Robert Rodger
Drinfeld-Jimbo Algebras

Introduction

Today's talk is similar to the one I gave three weeks ago, in that we again will build a construction using reasonable data that will provide for us a solution to the Yang-Baxter Equation. As you will recall, last time we were able to build a solution using any finite-dimensional Hopf algebra with an invertible antipode. This time, all we will need is a complex semisimple Lie algebra. However, in order to do so, we will have to forgo the implicit restriction we have imposed upon ourselves up until now; namely, that elements in the algebra underlying our module must be polynomials in the generators. We will, instead, allow elements in our algebra to be formal series. Doing so will require us to essentially start from scratch and rebuild our concept of what an algebra, and ultimately a braided bialgebra, is. In so doing, we will introduce the idea of a topology on our algebra and our tensor product, which will happen to also be a metric topology and thus can give us some amount of reassurance that we will retain some notion of convergence. (However, we will not explicitly need this metric in what follows, and I leave you to investigate it in the exercise.)

The Ring of Formal Series and h-Adic Topology

We consider the complex algebra $K = \mathbb{C}[[h]]$ of complex formal series in the variable h. Any element $f \in K$ is of the form $f = \sum_{n\geq 0} a_n h^n$, where $(a_0, a_1, ...)$ is a family of complex numbers indexed by the naturals. Summation is component-wise and multiplication is given by

$$ff' = \sum_{n\geq 0}\left(\sum_{p+q=n} a_p a'_q\right) h^n$$

Any polynomial in h can be considered as an element in K; in particular, the constant polynomial 1 is the multiplicative unit. Immediately, we see

LEM: A formal series f is invertible in $\mathbb{C}[[h]]$ iff $a_0 \neq 0$ in $\mathbb{C}$.

Now, for any integer $n > 0$, consider the algebra $K_n = \mathbb{C}[h]/(h^n)$ of truncated polynomials. There is a surjective morphism of algebras $\pi_n^K : K \twoheadrightarrow K_n$ sending

$$f = \sum_{n\geq 0} a_n h^n \mapsto \text{(the class of)} \sum_{k=0}^{n-1} a_k h^k \text{mod}(h^n)$$

whose kernel is $h^n K$. By the first isomorphism theorem (of rings), we have

$$\mathbb{C}[[h]]/(h^n) \cong \mathbb{C}[h]/(h^n)$$

For $n > 0$, we also have a surjective morphism of algebras $p_n : K_n \to K_{n-1}$ induced by the inclusion of ideals $(h^n) \subset (h^{n-1})$.

DEF: An **inverse system of abelian groups** (A_n, p_n) is a family $(A_n)_{n\in\mathbb{N}}$ of abelian groups and of morphisms of groups $(p_n : A_n \to A_{n-1})_{n>0}$.

Note: we write "abelian groups" above, but everything here also applies to commutative rings or modules. In particular, we're going to apply it in a bit to our algebras K_n.

DEF: Given (A_n, p_n) as above, its **inverse limit** $\varprojlim_n A_n$ is

$$\varprojlim_n A_n = \left\{ (x_n)_{n\geq 0} \in \prod_{n\geq 0} A_n \mid p_n(x_n) = x_{n-1} \ \ \forall n > 0 \right\}$$

The inverse limit has an abelian group structure as a subset of the direct product $\prod_{n\geq 0} A_n$, whose group structure is defined component-wise. Additionally, the natural projection $\pi : \prod_{n\geq 0} A_n \to A_k$ restricts to a morphism of groups $\pi_k^{lim} : \varprojlim_n A_n \to A_k$; it is defined by $\pi_k^{lim}((x_n)_n) = x_k$. If all maps p_n are surjective, then so are the maps π_n^{lim}. Note that $p_n \circ \pi_n^{lim} = \pi_{n-1}^{lim}$.

