Loop Groups & Conformal Nets

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0 Introduction

In these lecture notes we will see an example of a chiral conformal. We start by looking at a chiral algebra: the loop algebra. Next we classify all irreducible representations of this algebra. Finally we see that for each such representation we get a conformal net. Since we are looking at chiral CFT's the conformal net will assign an algebra to every open interval on the circle.

These notes are structured as follows. The first section introduces the loop algebra and the associated central extension. Then we move on to the classification of the irreducible representations. We will find such a representation for each dominant integral weight of our algebra. Finally we show that this construction gives us a conformal net.

1 The Affine Lie Algebra

1.1 Some semi-simple Lie algebra facts

The first part of this section is treated in any introductory text on semi-simple Lie algebras (for example [1]). I will assume them as facts, and not give any proofs for the statements.

The chiral algebra we will be working with is the **affine Lie Algebra**. To construct this we start with a compact Lie group G_0 which is simple, simply connected and simply laced. Denote the corresponding Lie algebra by $\mathcal{G}_0$ and its complexification by $\mathcal{G}$.

We choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathcal{G}$ and let Δ be the set of roots of $\mathfrak{h}$ on $\mathcal{G}$. They are the elements $\lambda \in \mathfrak{h}^*$ such that the set $\mathcal{G}_\lambda := \{a \in \mathcal{G} | [h,a] = \lambda(h)a \quad \forall h \in \mathfrak{h}\}$ is non-trivial. Then we get a decomposition of our Lie algebra into the Cartan subalgebra and the one-dimensional root spaces $\mathcal{G}_\alpha$:

$$\mathcal{G} = \mathfrak{h} \bigoplus_{\alpha \in \Delta} \mathcal{G}_\alpha$$

Now we choose a set of positive roots Δ_+ and set of simple roots $\{\alpha_1, \ldots, \alpha_l\}$ where l is the dimension of $\mathfrak{h}$. Thus we have a highest root in Δ_+ which we call θ. It is the root from which you can obtain any other root by subtracting positive roots. We now get the triangular decomposition of $\mathcal{G}$:

$$\mathcal{G} = n_+ \oplus \mathfrak{h} \oplus n_-$$

Here $n_+ = \bigoplus_{\alpha \in \Delta_+} \mathcal{G}_\alpha$ and n_- is defined similarly.

We choose a non-degenerate, invariant, symmetric bilinear form on $\mathcal{G}$ (for example the Killing form). Since this form is non-degenerate we get an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ and a corresponding bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$. We normalize this form by setting $(\theta|\theta) = 2$. Define the coroots as $h_{\alpha_i} := \nu^{-1}(\alpha_i)$.

Next we choose a *Chevalley* basis of $\mathcal{G}$ which has the following nice properties. The basis consist of $\{e_\alpha\}_{\alpha \in \Delta} \cup \{h_{\alpha_i} | i = 1 \ldots l\}$ where the e_α's are elements of $\mathcal{G}_\alpha$. Furthermore we have:

$$\begin{aligned} \left[e_{\alpha_i}, h_{\alpha_j}\right] &= 0 \\ [e_\alpha, e_{-\alpha}] &= h_\alpha \\ \left[h_{\alpha_i}, e_{\alpha_j}\right] &= \alpha_j(h_{\alpha_i}) e_{\alpha_j} \end{aligned}$$

If $\alpha + \beta$ is also a root and n is the maximal integer such that $\alpha - n\beta$ is a root then we also have:

$$[e_\alpha, e_\beta] = \pm(n+1) e_{\alpha+\beta}$$

We now define $\mathcal{G}_k$ as the real span of $\{ih_\alpha; e_\alpha - e_{-\alpha}; i(e_\alpha + e_{-\alpha})\}_{\alpha \in \Delta}$. $\mathcal{G}_k$ is isomorphic to $\mathcal{G}_0$ and we will identify them. Finally we let τ be the conjugation of $\mathcal{G}$ induced by $\mathcal{G}_k$ and set $x^* = -\tau(x)$ for $x \in \mathcal{G}$.

1.2 The main construction

Now we actually construct the affine Lie algebra, which is a central extension of the loop algebra. A loop in $\mathcal{G}$ is a map $f : S^1 \to \mathcal{G}$. If we assume that is polynomial we can write it as $f(\theta) = \sum_{n=-N}^{N} f_n e^{in\theta}$ where $f_n \in \mathcal{G}$. Let $\tilde{\mathcal{G}}$ be the Lie algebra of all such polynomial loops. We call this algebra the **loop algebra**. The Lie bracket is defined as $[e^{im\theta} f, e^{in\theta} g]_0 = e^{i(m+n)\theta}[f,g]_{\mathcal{G}}$ where $[\cdot,\cdot]_{\mathcal{G}}$ is the original Lie bracket of $\mathcal{G}$.

