1 Introduction

The main goal of this talk is to proof the Reeh-Schlieder theorem for local Möbius covariant nets of von Neumann algebras. The theorem states that the vacuum is a cyclic vector for any von Neumann algebra on an open set.

2 Preliminaries

2.1 The Möbius group

Recall that the group $SL(2,\mathbb{R})$ of real 2×2-matrices with determinant 1 acts on the compactified line $\mathbb{R} \cup \infty$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .x = \frac{ax+b}{cx+d}.$$

The kernel of this action is $\{\pm \mathbb{1}_2\}$. We may identify $\mathbb{R} \cup \infty$ and S^1 and identify $SL(2,\mathbb{R})$ with $SU(1,1)$ acting on S^1 by

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} .z = \frac{\alpha z+\beta}{\bar{\beta} z+\bar{\alpha}}.$$

Now $G = SU(1,1)/\pm\mathbb{1}_2$ is identified with a group of diffeomorphisms of S^1, which is called the Möbius group. We consider the following three one-parameter subgroups

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad \delta(s) = \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{s/2} \end{pmatrix}, \quad \tau(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Or,

$$\begin{aligned} R(\theta)z &= e^{i\theta} z && \text{on } S^1 \\ \delta(s)x &= e^s x && \text{on } \mathbb{R} \\ \tau(t)x &= x + t && \text{on } \mathbb{R}. \end{aligned}$$

Definition 1. *An interval I of S^1 is an open, connected, non-empty, non-dense subset of $S1$.*

The set of all intervals is denoted by $\mathcal{I}$. If I is an interval, we write $I'$for the interior of the complement of I in S^1. Given an interval I we want to define the one-parameter subgroups δ_I and τ_I associated with I. Let I_1 be the interval that corresponds to the upper half circle. Then define $\tau_{I_1} := \tau$. Now let $g \in G$ be such that $I = gI_1$, then set $\tau_I := g\tau_{I_1}g^{-1}$.

Exercise 1. *Show that $\tau_I(t)$ and $\tau_{I'}(s)$, $s,t \in \mathbb{R}$ generate G. Show also that if $t \leqslant 0$, $\tau_{I'}(t)$ maps I into itself.*

3 Möbius covariant nets of standard subspaces

Let $\mathcal{H}$ be a complex Hilbert space. A local Möbius covariant net H of real linear subspaces of $\mathcal{H}$ on the intervals of $S^1$is a map

$$I \to H(I),$$

that maps each interval $I \in \mathcal{I}$ to a closed real linear subspace of $\mathcal{H}$ such that it satisfies the following properties

1. ISOTONY: Let I_1 and I_2 be intervals such that $I_1 \subset I_2$, then

$$H(I_1) \subset H(I_2).$$

2. MÖBIUS COVARIANCE: There exists a unitary representation U of G, the Möbius group, on $\mathcal{H}$ such that

$$U(g)H(I) = H(gI),$$

where $g \in G$ and $I \in \mathcal{I}$.

3. POSITIVE ENERGY: The representation U is a positive energy representation.

4. CYCLICITY: The complex linear span of al $H(I)$'s is dense in $\mathcal{H}$.

5. LOCALITY: If I_1 and I_2 are disjoint intervals then

$$H(I_1) \subset H(I_2)'.$$

Let $(,)$ be the hermitian form of $\mathcal{H}$, let H be a real linear subspace of $\mathcal{H}$, then we define the symplectic complement H' of H by

$$H' := \{x \in \mathcal{H}, \text{ s.t. } \mathrm{Im}(x, \eta) = 0, \quad \forall \eta \in H\}.$$

Exercise 2. *Show that H is cyclic if and only if H' is separating.*

4 Möbius covariant nets of von Neumann algebras

Definition 2. *A net $\mathcal{A}$ of von Neumann algebras on S^1 is a map*

$$I \to \mathcal{A}(I),$$

from $\mathcal{I}$, to the set of von Neumann algebras on a Hilbert space $\mathcal{H}$ that verifies the isotony property:

ISOTONY: Let I_1 and I_2 be intervals such that $I_1 \subset I_2$, then

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2).$$

1. MÖBIUS COVARIANCE: There exists a strongly continuous unitary representation U of G, the Möbius group, on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI),$$

where $g \in G$ and $I \in \mathcal{I}$.

