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The Drinfeld Quantum Double Construction

Introduction

Last week, Stephan introduced the *Yang-Baxter equation* (YBE) and showed us one particular method for finding solutions to this equation, due to Vladimir Drinfeld. Namely, if we have a bialgebra $(H, \mu, \eta, \Delta, \epsilon)$ and if we can determine a universal R-matrix R for H, then, given two modules V,W over this *braided algebra* $(H, \mu, \eta, \Delta, \epsilon, R)$, we can construct a natural isomorphism $c^R_{V,W}$ of H-modules between $V \otimes W$ and $W \otimes V$, which in turn is a solution to the YBE on when $V = W$. The question then becomes: how to find braided algebras? Mercifully, Drinfeld was not a tease, and also devised a method for constructing a braided Hopf algebra out of any finite-dimensional Hopf algebra with an invertible antipode. These objects are called (Drinfeld) *quantum doubles*, the construction of which is the subject of today's lecture. To proceed, we'll need an operation called the *bicrossed product*, which I will also demonstrate, first on groups (to help us get a feel for the structure) and then on bialgebras (which we'll need to construct our quantum double).

Matched Groups → Bicrossed Product of Groups

The bicrossed product is a certain group structure we can place on the set-theoretic product of two groups H, K, given that these two groups satisfy certain conditions that make them so-called *matched.* What, exactly, these conditions are might seem a bit arbitrary, so instead of just listing them, I'm going to start with a hypothetical situation and derive them working backwards.

Let's say we have a group G with subgroups H and K such that, for any element $x \in G$, there exists a *unique* pair $(y, z) \in H \times K$ such that

$$x = yz.$$

(I use a suggestive notation, but at this point I place no further restrictions on either H or K; for instance, K does not have to be normal.) This condition allows us to pair with any $(y, z) \in H \times K$ a unique element in H, denoted $z \cdot y$, and a unique element in K, denoted z^y, such that

$$G \ni zy = (z \cdot y)z^y.$$

This new notation might seem at first cumbersome, but it turns out that it can be made less so with a number of distributive relations, which we will now derive. Let $y, y' \in H$ and $z, z' \in K$. In G, we have the associative relations

$$(zz')y = z(z'y) \qquad z(yy') = (zy)y'$$

Expanding these out gives us our desired distributive relations. (I'll do the first one, you can do the second.)

$$\begin{aligned}
\underbrace{(zz')}_{K}\underbrace{y}_{H} &= \underbrace{(zz' \cdot y)}_{H}\underbrace{(zz')^y}_{K} \\
\underbrace{z}_{K}(\underbrace{z'}_{K}\underbrace{y}_{H}) &= \underbrace{z}_{K}\underbrace{(z' \cdot y)}_{H}\underbrace{z'^y}_{K} \\
&= \underbrace{(z \cdot (z' \cdot y))}_{H}\underbrace{\underbrace{z^{z' \cdot y}}_{K}\underbrace{z'^y}_{K}}_{K}
\end{aligned}$$

which, because of the unique decomposition of elements in G, gives us the relations

$$\begin{aligned}
(zz' \cdot y) &= z \cdot (z' \cdot y) \\
(zz')^y &= z^{z' \cdot y} z'^y
\end{aligned}$$

Likewise, the expansion of the second associative relation gives us

$$\begin{aligned}
z \cdot (yy') &= (z \cdot y)(z^y \cdot y') \\
z^{yy'} &= (z^y)^{y'}
\end{aligned}$$

while expanding $z = z1$ and $y = 1y$ gives us

$$
\begin{aligned}
z \cdot 1 &= 1 \\
z^1 &= z \\
1 \cdot y &= y \\
1^y &= 1
\end{aligned}
$$

Notice that the relations $(zz' \cdot y) = z \cdot (z' \cdot y)$ and $1 \cdot y = y$ tell us that the map $\alpha : K \times H \to H$, defined by

$$\alpha(z, y) = z \cdot y$$

is a *left* action of the *group* K on the *set* H. Similarly, the relations $z^{yy'} = (z^y)^{y'}$ and $z^1 = z$ tell us that the map $\beta : K \times H \to K$, defined by

$$\beta(z, y) = z^y$$

is a *right* action of the *group* H on the *set* K. All of these conditions together are precisely what we'll use to define the notion of matched groups.