For convenience we replace $e^{i\theta}$ by t. Now as a vector space, $\tilde{\mathcal{G}} = \mathcal{G} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ where $\mathbb{C}[t, t^{-1}]$ is the algebra of Laurent polynomials in t. As an abbreviation we write $x(n)$ for the element $x \otimes t^n$.

We now want to find a central extension of the loop algebra. First we extend the bilinear form $(\cdot|\cdot)$ of previous section to $\tilde{\mathcal{G}}$ by setting:

$$(x \otimes P(t) | y \otimes Q(t)) := (\frac{1}{2\pi} \int_0^{2\pi} P(e^{i\theta}) Q(e^{i\theta}) d\theta)(x|y)$$

Now let $d = t\frac{d}{dt}$ which is a derivation of $\mathbb{C}[t, t^{-1}]$. Then we can define a cocycle ψ on $\tilde{\mathcal{G}}$ as follows:

$$\psi(x \otimes P(t), y \otimes Q(t)) := (x \otimes dP(t) | y \otimes Q(t))$$

Exercise: *Check that ψ is a 2-cocycle. $\psi : \tilde{\mathcal{G}} \times \tilde{\mathcal{G}} \to \mathbb{C}$ is called a cocycle if $\psi(x,y) = -\psi(y,x)$ and the cocycle condition holds:*

$$\psi([x,y], z0 + \psi([y,z], x) + \psi([z,x], y) = 0$$

Hence we get a corresponding central extension $\hat{\mathcal{G}}$ of $\tilde{\mathcal{G}}$. As a vector space we have $\hat{\mathcal{G}} = \tilde{\mathcal{G}} \oplus \mathbb{C} \cdot c$. Its bracket is defined as:

$$[f \oplus \lambda c, g \oplus \mu c] = [f,g]_0 + \psi(f,g) \cdot c$$

We can also extend the map $x \mapsto x^*$ by setting $c^* = c$ and $x(n)^* = x^*(-n)$. So we get a corresponding extension $\hat{\mathcal{G}}_k := \{c \in \hat{\mathcal{G}} | x^* = -x\}$. Again $\hat{\mathcal{G}}$ is the complexification of $\hat{\mathcal{G}}_k$. We call $\hat{\mathcal{G}}$ the **affine Lie algebra** and $\hat{\mathcal{G}}_k$ its **compact form**.

1.3 A final extension

To study the representations of $\hat{\mathcal{G}}$ it will be convenient to extend it by one extra dimension. The corresponding extension of our bilinear form will the be non-degenerate which is useful. As a vector space this extension is defined as $\hat{\mathcal{G}}^e := \hat{\mathcal{G}} \oplus \mathbb{C} \cdot d$ where again $d = t\frac{d}{dt}$. On $\hat{\mathcal{G}}^e$ we can define a Lie bracket by:

$$[x \otimes P(t) + \lambda c + \mu d, y \otimes Q(t) + \nu c + \sigma d]_e = [x \otimes P(t) + \lambda c, Q(t) \otimes y + \nu c] + \mu d(Q(t)) \otimes y - \sigma d(P(t)) \otimes x$$

We can also extend our bilinear, symmetric form to a form on $\hat{\mathcal{G}}^e$ by defining:

$$\begin{aligned} (x \otimes P(t)|c) &= (x \otimes P(t)|d) = 0 \\ (c|c) &= (d|d) = 0 \\ (d|c) &= (c|d) = 1 \end{aligned}$$

Now we look at the root space decomposition of $\hat{\mathcal{G}}^e$ relative to the abelian subalgebra $\hat{\mathfrak{h}}^e = 1 \otimes \mathfrak{h} \oplus \mathbb{C} \cdot c \oplus \mathbb{C} \cdot d$. First note that you can view $\mathfrak{h}^*$ as a subalgebra of $\hat{\mathfrak{h}}^{e*}$ by defining $\lambda(c) = \lambda(d) = 0$ for $\lambda \in \mathfrak{h}^*$. Furthermore define $\delta \in \hat{\mathfrak{h}}^{e*}$ by setting $\delta(\mathfrak{h}) = \delta(c) = 0$ and $\delta(d) = 1$. Now note that if $x_\alpha \in \mathcal{G}_\alpha$ we have:

$$[1 \otimes h + d, x_\alpha(n)]_e = (\alpha(h) + n) x_\alpha(n)$$

Using this you can quickly see that the roots of $\hat{\mathcal{G}}^e$ are:

$$\hat{\Delta} = \{k\delta + \alpha | k \in \mathbb{Z}, \alpha \in \Delta\} \cup \{k\delta | k \in \mathbb{Z} - \{0\}\}$$