2. POSITIVE ENERGY: The representation U is a positive energy representation.

3. EXISTENCE & UNIQUENESS OF THE VACUUM: Up to a phase there exists a unique unit U-invariant vector Ω (vacuum vector) and Ω is cyclic for the von Neumann algebra $\vee_{I\in\mathcal{I}}\mathcal{A}(I)$.

4. LOCALITY: If I_1 and I_2 are disjoint intervals then

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2)'.$$

5 Relation between the two types of nets

Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $\Omega \in \mathcal{H}$ a vector. Let M_{sa} denote the selfadjoint part of M and define

$$H_M := \overline{M_{sa}\Omega}.$$

Clearly H_M is a real Hilbert subspace of $\mathcal{H}$.
Recall that a closed real subspace $H \subset \mathcal{H}$ is

1. cyclic if $H + iH$ is dense in $\mathcal{H}$,

2. separating if $H \cap iH = \varnothing$.

The vector Ω is

1. cyclic if $\overline{M\Omega} = \mathcal{H}$,

2. separating if $m\Omega = 0 \Rightarrow m = 0$.

It follows that

$$\Omega \text{ is cyclic} \Leftrightarrow H_M \text{ is cyclic},$$
$$\Omega \text{ is separating} \Leftrightarrow H_M \text{ is separating}.$$

A standard subspace of $\mathcal{H}$ is a closed real linear subspace of $\mathcal{H}$ that is both cyclic and separating.

6 Reeh-Schlieder for local Möbius covariant nets of standard subspaces

Theorem 1 (Reeh-Schlieder). *Let H be a local Möbius covariant net of real linear subspaces of $\mathcal{H}$ on S^1, then $H(I)$ is a standard subspace of $\mathcal{H}$ for all $I \in \mathcal{I}$.*

Proof. Let $I \in \mathcal{I}$ be an interval. We need to show that $H(I)$ is cyclic and separating. Recall that the real linear subspace $H(I)$ is cyclic if $H(I) + iH(I)$ is dense in $\mathcal{H}$. This is equivalent to requiring that the complex span of $H(I)$ is dense in $\mathcal{H}$. This is equivalent to requiring that the only vector in $\mathcal{H}$ that is orthogonal to $H(I)$ is zero. Let $\eta \in \mathcal{H}$ sucht that η orthogonal to $H(I)$. Let I_0 be an interval such that $\bar{I_0} \subset I$. Now, for all $t \in O$, O a small neighbourhood of zero such that

$$\tau_I(t)I_0 \subset I \text{ and } \xi \in H(I_0),$$

we define

$$f(t) := (\eta, U(\tau_I(t))\xi).$$

Möbius covariance

$$U(\tau_i(t))\xi \in H(\tau_i(t)I_0) = H(I)$$

implies

$$f(t) = 0 \text{ for } t \in O.$$

$$U(\tau_I(t)) = \exp(itT),$$

where T is the generator of translations. We may define:

$$U(x + iy) := \exp(i(x + iy)T),$$

for y positive real this is bounded, since T is a positive operator. Now we define

$$f(z) := (\eta, U(\tau_I(z))\xi),$$

for $z \in O + i\mathbb{R}_+$, continuous on $\mathrm{Im}(z) \geqslant 0$, holomorphic on $\mathrm{im}(z) > 0$ and

$$g(z) := \overline{f(\bar{z})},$$

for $z \in O - i\mathbb{R}_-$ continuous on $\mathrm{Im}(z) \leqslant 0$, holomorphic on $\mathrm{im}(z) < 0$. The functions f and g agree on O, namely they are zero. Now the reflection principle by Schwartz implies that f and g are branches of a unique holomorphic function that is zero on O, holomorphic on a complex neighbourhood of O, hence zero everywhere. It follows that $f(t) = 0$ for all $t \in \mathbb{R}$. In the exercise we showed that for $t < 0$ $\tau_{I'}(t)$ maps I into itself, so we may repeat the argument for $\tau_{I'}$ instead of τ_I, we see that η is orthogonal to $H(\tau_{I'}I_0)$ for all $t \in \mathbb{R}$. Since η is orthogonal to all $H(\tau_I(s)I)$, take $\tau_I(s)I_0 \subset \tau_I(s)I$ and repeat the argument. We find that

$$\eta \text{ is orthogonal to } H(\tau_{I'}(t)\tau_I(s)I_0) \text{ for all } t, s \text{ in } \mathbb{R}.$$