DEF: The **inverse limit topology** is defined as follows. Put the discrete topology on each A_n (i.e. the topology for which every subset is an open). The inverse limit topology on $\varprojlim_n A_n$ is then the restriction of the direct product topology on $\prod_{n\geq 0} A_n$ (i.e. a basis of open sets of the inverse limit is given by the family of all subsets $(\pi^{lim})_n^{-1}(U_n)$, where U_n is any open subset of A_n).

A map f from a topological set to $\varprojlim_n A_n$ is continuous w.r.t. the inverse limit topology iff the map $\pi_n^{lim} \circ f$ into A_n is continuous $\forall n > 0$. In particular, the π_n^{lim} are continuous.

The inverse limit has the following universal property:

PROP: Given (A_n, p_n) as above, for any abelian group C and any given family $(f_n : C \to A_n)_{n\geq 0}$ of morphisms of groups s.t. $p_n \circ f_n = f_{n-1}$ for all $n > 0$, there exists a unique morphism of groups

$$f : C \to \varprojlim_n A_n$$

s.t. $\pi_n^{lim} \circ f = f_n$ for all $n \geq 0$.

Let us now return to our algebras K_n. We form the inverse system of algebras $(K_n, p_n)_n$ and the corresponding inverse limit $\varprojlim_n K_n$. Hence, objects in $\varprojlim_n K_n$ look like

$$(0, c_0, c_0 + c_1 h, c_0 + c_1 h + c_2 h^2, ...)$$

Since $p_n \circ \pi_n^K = \pi_{n-1}^K$, from the above proposition, there exists a unique morphism of algebras π from K to $\varprojlim_n K_n$ s.t. $\pi_n^{lim} \circ \pi = \pi_n^K$.

PROP: The map $\pi : K \to \varprojlim_n K_n$ is an isomorphism. (To use language similar to what was earlier presented, $\mathbb{C}[[h]] \cong \varprojlim_n \mathbb{C}[h]/(h^n)$.)
Proof: π is injective because its kernel, the intersection of all (h^n), is zero. We demonstrate surjectivity by constructing a right inverse. Let $(f_n)_{n>0} \in \varprojlim_n K_n$; thus $f_n \in K_n$ can be represented as

$$f_n = \sum_{k=0}^{n-1} a_k^{(n)} h^k$$

and we have $p_n(f_n) = f_{n-1}$ for all $n > 0$. Hence $a_k^{(n)} = a_k^{(n-1)}$ for $0 \leq k \leq n-2$. We can therefore define a formal series $f = \sum_{n\geq 0} a_n h^n$ by $a_n = a_n^{(n+2)} = a_n^{(n+3)} = ...$ We have $\pi(f) = (f_n)_n$. $\square$

The above proposition allows us to equip K with the inverse limit topology, except now we refer to it as the **h-adic topology**. For $U_n \subset K_n$ open, we have that $(\pi_n^K)^{-1}(U_n)$ is open; in particular, $(\pi_n^K)^{-1}(0) = (h^n)$ is open, and I think this is the reason for the re-naming of the topology.

Topologically Free Modules
Let M be a left module over K (so that, for $m \in M$, things like hm, $h^2 m$, etc. are also in M). Consider the family of submodules $(h^n M)_{n>0}$ and the canonical K-linear projections

$$p_n : M_n = M/h^n M \to M_{n-1} = M/h^{n-1} M$$

They form an inverse system of K-modules, and we may consider the inverse limit

$$\widetilde{M} = \varprojlim_n M_n$$

This has a natural structure as a K-module (namely, if $f \in K$ and $\widetilde{m} = (m_0, m_1, ...) \in \widetilde{M}$, then $f\widetilde{m} = (fm_0, fm_1, ...)$) and has a natural topology, the inverse limit topology, for which the family of submodules $(h^n \widetilde{M})_n$ is a family of open neighborhoods.

DEF: $\widetilde{M}$ is called the **h-adic completion** of M.