DEF: A pair (H, K) of groups is said to be **matched** if there exist a left action α of the group K on the set H and a right action β of the group H on the set K such that, for all $y, y' \in H$ and all $z, z' \in K$, we have

$$
\begin{aligned}
(zz')^y &= z^{z' \cdot y} z'^y \\
z \cdot (yy') &= (z \cdot y)(z^y \cdot y') \\
z \cdot 1 &= 1 \\
1^y &= 1
\end{aligned}
$$

where $\alpha(z, y) = z \cdot y$ and $\beta(z, y) = z^y$.

Now we can define the bicrossed product of groups.

DEF: Let (H, K) be a matched pair of groups. The **bicrossed product** $H \bowtie K$ of H and K is the group structure on the set-theoretic product $H \times K$ with unit $(1, 1)$ such that

$$(y, z)(y', z') = (y(z \cdot y'), z^{y'} z')$$

for all $y, y' \in H$ and $z, z' \in K$.

PROP:

(i) The bicrossed product group structure exists. Furthermore, the groups H and K can be identified with, respectively, the subgroups $H \times \{1\}$ and $\{1\} \times K$ of $H \bowtie K$, and every element $(y, z) \in H \bowtie K$ can be written uniquely as the product of an element of $H \times \{1\}$ and an element of $\{1\} \times K$:

$$(y, z) = (y, 1)(1, z)$$

where $y \in H$ and $z \in K$. Hence this structure is unique.

(ii) Conversely, let G be a group and H, K be subgroups of G such that the multiplication on G induces a set-theoretic bijection from $H \times K$ onto G. Then the pair (H, K) is necessarily matched and the previous bijection induces a group isomorphism from the bicrossed product $H \bowtie K$ onto G.

Proof of (i): It is straightforward enough to prove associativity for the above-defined product and to prove that $(1, 1)$ is both a left- and right-unit. What remains is demonstrate the existence of an inverse for every element $(y, z) \in H \bowtie K$. That is, given (y, z), we want to find a $y' \in H$, $z' \in K$ such that

$$(y, z)(y', z') = (1, 1)$$

By definition of the bicrossed product, we have

$$y(z \cdot y') = 1 \qquad z^{y'} z' = 1$$

Using the identities, the first equality gives us

$$y' = 1 \cdot y' = (z^{-1} z) \cdot y' = z^{-1} \cdot (z \cdot y') = z^{-1} \cdot y^{-1}$$

and using this result, the second equality gives us

$$z' = (z^{z^{-1} \cdot y^{-1}})^{-1}$$

Set $(y',z')(y,z) = (Y,Z)$. We need to show that $(Y,Z) = (1,1)$. Multiplying the above identity by (y,z) on the left, we find

$$(y,z) = (y,z)(Y,Z) = (y(z \cdot Y), z^Y Z)$$

implying

$$Y = z^{-1} \cdot (z \cdot Y) = z^{-1} \cdot 1 = 1 \qquad Z = z^Y z^{-1} = z^1 z^{-1} = zz^{-1} = 1$$

Hence, the element (y,z) is invertible, with left- and right-inverse

$$(y,z)^{-1} = \left(z^{-1} \cdot y^{-1}, (z^{z^{-1} \cdot y^{-1}})^{-1} \right)$$

Lastly, we check that

$$(y,1)(y',1) = (y(1 \cdot y'), 1^{y'} 1) = (yy',1) \quad (1,z)(1,z') = (z \cdot 1, z^1 z') = (1,zz') \quad (y,1)(1,z) = (y(1 \cdot 1), 1^1 z) = (y,z)$$

proving the subgroup and uniqueness claims.

Proof of (ii): Follows from the discussion leading to the definition of a matched pair of groups.

Examples:

(i) Product of groups: Let H and K be groups. If we let each act trivially on the other (i.e. $z \cdot y = y$, $z^y = z$), then (H,K) is a matched pair, and $H \bowtie K$ is isomorphic to $H \times K$.

(ii) Semidirect product of groups: Let H and K be groups. We suppose that H acts trivially on K ($z^y = z$) and that K acts on H by group automorphisms (i.e. $z \cdot (yy') = (z \cdot y)(z \cdot y')$ and $z \cdot 1 = 1$ for all $y, y' \in H$ and $z \in K$). Then again (H,K) is a matched pair and $H \bowtie K$ is isomorphic to $H \rtimes K$. One can show that $(1,z)(y,1)(1,z)^{-1} = ((z \cdot y),1)$, proving that $H \times (1)$ is a normal subgroup of $H \bowtie K$ and that the action of K on H corresponds to the conjugation in the bicrossed product.