We can then define the positive roots as:

$$\hat{\Delta}_+ = \{\alpha + n\delta | \alpha \in \Delta, n > 0\} \cup \{n\delta | n > 0\} \cup \Delta_+$$

Then the set of simple roots is $\{\alpha_0, \alpha_1, \ldots, \alpha_l\}$, where $\alpha_0 = \delta - \theta$. The corresponding coroots are $\hat{h}_{\alpha_0} = c - h_\theta$ and $\hat{h}_{\alpha_i} = h_{\alpha_i}$.

Finally you can show that $\hat{\mathcal{G}}^e$ admits a **triangular decomposition**: $\hat{\mathcal{G}}^e = \hat{n}_+ \oplus \hat{\mathfrak{h}}^e \oplus \hat{n}_-$ where:

$$\begin{aligned} \hat{n}_+ : &= \left(\sum_{\alpha \in \Delta_+}^{\oplus} \mathcal{G}_\alpha \otimes 1 \right) \oplus \mathcal{G} \otimes \mathbb{C}[t] \\ \hat{n}_- : &= \left(\sum_{-\alpha \in \Delta_+}^{\oplus} \mathcal{G}_\alpha \otimes 1 \right) \oplus \mathcal{G} \otimes \mathbb{C}[t^{-1}] \end{aligned}$$

2 Conformal Nets for the Loop Group

The following sections are all completely based on the article [3] by Gabbiano and Fröhlich.
Now we will use loop groups to construct conformal nets. The idea is the following. Let LG_0 be the group of smooth maps $f : S^1 \to G_0$ under pointwise multiplication. Then we can define for each open interval I in S^1 the algebra:

$$\bar{\mathcal{A}}(I) := \{f \in LG_0 | \ \mathrm{supp} f \subset I\}$$

This gives us the assignation of an algebra to a piece of space-time. We also need to find the Hilbert space on which we can represent this. We will find that for every *dominant integral weight* there is such a Hilbert space. This will give us a complete classification of all Hilbert spaces on which we can represent our algebras.

2.1 The loop group

As said above LG_0 is the group of smooth loops in G_0. Let $L\mathcal{G}_0$ be the corresponding Lie algebra of smooth maps $f : S^1 \to \mathcal{G}_0$. By $L\mathcal{G}_0^{pol}$ we denote the subalgebra of loops having finite fourier series. We can define a skew-symmetric bilinear form on $L\mathcal{G}_0$ by:

$$\omega(f; g) = \frac{1}{2\pi} \int_0^{2\pi} (f(\theta)|g'(\theta))d\theta$$

This is a cocycle and thus we get a central extension $\widehat{L\mathcal{G}_0}$. In fact you can show that the corresponding extension $\widehat{L\mathcal{G}_0}^{pol}$ coincides with $\hat{\mathcal{G}}_k$. Hence $\hat{\mathcal{G}}$ is the complexified Lie algebra of $\widehat{L\mathcal{G}_0}^{pol}$.

There is a corresponding extension for the loop group LG_0. This extension was described in a useful way in [2] by Mickelsson. Let D be the closed unit disc in $\mathbb{C}$ and DG_0 the group of smooth maps from D to G_0 with radial derivative vanishing at the boundary. Furthermore let N be the normal subgroup consisting of maps $f : D \to G_0$ such that $f_{S^1} \equiv e$. Next define a real-valued cocycle by:

$$\gamma(f; g) = \frac{1}{8\pi^2} \int_D (f^{-1}df|dgg^{-1})$$

Then you can look at the group $DG_0 \times_\gamma U(1)$ with multiplication given by $(f;\lambda)\cdot(g;\mu) = (f \cdot g; \lambda\mu e^{2\pi i\gamma(f;g)})$. Mickelsson now showed that there is a homomorphism $\phi : N \to DG_0 \times U(1)$ such that its image is a normal subgroup. He then defined $\widehat{LG_0} := (DG_0 \times_\gamma U(1))/\phi(N)$ and showed that this coincides with the previous definition. Thus obtaining an exact sequence:

$$1 \to U(1) \to \widehat{LG_0} \xrightarrow{p} LG_0 \to 1$$

Now we can define our algebras and prove a lemma. For an open interval $I \subset S^1$ let $\bar{\mathcal{A}}(I) := \{f \in LG_0 | \ \mathrm{supp} f \subset I\}$ and define $\hat{\mathcal{A}}(I) := p^{-1}(\bar{\mathcal{A}}(I)) \subset \widehat{LG_0}$.

