Kawahigashi and Longo's classification of conformal nets with $c < 1$

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8th June 2013

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Abstract

These are notes for the talks of 29 May of the Utrecht University student seminar on algebraic quantum field theory of Spring 2013. Every conformal net has an invariant $c \in \mathbb{R}$ associated to it, called its central charge. Kawahigashi and Longo have classified conformal nets with $c < 1$ and we will discuss some ideas and tools that go into their proof.

1 Introduction

Recall from Joost's talk of 1 May the construction of certain conformal nets, called *Virasoro nets* from projective unitary representations of $\mathrm{Diff}^+(S^1)$. Joost denoted them by $\mathcal{A}_{\mathrm{Vir},c}$, but we will write Vir_c. These turn out to play a central role in the general theory, because every conformal net contains a copy of a Virasoro net.

If $\mathcal{A}$ is a conformal net and U its unitary representation of $\mathrm{Diff}^+(S^1)$, then we call a family of von Neumann algebras $\{\mathcal{B}(I)\}_{I\in\mathcal{I}}$ with $\mathcal{B}(I) \subseteq \mathcal{A}(I)$ a ***(conformal) subnet*** of $\mathcal{A}$ if $\mathcal{B}$ is also isotonous, that is, $\mathcal{B}(I) \subseteq B(J)$ if $I \subseteq J$, and $\mathcal{B}$ is also diffeomorphism covariant, that is, $U(g)\mathcal{B}(I)U(g)^* = \mathcal{B}(gI)$ for all $g \in \mathrm{Diff}^+(S^1)$. So this definition is what you would expect, except that there is a slight subtility: if the vacuum vector Ω of $\mathcal{A}$ is also cyclic for the whole Hilbert space for all $\mathcal{B}(I)$, then $\mathcal{B}$ and $\mathcal{A}$ must agree. This can be proven with the theorem by Takesaki that was discussed by André at the beginning of his talk on 10 May. See [1, Lemma 3.3] for a proof.

Recall that a Möbius covariant net $\mathcal{A}$ with Hilbert space $\mathcal{H}$ is called ***irreducible*** if $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I) = B(\mathcal{H})$. Alternative definitions may be given by the demand that its vacuum vector Ω is up to scalar multiples the unique vector invariant under the representation of the Möbius group, or the demand that all von Neumann algebras $\mathcal{A}(I)$ are factors. In this case, they are either $\mathbb{C}$ or a type III_1-factor.

Proposition 1.1. *Let $\mathcal{A}$ be an irreducible conformal net. Then it contains canonically a subnet that is isomorphic to* Vir_c *for some central charge c If $c < 1$, then this subnet is an irreducible subnet with finite index.*

If U is the projective unitary representation of $\mathrm{Diff}^+(S^1)$ that implements the diffeomorphism covariance of $\mathcal{A}$, then the candidate for Vir_c is the subnet $U(\mathrm{Diff}^+(I))''$. See [6, Proposition 3.5] for the rest of the proof.

In turn, Virasoro nets do not contain any proper conformal nets themselves. In particular, applying the above construction will give you the Virasoro net back. It is for this reason that they sometimes go under the name *minimal models* in the physics literature.

Definition 1.2. The ***central charge*** of an irreducible conformal net $\mathcal{A}$ is the scalar $c \in \mathbb{R}$ such that $\mathrm{Vir}_c \subseteq \mathcal{A}$.

When trying to classify conformal nets, it turns out to be a good idea to do this by their central charge. The nets that have a given central charge c are namely by definition the extensions of the Virasoro net Vir_c. As André explained, inclusions of conformal nets behave the other way around compared to for example inclusions of groups: for a given conformal net, its family of subnets might be very large, and it is the family of extensions on which restrictive conditions hold.

In 2004, Yasuyuki Kawahigashi and Roberto Longo succeeded in classifying all conformal nets with $c < 1$. This is the topic of this talk. Note that for example loop group nets and lattice nets will fall outside this classification. The net associated to an even lattice Λ for example will have central charge equal to the rank of Λ. Their result is as follows:

Theorem 1.3 (Kawahigashi, Longo [6])**.** *The irreducible conformal nets with central charge $c < 1$ are*

- *the Virasoro nets* Vir_c *with $c < 1$,*
- *two infinite families, namely the simple current extensions with index* 2 *of the above Virasoro nets,*
- *four exceptional nets, which have central charge* $c = \frac{21}{22}, \frac{25}{26}, \frac{144}{145}$ *and* $\frac{154}{155}$.

More precisely, this list turns out to correspond to the list of pairs of A-D_{2n}-$E_{6,8}$ Dynkin diagrams such that the difference of the Coxeter numbers is 1. A little bit more on this later.