Bicrossed Product of Groups → Bicrossed Product of Bialgebras

Reinier showed us back in the third lecture that, if G is a group, then $k[G]$ has a natural Hopf algebra structure. One question we could ask ourselves today is, given a matched pair (H,K) of groups, can we build the algebra of the bicrossed product $H \bowtie K$ out of the group algebras $k[H]$ and $k[K]$? To answer the question, we will first need the concept of *matched bialgebras*, and for that we will need the concept of a *module-coalgebra over a bialgebra*, which is a generalization of the action of a group on a set in the following sense. (Recall: if H is a bialgebra, an algebra A is a *module-algebra over H* if (i) the vector space underlying A is an H-module, and (ii) multiplication μ and the unit η of A are morphisms of H-modules.)

Let G be a group, X a set, and α an action of G on X:

$$\alpha : G \times X \to X$$

Linearizing, this gives us a morphism of coalgebras:

$$\alpha : k[G \times X] \to k[X]$$

(Remember, if X is a set, then $k[X] = \bigoplus_{x \in X} kx$ is a vector space, which in turn becomes a coalgebra if we define $\Delta(x) = x \otimes x$ and $\epsilon(x) = 1$.) Composing this with the natural isomorphism

$$k[G] \otimes k[X] \cong k[G \times X], \qquad g \otimes x \mapsto (g,x)$$

which, by Ch.III, is a coalgebra isomorphism, we see that the group action of G on X gives rise to an action of the Hopf algebra $k[G]$ on the coalgebra $k[X]$ such that the map

$$k[G] \otimes k[X] \to k[X]$$

is a morphism of coalgebras.

DEF: Let H be a bialgebra and C be a coalgebra. Then C is a **module-coalgebra over H** if there exists a morphism of coalgebras $H \otimes C \to C$ inducing an H-module structure on C.

Thus, in the above, we have that the coalgebra $k[X]$ is a module-coalgebra over the Hopf algebra $k[G]$. This leads us to the following definition.

DEF: A pair (X, A) of bialgebras is **matched** if there exist linear maps $\alpha : A \otimes X \to X$ and $\beta : A \otimes X \to A$ turning X into a module-coalgebra over A and turning A into a right module-coalgebra over X such that, if we set

$$\alpha(a \otimes x) = a \cdot x \qquad \beta(a \otimes x) = a^x$$

the following conditions are satisfied for all $a, b \in A$ and all $x, y \in X$:

$$\begin{aligned}
(ab)^x &= \sum_{(b)(x)} a^{b' \cdot x'} b''^{x''} \\
a \cdot (xy) &= \sum_{(a)(x)} (a' \cdot x')(a''^{x''} \cdot y) \\
a \cdot 1 &= \epsilon(a)1 \\
1^x &= \epsilon(x)1 \\
\sum_{(a)(x)} a'^{x'} \otimes a'' \cdot x'' &= \sum_{(a)(x)} a''^{x''} \otimes a' \cdot x'
\end{aligned}$$

Note the similarity between the definitions for a matched pair of groups and for a matched pair of bialgebras. The only new condition is the last one, which arises from the comultiplication structure of a bialgebra that groups lack. (Note that it is automatically satisfied with both A and X are cocommutative.) It can be shown that, if (H, K) is a matched pair of groups, then $(k[H], k[K])$ is a matched pair of bialgebras. Also, it follows from the fact that α and β are coalgebra morphisms that

$$\Delta(a \cdot x) = \sum_{(a)(x)} a' \cdot x' \otimes a'' \cdot x'', \qquad \epsilon(a \cdot x) = \epsilon(a)\epsilon(x)1$$

in X, and

$$\Delta(a^x) = \sum_{(a)(x)} a'^{x'} \otimes a''^{x''}, \qquad \epsilon(a^x) = \epsilon(a)\epsilon(x)1$$

in A.