Lemma:

- If $I_1 \subseteq I_2$ then also $\hat{\mathcal{A}}(I_1) \subseteq \hat{\mathcal{A}}(I_2)$.
- Let $\hat{f} \in \hat{\mathcal{A}}(I_1)$ and $\hat{g} \in \hat{\mathcal{A}}(I_2)$. If $I_1 \cap I_2 = \emptyset$ then $\hat{f} \cdot \hat{g} = \hat{g} \cdot \hat{f}$ in $\widehat{LG_0}$.

Proof. First of note that the first claim of the lemma immediately follows from the definition of $\hat{\mathcal{A}}(I)$.

Let $f := p(\hat{f})$ and $g := p(\hat{g})$. The construction of Mickelsson shows that there is a representative of $\hat{f}$ in $DG_0 \times U(1)$ of the form $(\tilde{f}; \lambda)$ where $\tilde{f}$ is constructed as follows. Denote by C_1 the open cake slice of D having I_1 and 0 on its boundary. Let C_1' be the complement of C_1 in D.

Now extend f to a map $\tilde{f}$ on D such that $\tilde{f}|_{C_1'} \equiv e$. This extension exist because G_0 is simply connected. Of course we can do the same for g. Now since $C_1 \cap C_2 = \emptyset$ we see that $\tilde{f} \cdot \tilde{g} = \tilde{g} \cdot \tilde{f}$. Hence we also get $\gamma(\tilde{f}; \tilde{g}) = 0$. This in turn means that $(\tilde{f}; \lambda)(\tilde{g}; \mu) = (\tilde{g}; \mu)(\tilde{f}; \lambda)$. So we can conclude that in fact $\hat{f} \cdot \hat{g} = \hat{g} \cdot \hat{f}$ in $\widehat{LG_0}$ as required.

□

2.2 Highest weight modules

Now we will classify all irreducible representations of $\hat{\mathcal{G}}_k$. For this we need the *dominant integral weights* of $\hat{\mathcal{G}}^e$. First of all we define the **fundamental weights** of $\hat{\mathfrak{h}}^{e*}$. They are the elements $\hat{\Lambda}_0, \ldots, \hat{\Lambda}_l \in \hat{\mathfrak{h}}^{e*}$ such that $\hat{\Lambda}(\hat{h}_{\alpha_j}) = \delta_{ij}$ for $i, j = 0, \ldots, l$ and $\hat{\Lambda}_i(d) = 0$.

We use the fundamental weights to define the **weight lattice** P of $\hat{\mathcal{G}}^e$ as $P := \sum_{i=0}^{l} \mathbb{Z} \cdot \hat{\Lambda}_i$. The set $P_+ := \sum_{i=0}^{l} \mathbb{Z}_+ \cdot \hat{\Lambda}_i$ is the set of **dominant integral weights**. Given an element $\hat{\Lambda} \in P_+$, the positive number $\hat{\Lambda}(c)$ is the **level** of $\hat{\Lambda}$. Note that for each level $m \in \mathbb{Z}_+$ the set of dominant integral weights of level m is a finite set.

The proofs of the following statements can be found in the article [4] by Goodman and Wallach.
Given an element $\hat{\Lambda} \in P_+$ we can find a unique (up to isomorphism) irreducible **highest weight module** $(L(\hat{\Lambda}), \pi_{\hat{\Lambda}})$. A $\hat{\mathcal{G}}^e$-module $(V; \pi)$ is called a highest weight module with highest weight $\hat{\Lambda}$, if there exists a vector $v_{\hat{\Lambda}} \in V$ such that:

$$\begin{aligned} \pi(\hat{n}_+) \cdot v_{\hat{\Lambda}} &= 0 \\ \pi(h) \cdot v_{\hat{\Lambda}} &= \hat{\Lambda}(h) \cdot v_{\hat{\Lambda}} \quad \text{for } h \in \hat{\mathfrak{h}}^e \\ \pi(U(\hat{\mathcal{G}}^e)) \cdot v_{\hat{\Lambda}} &= V \end{aligned}$$

Here $U(\hat{\mathcal{G}}^e)$ is the enveloping algebra of $\hat{\mathcal{G}}^e$.