The projections $i_n : M \to M_n$ induce a unique K-linear map $i : M \to \widetilde{M}$ s.t. $\pi_n^{lim} \circ i = i_n$ for all n. That is, for $m \in M$, $i(m) = (0, i_1(m), i_2(m), ...)$. The kernel of i is given by

$$\mathrm{Ker}(i) = \bigcap_{n>0} h^n M$$

DEF: A K-module M is **separated** if $\mathrm{Ker}(i) = \{0\}$. It is **complete** if i is surjective. (In particular, if M is isomorphic to its completion $\widetilde{M}$, it is complete.)

For any module M, the module $M/(\bigcap_{n\geq 0} h^n M)$ is, by definition, separated and the completion $\widetilde{M}$ is complete. Why? Consider the projection $\pi_n^{lim} : \widetilde{M} \to M_n$: its kernel is $h^n \widetilde{M}$, and therefore

$$\widetilde{M}/h^n\widetilde{M} \cong M_n = M/h^n M$$

and if we take the inverse limit of both sides, we have $\widetilde{\widetilde{M}} \cong \widetilde{M}$.

Any separated, complete K-module will be equipped with the h-adic topology coming from the inverse limit topology on $\widetilde{M}$.

DEF: Given any complex vector space V, the **topologically free module** $V[[h]]$ is the set of all formal series $\sum_{n\geq 0} v_n h^n$, where $(v_0, v_1, ...)$ is an infinite family of elements of V, with the left K-module structure. For instance, if $f \in K$ and $v \in V[[h]]$:

$$fv = \left(\sum_{n\geq 0} a_n h^n\right)\left(\sum_{m\geq 0} v_m h^m\right) = \sum_{n\geq 0}\left(\sum_{p+q=n} a_p v_q\right) h^n$$

Setting $V = \mathbb{C}$ allows us to recover K.

PROP: Any topologically free module is separated and complete.
Proof: The submodule $h^n V[[h]]$ is the set of all elements $\sum_{n\geq 0} v_n h^h$ s.t. $v_0 = ... = v_{n-1} = 0$. It follows that $\bigcap_{n\geq 0} h^n V[[h]] = 0$, so $V[[h]]$ is separated. The proof of completeness follows the one showing $K \cong \varprojlim_n K_n$; hence $V[[h]] \cong \varprojlim_n (V[[h]]/h^n V[[h]])_{n>0}$, which is complete.

Hence, any topologically free module can be endowed with the h-adic topology. Additionally, we can strengthen the above proposition: A left K-module is topologically free iff it is separated, complete, and torsion-free. (Recall: a K-module is **torsion-free** if $hm \neq 0$ when $M \ni m \neq 0$.)

PROP: For any separated, complete K-module N, there is a natural bijection

$$\mathrm{Hom}_K(V[[h]], N) \cong \mathrm{Hom}(V, N)$$

where Hom_K denotes the space of K-linear maps.
Proof: The proof relies on the idea of an inverse limit of a family of K-linear maps, which I don't have time to introduce. But the result is necessary for the description of Quantum Enveloping Algebras, a general class of objects of which our Drinfeld-Jimbo algebras are particular example.

Topological Tensor Product
Let M and N be left-modules over the algebra $K = \mathbb{C}[[h]]$. Consider the K-module $M \otimes_K N$ obtained as the quotient of the vector space $M \otimes N$ by the subspace spanned by all elements of the form $fm \otimes n - m \otimes fn$, where $f \in K$, $m \in M$, $n \in N$.