DEF: Let (X, A) be a matched pair of bialgebras. The **bicrossed product of X and A**, denoted $X \bowtie A$, consists of a bialgebra structure on the vector space $X \otimes A$ with unit equal to $1 \otimes 1$ such that its product is given by

$$(x \otimes a)(y \otimes b) = \sum_{(a)(y)} x(a' \cdot y') \otimes a''^{y''} b$$

its coproduct by

$$\Delta(x \otimes a) = \sum_{(a)(x)} (x' \otimes a') \otimes (x'' \otimes a'')$$

and its counit by

$$\epsilon(x \otimes a) = \epsilon(x)\epsilon(a)$$

for all $a \in A$ and $x \in X$.

THM: The bicrossed algebra structure exists and is unique. Furthermore, the injective maps $i_X(x) = x \otimes 1$ resp. $i_A(a) = 1 \otimes a$ from X resp. A into $X \bowtie A$ are bialgebra morphisms. We also have

$$x \otimes a = (x \otimes 1)(1 \otimes a)$$

for all $a \in A$ and $x \in X$. If the bialgebras X resp. A have antipodes S_X resp. S_A, then the bicrossed product is a Hopf algebra with antipode S given by

$$S(x \otimes a) = \sum_{(x)(a)} S_A(a'') \cdot S_X(x'') \otimes S_A(a')^{S_X(x')}$$

Proof: It is clear from the above formulas that we have equipped the bicrossed product structure with the coalgebra structure of the tensor product of coalgebras X and A. (Recall from Reinier's lecture: given two coalgebras $(X, \Delta_X, \epsilon_X)$, $(A, \Delta_A, \epsilon_A)$ the tensor product $(X \otimes A)$ endowed with the maps $\Delta = (id_X \otimes \tau_{X,A} \otimes id_A) \circ (\Delta_X \otimes \Delta_A)$ and $\epsilon = \epsilon_X \otimes \epsilon_A$ is a coalgebra.) Hence i_X and i_A are coalgebra morphisms and what remains to be shown is that $X \bowtie A$ has an algebra structure and that i_X, i_A, Δ and ϵ are algebra morphisms. This follows from straightforward Sweedler-style calculations and can be found in Kassel. As for the antipode claim, this is not the case. Recall that from the definition of the antipode, we need to show that

$$\sum_{(x)(a)} (x' \otimes a')S(x'' \otimes a'') = \epsilon(x \otimes a)1 \otimes 1 = \sum_{(x)(a)} S(x' \otimes a')(x'' \otimes a'')$$

I'll do the LHS; what follows is an elaboration of that which is in the book.

$$\begin{aligned}
\sum_{(x)(a)} (x' \otimes a')S(x'' \otimes a'') &= \sum_{(x)(a)} (x' \otimes a')(S_A(a''') \cdot S_X(x''') \otimes S_A(a'')^{S_X(x'')}) \\
&= \sum_{\substack{(x)(a)(a') \\ (S_A(a''') \cdot S_X(x'''))}} x' \left((a')' \cdot (S_A(a''') \cdot S_X(x'''))'\right) \otimes (a')''^{\left(S_A\left(a'''\right) \cdot S_X\left(x'''\right)\right)''} S_A(a'')^{S_X\left(x''\right)} \\
&= \sum_{\substack{(x)(a) \\ (S_A(a''''))(S_X(x'''))}} x' \left(a' \cdot (S_A(a''''))' \cdot (S_X(x'''))'\right) \otimes a''^{\left(S_A\left(a''''\right)\right)'' \cdot \left(S_X\left(x'''\right)\right)''} S_A(a''')^{S_X\left(x''\right)} \\
&= \sum_{(x)(a)} x' \left(a' \cdot S_A(a''''') \cdot S_X(x'''')\right) \otimes a''^{S_A\left(a''''\right) \cdot S_X\left(x'''\right)} S_A(a''')^{S_X\left(x''\right)} \\
&= \sum_{\substack{(x)(a) \\ (S_A(a'''))(S_X(x''))}} x' \left(a' \cdot S_A(a'''') \cdot S_X(x''')\right) \otimes a''^{\left(S_A\left(a'''\right)\right)' \cdot \left(S_X\left(x''\right)\right)'} \left(S_A(a''')\right)''^{\left(S_X\left(x''\right)\right)''} \\
&= \sum_{(x)(a)} x' \left((a' S_A(a'''')) \cdot S_X(x''')\right) \otimes \left(a'' S_A(a''')\right)^{S_X(x'')} \\
&= \sum_{(x)(a)} x' \left((a' S_A(a''')) \cdot S_X(x''')\right) \otimes \epsilon(a'')\,(1)^{S_X(x'')} \\
&= \sum_{(x)(a)} x' \left((a' S_A(a''')) \cdot S_X(x''')\right) \otimes \epsilon(a'')\epsilon(S_X(x''))1 \\
&= \sum_{(x)(a)} x' \left(((a'\epsilon(a''))\, S_A(a''')) \cdot (\epsilon(S_X(x''))\, S_X(x'''))\right) \otimes 1 \\
&= \sum_{(x)(a)} x' \left((a' S_A(a'')) \cdot S_X(x'')\right) \otimes 1 \\
&= \epsilon(a) \left(\sum_{(x)} x' \left(1 \cdot S_X(x'')\right) \otimes 1\right) \\
&= \epsilon(a) \left(\sum_{(x)} x' S_X(x'') \otimes 1\right) \\
&= \epsilon(a)\epsilon(x) 1 \otimes 1 \\
&= \epsilon(x \otimes a) 1 \otimes 1
\end{aligned}$$