The plan is to construct for a fixed Virasoro net Vir_c with $c < 1$ a map

$$\left\{ \begin{array}{c} \text{irreducible extensions} \\ \mathcal{B} \supseteq \mathrm{Vir}_c \text{ of finite index} \end{array} \right\} \to \left\{ \begin{array}{c} \text{type I modular} \\ \text{invariants of } \mathrm{Vir}_c \end{array} \right\}.$$

The right hand side is a certain finite set of matrices associated to Vir_c that has been completely classified: it corresponds to the above mentioned list of pairs of Dynkin diagrams. The heart of Kawahigashi and Longo's proof is then showing that this map is a bijection, which therefore implies a classification of the left hand side. These notes will discuss the proof of the bijectivity though.

2 Completely rational conformal nets

This, and the next section will discuss somewhat more general theory of conformal nets than what only applies to Vir_c. We will restrict ourselves to a particular class of 'nice' conformal nets. We will only define what this means for an irreducible net, because otherwise one of the conditions becomes slightly more technical.

Definition 2.1. An irreducible conformal net $\mathcal{A}$ is called ***completely rational*** if it is

- ***split***: if I_1 and I_2 are two intervals such that $\overline{I_1} \subseteq I_2^\circ$, then $\mathcal{A}(I_1) \vee \mathcal{A}(I_2') = \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$,
- ***strongly additive*** ('removing a single point from an interval does not matter'): if I_1 and I_2 are two intervals that are obtained by removing a single point from an interval I, then $\mathcal{A}(I) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2)$,
- of ***finite index***: if we split the S^1 into four disjoint intervals $I_1, \ldots, I_4$, where I_1 and I_3 do not touch and I_2 and I_4 do not touch, then the inclusion of factors $\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \subseteq (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))'$ (that we have by locality of $\mathcal{A}$) is of finite Jones-Kosaki index.

What we will be interested in is that these three quite technical conditions on a net turn out to imply that its representation theory is especially nice. For the rest of this section, $\mathcal{A}$ will be an irreducible, completely rational conformal net. Recall that for two representations λ and μ, we denoted their braiding by $\beta_{\lambda\mu} \colon \lambda \circ \mu \to \mu \circ \lambda$. See for details on this braiding André's talk of 10 May and [5, Section IV.4].

Theorem 2.2. *The category of representations* $\mathrm{Rep}\,\mathcal{A}$ *of* $\mathcal{A}$ *is a* ***modular tensor category***. *The most important properties of such a category are*

- *there is a notion of* ***simplicity*** *of an object, and every object is* ***semisimple****, that is, isomorphic to a finite direct sum of simple objects,*
- *up to isomorphism, there are only finitely many simple objects, say* $\lambda_0, \ldots, \lambda_n$*, where* $\lambda_0 := \mathcal{H}_0$*, the vacuum representation,*
- *the braiding that we have on the tensor category* $\mathrm{Rep}\,\mathcal{A}$ *is* ***non-degenerate****, that is, if* $\lambda \in \mathrm{Rep}\,\mathcal{A}$ *is a simple object such that*

$$\beta_{\lambda\mu} \circ \beta_{\mu\lambda} = \mathrm{id}_{\mu\lambda}$$

for all simple objects $\mu \in \mathrm{Rep}\,\mathcal{A}$*, then* λ *must be the vacuum representation* $\lambda_0 = \mathcal{H}_0$*.*

This second property can be interpreted as saying that when a strand labeled by λ is braided with another strand labeled with μ, then this is a genuine braiding. One can not pull the λ-strand through the μ-strand to obtain two unbraided strands going straight down.

It has been shown that the loop group nets $SU(2)_k$ are completely rational. For low levels, their simple objects have appeared several times in André's talks in this seminar. Furthermore, this result can be used to show that also Virasoro nets Vir_c with $c < 1$ are completely rational. See [5, Corollary 3.4].

We will now start the construction of two matrices S and T associated to $\mathcal{A}$.

Definition 2.3. The ***Verlinde***, or also called ***fusion coefficients*** $N_{\mu\nu}^{\lambda}$ of $\mathcal{A}$ are defined as the dimensions of certain intertwiner spaces between representations of $\mathcal{A}$

$$N_{\mu\nu}^{\lambda} := \dim \mathrm{Hom}(\lambda, \mu \circ \nu),$$

where μ, ν and λ run over all simple objects in $\mathrm{Rep}\,\mathcal{A}$.

