $\mathcal{O}_K \hookrightarrow K$-$\mathsf{spaces}$.

Let repr be the category of orbispaces and representable maps, and consider the the full subcategory of "orbipoints" $BG := (K(G,1), pt)$. Taking the standard simplicial model for BG and using the realization of the simplicial mapping space instead of all continuous maps, we obtain an equivalent subcategory $\mathcal{B}$. Its morphisms are given by

$$\mathrm{Hom}_{\mathcal{B}}(BG, BH) = \big(\mathrm{Mono}(G, H) \times EH\big)/H. \tag{6.38}$$

Let $\mathcal{B}$-spaces be the category of continuous contravariant functors $\mathcal{B} \to \mathsf{spaces}$. One has two functors similar to (6.36) and (6.37) given by

$$(E, X) \mapsto \big(BG \mapsto \mathrm{Map}_{\mathsf{repr}}(BG, (E, X))\big) \tag{6.39}$$

and

$$F \mapsto B(F, \mathcal{B}, \iota), \tag{6.40}$$

where ι is the inclusion $\mathcal{B} \hookrightarrow \mathsf{repr}$. It is important that we only have monomorphisms in (6.38), otherwise the right hand side of (6.40) would fail the third condition of Theorem 3.5, and therefore wouldn't be an orbispace[1]. The proof of Elmendorf's theorem applies, and we get that the functors (6.39) and (6.40) are homotopy inverses.

In order to show that K-spaces and repr are equivalent, it's enough to show that $\mathcal{O}_K$-spaces and $\mathcal{B}$-spaces are equivalent. To see that, we compare the categories $\mathcal{O}_K$ and $\mathcal{B}$. The hom-spaces in $\mathcal{O}_K$ are given by

$$\mathrm{Hom}_{\mathcal{O}_K}(K/G, K/H) = \big\{k \in K \,\big|\, Ad(k)G \subset H\big\}/H. \tag{6.41}$$

By (6.33), the space $\{k \in K | Ad(k)G \subset H\}$ is homotopy equivalent to the set of monomorphisms from G to H. So (6.41) and (6.38) are homotopy equivalent.

We have shown that $\mathcal{O}_K$ and $\mathcal{B}$ are homotopy equivalent categories. Therefore, the same holds for $\mathcal{O}_K$-spaces and $\mathcal{B}$-spaces, and also for K-spaces and repr. $\square$

Remark 6.17 The statement in Theorem 6.16 still holds if we replace finite groups

[1] We should also replace right hand side of (6.40) by a stratified fibration.

by any class of topological groups that admits a set of representatives.