Here we used extensively the fact that the product, α, and β are coalgebra morphisms (the product because Δ, ϵ are algebra morphisms) and that, for f a coalgebra morphism,

$$\sum_{(f(x))} (f(x))' \otimes (f(x))'' = \sum_{(x)} f(x') \otimes f(x'')$$

while for S,

$$\sum_{(S(x))} (S(x))' \otimes (S(x))'' = \sum_{(x)} S(x'') \otimes S(x')$$

Two notes:

(i) If (H, K) is a matched pair of groups, $(k[H], k[K])$ is a matched pair of bialgebras and $k[H \bowtie K] \cong k[H] \bowtie k[K]$, which I think follows from

(ii) If X, A are bialgebras and we allow each to act trivially on the other via $a \cdot x = \epsilon(a)x$, $a^x = \epsilon(x)a$ for all $x \in X, a \in A$, then $X \bowtie A$ is isomorphic to $X \otimes A$.

Representations, or Getting to H and $(\mathbf{H^{op}})^*$ as Module-Coalgebras over One Another
Let $(H, \mu, \eta, \Delta, \epsilon, S)$ be a Hopf algebra and $a, x \in H$. Set

$$a \cdot x = \sum_{(a)} a' x S(a''), \qquad x^a = \sum_{(a)} S(a') x a''$$

PROP + DEF: The map $(a, x) \mapsto a \cdot x$ endows H with the structure of a left module-algebra on the bialgebra H. The thus-defined H-module is denoted by $\mathbf{{}_{ad}H}$, and the action is called the **left adjoint representation of H**. Similarly, the map $(x, a) \mapsto x^a$ endows H with the structure of a right module-algebra on the bialgebra H. The H-module defined this way is denoted $\mathbf{H_{ad}}$, and the action is called the **right adjoint representation of H**.

PROOF: The proof for ${}_{ad}H$ is in the book; I'll do the proof for H_{ad}. We first check that $(x, a) \mapsto x^a$ puts an H-module structure on H. Indeed, we have $x^1 = x$ and

$$\begin{aligned}
(x^a)^b &= \sum_{(a)} (S(a')xa'')^b \\
&= \sum_{(a)(b)} S(b')S(a')xa''b'' \\
&= \sum_{(a)(b)} S(a'b')xa''b'' \\
&= \sum_{(ab)} S((ab)')x(ab)'' = x^{(ab)}
\end{aligned}$$

for all $a, b \in H$. And it is a right module-algebra over H:

$$1^a = \sum_{(a)} S(a')a'' = \epsilon(a)1$$

and

$$\begin{aligned}
\sum_{(a)} x^{a'} y^{a''} &= \sum_{(a)} S(a')xa''S(a''')ya'''' \\
&= \sum_{(a)} S(a')x\epsilon(a'')ya''' \\
&= \sum_{(a)} S(a')xya'' \\
&= (xy)^a
\end{aligned}$$

Examples:
(i) Group conjugation: If G is a group and $k[G]$ the corresponding Hopf algebra, then the left adjoint representation of $k[G]$ is given by the formula $a \cdot x = axa^{-1}$ for all $a, x \in G$.
(ii) Adjoint representation of a Lie algebra: Let $\mathfrak{g}$ be a Lie algebra and $U(\mathfrak{g})$ be its enveloping algebra equipped with its canonical Hopf algebra structure. The left adjoint representation of $U(\mathfrak{g})$ is given by $a \cdot x = ax - xa$ for all $a, x \in \mathfrak{g}$. The corresponding representation of $\mathfrak{g}$ is called the adjoint representation of the Lie algebra $\mathfrak{g}$.