Alternatively, we may observe that the Grothendieck group $K_0(\mathrm{Rep}\,\mathcal{A})$ of the category $\mathrm{Rep}\,\mathcal{A}$ is a finite-dimensional algebra over $\mathbb{C}$ with basis the classes $[\lambda]$ of the simple objects λ of $\mathrm{Rep}\,\mathcal{A}$ and product given by $[\mu] \cdot [\nu] := [\mu \circ \nu]$, where μ and ν are simple objects. Then the Verlinde coefficients are nothing but the structure coefficients with respect to this basis:

$$[\mu] \cdot [\nu] := [\mu \circ \nu] =: \sum_{\lambda \text{ simple}} N_{\mu\nu}^{\lambda} [\lambda].$$

Recall from Bram's last talk of 15 May that we associated to every simple object $\lambda \in \mathrm{Rep}\,\mathcal{A}$ its *statistical dimension* $d_\lambda \in \mathbb{C}$ and its *statistics phase* $\kappa_\lambda \in S^1 \subseteq \mathbb{C}$. This statistics phase can be interpreted as the value of the twist isomorphism $\lambda \to \lambda$.

Definition 2.4. We define ***Rehren's Y-matrix*** associated to $\mathcal{A}$ as the matrix with coefficients

$$Y_{\mu\nu} := \sum_{\lambda \text{ simple}} \frac{\kappa_\mu \kappa_\nu}{\kappa_\lambda} N_{\mu\nu}^{\lambda} d_\lambda \in \mathbb{C},$$

where μ and ν are simple objects of $\mathrm{Rep}\,\mathcal{A}$. We furthermore define the following two scalars z and c associated to $\mathcal{A}$:

$$z := \sum_{\lambda \text{ simple}} d_\lambda^2 \kappa_\lambda$$

and

$$c := 4 \cdot \arg(z)/\pi.$$

The expression for $Y_{\mu\nu}$ is complicated, but it turns out to be nothing but $\mathrm{tr}(\beta_{\mu\nu} \circ \beta_{\nu\mu})$. That is, it is the scalar assigned to the Hopf link labeled with μ and ν. See [2, Section 4] for some pictures. Note that c is only well-defined up to multiples of 8. This will not matter though for the following definition.

Definition 2.5. The ***S and T matrices*** associated to $\mathcal{A}$ are given by the coefficients

$$S_{\mu\nu} := |z|^{-1} Y_{\mu\nu}$$

and

$$T_{\mu\nu} := e^{-\pi i c/12} \kappa_\mu \delta_{\mu\nu},$$

where μ and ν are simple objects of $\mathrm{Rep}\,\mathcal{A}$.

So S is a certain normalisation of Rehren's Y-matrix and T is a certain normalisation of the diagonal matrix of the values κ_μ of the twists on the simple objects.

These matrices turn out to afford a representation of $\mathrm{SL}(2, \mathbb{Z})$ on $K_0(\mathrm{Rep}\,\mathcal{A})$.

3 α-Induction and modular invariants

Recall that if G is a finite group, $H \subseteq G$ a subgroup and $\rho\colon H \to \mathrm{GL}(W)$ a representation of H on a finite dimensional vector space W, then there is a method to lift ρ to a representation of G by defining $\mathrm{Ind}_H^G := \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$, where $\mathbb{C}[G]$ and $\mathbb{C}[H]$ are the group algebras of G and H respectively. This is known as the *induced representation* of W. The same thing works in the representation theory of Lie algebras for example, except that there the universal enveloping algebra takes on the role of group algebra.

Something similar is possible for representations of conformal nets, and this method is known as α-induction. There are important differences though. Firstly, for an extension of groups $H \subseteq G$, G has a larger symmetry and in some sense also a larger representation category. On the other hand, in an extension $\mathcal{B} \supseteq \mathcal{A}$ of conformal nets, $\mathcal{B}$ will have in some sense a smaller representation category than $\mathcal{A}$. The representations induced from representations of $\mathcal{A}$ will not be genuine representations of $\mathcal{B}$. Kawahigashi describes this as $\mathrm{Rep}\,\mathcal{B}$ being 'too small' to accept the image of α-induction.

Another difference is that α-induction applied to an extension $\mathcal{B} \supseteq \mathcal{A}$ will make use of the braiding structure on $\mathrm{Rep}\,\mathcal{A}$. Recall, however, that we defined this braiding by always choosing an equivalent representation localised in an interval to the right. We might as well have chosen to always localise to the left, and this would have given a different braiding. We denote the former with β^+ and the latter with β^-. This is the geometrical interpretation in our context of the abstract fact that in a tensor category equipped with a braiding $c_{V,W}\colon V \otimes W \to W \otimes V$, $c_{W,V}^{-1}$ again defines a braiding: braidings on tensor categories always come in pairs. Therefore, for each representation λ of $\mathcal{A}$, there will be two induced representations of $\mathcal{B}$: α_λ^+ and α_λ^-.

The definition of α-induction can probably be given in a manner very similar to that of induction of group or Lie algebra representations. This would require more machinery though, which has already been developed by André and his collaborators.