But $\tau_I(t)$ and $\tau_{I'}(s)$ generate G, so in fact η is orthogonal to $H(gI_0)$ for all $g \in G$. However, G acts transitively on $\mathcal{I}$, and the complex linear span of all the $H(I)$'s is $\mathcal{H}$, hence $\eta = 0$. Hence $H(I) + iH(I)$ is dense in $\mathcal{H}$, so $H(I)$ is cyclic. Since I' is an interval, by the same reasoning $H(I')$ is cyclic. But $H(I') \subset H(I)'$, by locality, so $H(I)'$ is also cyclic, hence $\overline{H(I)} = H(I)$ is separating. □

7 Reeh-Schlieder for local Möbius covariant nets of von Neumann algebras

Theorem 2 (Reeh-Schlieder)**.** *Let $\mathcal{A}$ be a local Möbius covariant net on S^1. For any $I \in \mathcal{I}$, the vector Ω is cyclic and separating for the von Neumann algebra $\mathcal{A}(I)$.*

Proof. Let $\mathcal{H}_0$ be the complex Hilbert subspace generated by all the $H(I) := \overline{\mathcal{A}(I)_{\mathrm{sa}}\Omega}$. The map

$$I \to H(I)$$

is a local Möbius covariant net of real subspaces of $\mathcal{H}_0$. By the Reeh-Schlieder theorem for nets of real linear subspaces, we have

$$\overline{H(I) + iH(I)} = \mathcal{H}_0$$

for every fixed interval I. Now let

$$E : \mathcal{H} \to \mathcal{H}_0 = \overline{H(I) + iH(I)}$$

denote the orthogonal projection. We have that $E \in \mathcal{A}(I)'$ and clearly, E is independent of I, so $E \in \cap_I \mathcal{A}(I)' = (\vee_i \mathcal{A}(I))'$. Let $v \in \mathcal{H}$ be such that $Ev = \Omega$, then

$$E\Omega = E^2 v = Ev = \Omega.$$

Let $w \in \mathcal{H}$, by cyclicity of Ω for $\vee_{I \in \mathcal{I}} \mathcal{A}(I)$, there exist $\{\omega_n\}_n$, $\omega_n \in \vee_{I \in \mathcal{I}} \mathcal{A}(I)$ such that

$$\mathrm{lim}_{n\to\infty} \omega_n \Omega = w.$$

Now

$$\begin{aligned} Ew &= \mathrm{lim}_{n\to\infty} E\omega_n\Omega \\ &= \mathrm{lim}_{n\to\infty} \omega_n E\Omega \\ &= \mathrm{lim}_{n\to\infty} \omega_n \Omega \\ &= w, \end{aligned}$$

hence $E = 1$. So $\mathcal{H} = \mathcal{H}_0$. Now for every I we have $H(I)$ is standard and hence by the equivalence Ω is both cyclic and separating for $\mathcal{A}(I)$. □

8 Consequences

If O is some open region of spacetime such that its complement is nonempty, then $\mathcal{A}(O)$ does not contain any operators that annihilate the vacuum. Let T be such an operator. Let $v \in \mathcal{H}$, then write

$$v = \lim_{n \to \infty} v_n \Omega,$$

for $v_n \in \mathcal{A}(O') \subset \mathcal{A}(O)'$. Hence

$$Tv = \lim_{n \to \infty} T v_n \Omega = \lim_{n \to \infty} v_n T \Omega = 0.$$

This implies that the stress-energy tensor, if localized to a bounded region, cannot be a positive operator, since its expectation value in the vacuum is zero, in fact it cannot be bounded from below. It also implies that there cannot be a local number operator.