The above is a representation of H on itself. We'll also want a representation of H on $(H^{op})^*$ later.

LEM: Consider a Hopf algebra H with invertible antipode S and an algebra A that is a left (resp. right) module-algebra over H. Let us put on the dual vector space A^* the left (resp. right) H-module structure given by

$$< a, xf > = < S^{-1}(x)a, f > \qquad (resp. < a, fx > = < aS^{-1}(x), f >)$$

for all $a \in A$, $x \in H$, and $f \in A^*$. If A is finite-dimensional, then the coalgebra $(A^{op})^*$ is a module-coalgebra over H.

PROOF: We need to check that A^* is a left H-module and that the left action of H on A^* defines an H-module-coalgebra structure on A^*. It suffices to check that the map from $H \otimes A^*$ to A^* which defines the action of H on A^* is a coalgebra morphism, i.e.

$$\epsilon(xf) = \epsilon(x)\epsilon(f)$$

and

$$\sum_{(xf)} (xf)' \otimes (xf)'' = \sum_{(x)(f)} x'f' \otimes x''f''$$

Checking that A^* is a left H-module is routine. As for the rest,

$$\epsilon(xf) = (xf)(1) = f(S^{-1}(x)1) = \epsilon(S^{-1}(x))f(1) = \epsilon(x)\epsilon(f)$$

since $\epsilon = \eta^*$ and $\epsilon \circ S = \epsilon$, and, checking the last identity on an element $a \otimes b \in A \otimes A$,

$$\begin{aligned}
< a \otimes b, \sum_{(xf)} (xf)' \otimes (xf)'' > &= \sum_{(xf)} < a, (xf)' >< b, (xf)'' > \\
&= < ba, xf > \\
&= < S^{-1}(x)(ba), f > \\
&= \sum_{(x)} < (S^{-1}(x)'b)(S^{-1}(x)''a), f > \\
&= \sum_{(x)(f)} S^{-1}(x'')b, f'' >< S^{-1}(x')a, f' > \\
&= \sum_{(x)(f)} a, x'f' >< b, x''f'' > \\
&= < a \otimes b, \sum_{(x)(f)} x'f' \otimes x''f'' >
\end{aligned}$$

Note that comultiplication on the (finite-dimensional) coalgebra $(A^{op})^*$ is the opposite comultiplication of the dual coalgebra A^*, that is

$$< ab, f >= \sum_{(f)} < b, f' >< a, f'' >$$

whenever $a, b \in A$ and $f \in (A^{op})^*$.

Plugging H into the above for A and using the first PROP from this section, we get:

CORO: Let $H = (H, \mu, \eta, \Delta, \epsilon, S, S^{-1})$ be a finite-dimensional Hopf algebra with invertible antipode S. There is a unique left (resp. right) H-module-coalgebra structure on $(H^{op})^* = (H^*, \Delta^*, \epsilon^*, (\mu^{op})^*, \eta^*, (S^{-1})^*, S^*)$ given for $a, x \in H$ and $f \in H^*$ by

$$< a, x \cdot f >= \sum_{(x)} < S^{-1}(x'')ax', f >$$

$$(resp. < a, f^x >= \sum_{(x)} < x''aS^{-1}(x'), f >)$$

where the definitions follow from the PROP above.

DEF: The above are called the **left and right coadjoint representations of H**.

Applying the above corollary to the Hopf algebra $(H^{cop})^* = (H^*, (\Delta^{op})^*, \epsilon^*, \mu^*, \eta^*, (S^{-1})^*, S^*)$, we get that there exists a unique left/right $(H^{cop})^*$-module-coalgebra structure on $(((H^{cop})^*)^{op})^* = (H^{**}, \mu^{**}, \eta^{**}, (((\Delta^{op})^*)^{op})^*, \epsilon^{**}, S^{**}, (S^{-1})^{**})$. Using the natural identification between H^{**} and H and the isomorphism between $(H^{cop})^*$ and $(H^{op})^*$ via S^*(remember, H is finite-dimensional), we get a right $(H^{op})^*$-module-coalgebra structure on the Hopf algebra H. (So long as we can convince ourselves that $(((\Delta^{op})^*)^{op})^* = \Delta$ under these circumstances...which seems reasonable, but I don't have a proof of it for you.) Formally,