DEF: The **topological tensor product** $M\widetilde{\otimes}N$ of M and N is the h-adic completion of $M \otimes_K N$:

$$M\widetilde{\otimes}N = (\widetilde{M \otimes_K N}) = \varprojlim_n (M \otimes_K N)/h^n(M \otimes_K N)$$

Since it is defined as a completion, the topological tensor product of two modules is always complete. The usual associativity and commutativity constraints induce the following K-linear isomorphisms:

$$(M\widetilde{\otimes}N)\widetilde{\otimes}P \cong M\widetilde{\otimes}(N\widetilde{\otimes}P)$$

$$M\widetilde{\otimes}N \cong N\widetilde{\otimes}M$$

Also,

$$K\widetilde{\otimes}M \cong \widetilde{M} \cong M\widetilde{\otimes}K$$

i.e. K serves as a unit for completions.

PROP: If M and N are topologically free modules, then so is $M\widetilde{\otimes}N$. More precisely,

$$V[[h]]\widetilde{\otimes}W[[h]] = (V \otimes W)[[h]]$$

Topological Algebras

The intuitive way to work with topological algebras is basically to take what you know about non-topological algebras and put tildes over all your tensor products and replace your ground field k with our algebra K.

DEF: A **topological algebra** is a triple (A, μ, η), where A is a module over the ring $K = \mathbb{C}[[h]]$, $\mu : A\widetilde{\otimes}A \to A$, and $\eta : K \to A$ are K-linear maps s.t.

$$\mu \circ (\mu\widetilde{\otimes}\mathrm{id}_A) = \mu \circ (\mathrm{id}_A\widetilde{\otimes}\mu)$$

$$\mu \circ (\eta\widetilde{\otimes}\mathrm{id}_A) = \mathrm{id}_A = \mu \circ (\mathrm{id}_A\widetilde{\otimes}\eta)$$

Not surprisingly, if we identify $K\widetilde{\otimes}K$ with K, then $(K, \mathrm{id}_K, \mathrm{id}_K)$ is a topological algebra (i.e. K is akin to k in the non-topological case). Additionally, we can define the topological tensor product of two topological algebras simply by adding a tilde to all of the $\otimes$'s in our definitions.

Let (A, μ, η) be a topological algebra and $f(h) = \sum_{n\geq 0} c_n h^n$ a formal series with complex coefficients. For any $a \in A$ the formula

$$f(ha) = \sum_{n\geq 0} c_n a^n h^n$$

defines a unique element in the inverse limit $\widetilde{A} = \varprojlim_n A/h^n A$ (namely, $(0, c_0, c_0+c_1ah, c_0+c_1ah+c_2a^2h^2, ...)$), and if A is separated and complete, it defines an element, still denoted by $f(ha)$, in $A \cong \widetilde{A}$. We can use this to define

$$e^{ha} = \sum_{n\geq 0} \frac{a^n h^n}{n!}$$

so long as A is separated and complete. Further, if $a' \in A$ commutes with a,

$$e^{ha}e^{ha'} = e^{h(a+a')}$$

implies that e^{ha} is invertible. We will need this formalism later to define the R-matrices of our DJA's.

We didn't discuss quasi-bialgebras in the non-topological case, but we'll need their topological counterpart in order to define DJA's, so...

DEF: A **topological quasi-bialgebra** is a sextuple $(A, \mu, \eta, \Delta, \epsilon, \Phi)$, where (A, μ, η) is a topological algebra, $\Delta : A \to A\widetilde{\otimes}A$ and $\epsilon : A \to K$ are K-linear maps, and Φ is an invertible element in $A\widetilde{\otimes}A\widetilde{\otimes}A$ s.t.

$$(\mathrm{id}_A\widetilde{\otimes}\Delta)(\Delta(a)) = \Phi\left(\left(\Delta\widetilde{\otimes}\mathrm{id}_A\right)(\Delta(a))\right)\Phi^{-1}$$

for all $a \in A$,

$$(\epsilon\widetilde{\otimes}\mathrm{id}_A)\Delta = \mathrm{id}_A = (\mathrm{id}_A\widetilde{\otimes}\epsilon)\Delta$$

$$(\mathrm{id}_A\widetilde{\otimes}\mathrm{id}_A\widetilde{\otimes}\Delta)(\Phi)(\Delta\widetilde{\otimes}\mathrm{id}_A\widetilde{\otimes}\mathrm{id}_A)(\Phi) = \Phi_{234}(\mathrm{id}_A\widetilde{\otimes}\Delta\widetilde{\otimes}\mathrm{id}_A)(\Phi)\Phi_{123}$$

$$(\mathrm{id}_A\widetilde{\otimes}\epsilon\widetilde{\otimes}\mathrm{id}_A)(\Phi) = 1 \otimes 1$$