Recall the following global version of the canonical endomorphism of a subfactor from the first half of Laura's talk of 22 May. Let $\mathcal{B} \supseteq \mathcal{A}$ be an irreducible extension of finite index and $\iota\colon \mathcal{A} \to \mathcal{B}$ the inclusion. Then there exists for each interval I a map $\bar{\iota}\colon \mathcal{B} \to \mathcal{A}$ such that for all intervals $\widetilde{I} \supseteq I$, $\iota \circ \bar{\iota}|_{\mathcal{B}(I)}$ is a canonical endomorphism for $\mathcal{B}(\widetilde{I})$. We define

$$\gamma := \iota \circ \bar{\iota}, \qquad \theta := \bar{\iota} \circ \iota,$$

and call these the ***canonical***, and ***dual canonical endomorphism*** of $\mathcal{A} \subseteq \mathcal{B}$ respectively.

Definition 3.1. Let $\mathcal{B} \supseteq \mathcal{A}$ be an irreducible extension of finite index of $\mathcal{A}$ and $\lambda \in \mathrm{Rep}\,\mathcal{A}$. Then we define its two induced representations α_λ^+ and α_λ^- as

$$\alpha_\lambda^\pm := \bar{\iota}^{-1} \circ \mathrm{Ad}(\beta_{\lambda\theta}^\pm) \circ \lambda \circ \bar{\iota}.$$

We have $\alpha_\lambda^\pm|_{\mathcal{B}(I)} \in \mathrm{End}\,\mathcal{B}(I)$. Furthermore, $\alpha_\lambda^\pm \circ \iota = \iota \circ \lambda$, that is, its restriction to $\mathcal{A}$ gives you λ back again. Secondly, $\alpha_{\mathcal{H}_0^\mathcal{A}}^\pm = \mathcal{H}_0^\mathcal{B}$, that is, the vacuum representation of $\mathcal{A}$ lifts to that of $\mathcal{B}$.

Definition 3.2. A ***modular invariant (matrix)***, also called a ***coupling matrix***, of $\mathcal{A}$ is a matrix Z, whose columns and rows are labeled by the simple objects of $\operatorname{Rep}\mathcal{A}$, such that

- $Z_{\lambda\mu}$ is a non-negative integer for all λ and μ,
- $Z_{00} = 1$,
- Z commutes with the S and T matrices associated to $\mathcal{A}$.

Note that a net always has at least one modular invariant, namely the identity matrix. It might have more though. In André's talk of 29 March, he mentioned in the discussion of the construction of an anomaly-free full CFT that $SU(2)_1$, $SU(2)_2$ and $SU(2)_3$ only have a trivial modular invariant, while $SU(2)_4$ also has the matrix

$$\begin{pmatrix} 1 & & & & 1 \\ & 0 & & 0 & \\ & & 2 & & \\ & 0 & & 0 & \\ 1 & & & & 1 \end{pmatrix}.$$

In general, we have the following result:

Proposition 3.3. *A completely rational conformal net has only finitely many modular invariants.*

This number will often be 1, 2 or 3 for us. See the last paragraph on page 4 of [2] for a proof.

Theorem 3.4. *Let $\mathcal{B} \supseteq \mathcal{A}$ be an irreducible extension of finite index. Then the matrix Z with columns and rows labeled by the simple objects of* $\operatorname{Rep}\mathcal{A}$*, and defined by*

$$Z_{\lambda\mu} := \dim \operatorname{Hom}(\alpha^+_\lambda, \alpha^-_\mu),$$

is a type I modular invariant of $\mathcal{A}$.

See [3, Corollary 5.8] for a proof. So we have a map

$$\left\{ \begin{array}{c} \text{irreducible extensions} \\ \mathcal{B} \supseteq \mathcal{A} \text{ of finite index} \end{array} \right\} \to \left\{ \begin{array}{c} \text{type I modular} \\ \text{invariants of } \mathcal{A} \end{array} \right\}.$$

In general, this map will not be injective or surjective. Kawahigashi and Longo proved that it bijective for $\mathcal{A} = \mathrm{Vir}_c$ if $c < 1$.

The modular invariants of Vir_c with $c < 1$ have been classified in [4]. Those of type I turn out to be labeled by the following pairs of Dynkin diagrams. Write $c = 1 - 6/(m(m+1)) < 1$, $m = 2, 3, 4, \ldots$. Then

m	n	$4n+1$	$4n+2$	11	12	29	30
Label	(A_{n-1}, A_n)	(A_{4n}, D_{2n+2})	(D_{2n+2}, A_{4n+2})	(A_{10}, E_6)	(E_6, A_{12})	(A_{28}, E_8)	(E_8, A_{30})

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