PROP: Under the hypothesis of the above corollary, there exists a unique right $(H^{op})^*$-module-coalgebra structure on H given for $a \in H$ and $f \in H^*$ by

$$a^f = \sum_{(a)} f(S^{-1}(a''')a')a''.$$

THM: Let $(H, \mu, \eta, \Delta, \epsilon, S, S^{-1})$ be a finite-dimensional Hopf algebra with invertible antipode. Consider the Hopf algebra

$$X = (H^{op})^* = (H^*, \Delta^*, \epsilon^*, (\mu^{op})^*, \eta^*, (S^{-1})^*, S^*).$$

Let $\alpha : H \otimes X \to X$ and $\beta : H \otimes X \to H$ be the linear maps given by

$$\alpha(a \otimes f) = a \cdot f = \sum_{(a)} f(S^{-1}(a'')?a')$$

and

$$\beta(a \otimes f) = a^f = \sum_{(a)} f(S^{-1}(a''')a')a''$$

where $a \in H$ and $f \in X$ and the ? serves as a mute variable. Then the pair (H, X) of Hopf algebras is matched.

$(\mathbf{H}, (\mathbf{H^{op}})^*)$ matched → Bicrossed Product of H and $(\mathbf{H^{op}})^*$ (The Quantum Double)
We have to assume here that, if (H, X) as above is matched, then so is (X, H). This is probably true if we switch that last PROP from being about right module-coalgebras to being about left module-coalgebras, which should be provable using the same techniques.

DEF: Given a finite-dimensional Hopf algebra H with invertible antipode, the **quantum double** $D(H)$ is the bicrossed product of H and $X = (H^{op})^*$:

$$D(H) = X \bowtie H = (H^{op})^* \bowtie H$$

Since both H and $(H^{op})^*$ have antipodes, we know that $D(H)$ is a Hopf algebra. We can further prove that $D(H)$ is a braided Hopf algebra, but first we give a description.

As a vector space, $D(H) = X \otimes H$. The unit is $1 \otimes 1$ and the counit is given by $\epsilon(f \otimes a) = \epsilon(a)f(1)$. Comultiplication is $\Delta(f \otimes a) = \sum_{(a)(f)} (f' \otimes a') \otimes (f'' \otimes a'')$. Multiplication is, by definition,

$$(f \otimes a)(g \otimes b) = \sum_{(a)(g)} f(a' \cdot g') \otimes a''^{g''} b$$

and we can show that this in turn equals $\sum_{(a)} fg(S^{-1}(a''')?a') \otimes (a''b)$. $D(H)$ contains H and X as Hopf subalgebras via the embeddings i_H and i_X given by

$$i_H(a) = 1 \otimes a, \qquad i_X(f) = f \otimes 1$$

such that $f \otimes a = i_X(f)i_H(a)$ for all $f \in X$, $a \in H$.

Consider the map

$$\lambda_{H,H} : H \otimes X \to End(H), \qquad \lambda_{H,H}(a \otimes f)(b) = f(b)a$$

for all $a, b \in H$, $f \in X$. Since H is finite dimensional, by Coro II.2.3 we have that $\lambda_{H,H}$ is an isomorphism, so we can define

$$\rho = \lambda_{H,H}^{-1}(id_H) \in H \otimes X$$

which, in turn, allows us to define the universal R-matrix of D(H) as

$$R = (i_H \otimes i_x)(\rho) \in D(H) \otimes D(H)$$

Choosing a basis $\{e_i\}_{i \in I}$ of H (and thus a basis $\{e^i\}_{i \in I}$ of X) gives us explicit formulas for ρ and R:

$$\rho = \sum_{i \in I} e_i \otimes e^i, \qquad R = \sum_{i \in I} (1 \otimes e_i) \otimes (e^i \otimes 1)$$

This leads us to the principal claim of today's lecture:

THM: Under the previous hypotheses, the Hopf algebra D(H), equipped with the element $R \in D(H) \otimes D(H)$ as above, is braided.

EXERCISE: # 7 from chapter IX. For notation, see IX.4.1 formula (4.5) and IX.4.3, and for a review of what we already know about $k[G]$, see III.1 example 3, III.2 example 2, and III.3 example 2. Additionally, the material of section IX.5 might be useful to you.