On the unique irreducible highest weight module there exists a positive definite hermitian form $\langle \cdot | \cdot \rangle$ such that:

$$\langle \pi_{\hat{\Lambda}}(x) u | v \rangle = \langle u | \pi_{\hat{\Lambda}}(x^*) v \rangle \quad \forall u, v \in L(\hat{\Lambda}),\ x \in \hat{\mathcal{G}}^e$$

Now we can finally define the Hilbert space on which we will represent our algebra. It is $\mathcal{H}_{\hat{\Lambda}} := \overline{L(\hat{\Lambda})}$, the Hilbert space completion of $L(\hat{\Lambda})$ with respect to the hermitian $\langle \cdot | \cdot \rangle$. Since we can identify $\hat{\mathcal{G}}_k$ and $\widehat{L\mathcal{G}_0}^{pol}$ we in fact get a representation of $\widehat{LG_0}$ on $\mathcal{H}_{\hat{\Lambda}}$. Goodman and Wallach show that this is strongly continuous, irreducible unitary representation.

2.3 An example

As an example of some of the things discussed we can look at $\mathcal{G} = \mathfrak{sl}(n, \mathbb{C})$, the complex $n \times n$-matrices with trace equal to 0. The Lie bracket is just the commutator bracket. Now we choose as a Cartan subalgebra the algebra $\mathfrak{h}$ of diagonal matrices with trace 0. Let $e_i \in \mathfrak{h}^*$ be the element which sends a matrix H to its i,i-th entry H_{ii}. Then the roots are given by $\alpha_{i,j} = e_i - e_j$.

If we choose as positive roots the $\alpha_{i,j}$ for which $i < j$, then the triangular decomposition of $\mathcal{G}$ becomes $\mathcal{G} = n_+ \oplus \mathfrak{h} \oplus n_-$. Here n_+ are the strictly upper triangular matrices and n_- are the strictly lower triangular matrices. Hence also the name triangular decomposition.

Exercise: *Check that the $\alpha_{i,j}$ indeed define the roots of $\mathfrak{sl}(n,\mathbb{C})$. What are the corresponding coroots? What is the highest root and its corresponding coroot? How do the fundamental weights look like? And for $\widehat{\mathfrak{sl}(n,\mathbb{C})}^e$? How do the fundamental weights look like there?*

3 A conformal net

The setting in which we work is the following: $\mathcal{H}$ is a separable Hilbert space with $\pi : \mathrm{PSU}(2) \to \mathcal{U}(\mathcal{H})$ a strongly continuous projective representation of the Moebius group. Here $\mathrm{PSU}(2) = \mathrm{SU}(2)/\{\pm\mathbf{1}\}$.

Now let $\{\mathcal{A}(I)\}_{I\subset S^1}$ be a collection of von Neumann algebras acting on $\mathcal{H}$. They are indexed by open, non-dense intervals $I \subset S^1$. Denote $I' := S^1\backslash\bar{I}$. We call $\{\mathcal{A}(I)\}_{I\subset S^1}$ a **conformal net** if it satisfies the properties:

- **Isotony:** $I_1 \subseteq I_2 \to \mathcal{A}(I_1) \subseteq \mathcal{A}(I_2)$
- **Locality:** $I_1 \subseteq I_2' \to \mathcal{A}(I_1) \subseteq \mathcal{A}(I_2)'$
- **Moebius Covariance:** $\pi(A)\mathcal{A}(I)\pi(A)^* = \mathcal{A}(A\cdot I), \quad \forall A \in \mathrm{PSU}(2)$

Now we define $\mathcal{A}_{\hat{\Lambda}}(I) := \{\pi_{\hat{\Lambda}}(\hat{\mathcal{A}(I)})\}''$. This gives us a conformal net:

Theorem For every $\hat{\Lambda} \in P_+$, the collection $\{\mathcal{A}_{\hat{\Lambda}}(I)\}_{I\subset S^1}$ is a conformal net on $\mathcal{H}_{\hat{\Lambda}}$.
Proof. $\mathcal{A}_{\hat{\Lambda}}$ is the weak closure of $\pi_{\hat{\Lambda}}(\hat{\mathcal{A}}(I))$. Hence isotony and locality directly follow from the previous lemma. The proof of the Moebius covariance is a part of the article by Goodman and Wallach.

□

4 Exercises

The first exercise can be found on in section 1.2 on the top of page 3. The second exercise can be found in section 2.3 on the top of page 6.

5 Bibliography

[1] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, 3rd revised printing, Springer Verlag, New York, 1973

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