What you should take away from this is that there exists a generalization of topological (and non-topological) bialgebras, which includes an extra piece of data, Φ, which can potentially obstruct coassociativity much in the way that R can potentially obstruct cocommutativity.

DEF: When $\Phi = 1\widetilde{\otimes}1\widetilde{\otimes}1$, we call A a **topological bialgebra**, in accordance with the non-topological case. (In this case, we retain coassociativity.)

For the sake of completeness, I introduced (topological) quasi-bialgebras because the general theory allows for non-coassociative bialgebras. However, the specific example in which we are interested, the Drinfeld-Jimbo algebra, happens to be coassociative and so for the remainder of the talk, if it makes you more comfortable, you can just mentally replace every instance of "quasi-bialgebra" with just "bialgebra" and forget about Φ; all the remaining theory will remain true in this case.

DEF: A **topological braided quasi-bialgebra** $(A, \mu, \eta, \Delta, \epsilon, \Phi, R)$ is a topological quasi-bialgebra with an invertible element $R \in A\widetilde{\otimes}A$, called the **universal *R*-matrix of A**, satisfying

$$\Delta^{op}(a) = R\Delta(a)R^{-1}$$

$$(\mathrm{id}_A\widetilde{\otimes}\Delta)(R) = (\Phi_{231})^{-1}R_{13}\Phi_{213}R_{12}(\Phi_{123})^{-1}$$

$$(\Delta\widetilde{\otimes}\mathrm{id}_A)(R) = \Phi_{312}R_{13}(\Phi_{132})^{-1}R_{23}\Phi_{123}$$

EX: Let $A = (A, \mu, \eta, \Delta, \epsilon, \Phi, R)$ be a topological braided quasi-bialgebra. Since $(A\widetilde{\otimes}A)/h(A\widetilde{\otimes}A) \cong A/hA \otimes A/hA$, the K-linear maps $\mu, \eta, \Delta, \epsilon$ induce $\mathbb{C}$-linear maps (via restriction):

$$\bar{\mu} : A/hA \otimes A/hA \to A/hA, \quad \bar{\eta} : \mathbb{C} \to A/hA$$

$$\bar{\Delta} : A/hA \to A/hA \otimes A/hA, \quad \bar{\epsilon} : A/hA \to \mathbb{C}$$

Define $\bar{\Phi}$ as the class of Φ modulo $(A/hA)^{\otimes 3}$ and $\bar{R}$ as the class of R modulo $(A/hA)^{\otimes 2}$. Then $\bar{A} = (A/hA, \bar{\mu}, \bar{\eta}, \bar{\Delta}, \bar{\epsilon}, \bar{\Phi}, \bar{R})$ is a (non-topological!) braided quasi-bialgebra.

DEF: A **topological *A*-module** M over a topological algebra $A = (A, \mu, \eta)$ is a left K-module with a K-linear map $\mu_M : A\widetilde{\otimes}M \to M$ s.t.

$$\mu_M \circ (\mu\widetilde{\otimes}\mathrm{id}_M) = \mu_M \circ (\mathrm{id}_A\widetilde{\otimes}\mu_M), \quad \mu_M \circ (\eta\widetilde{\otimes}\mathrm{id}_M) = \mathrm{id}_M$$

If M and N are topological A-modules, then, in parallel with the non-topological case, we can put a topological A-module structure on their topological tensor product $M\widetilde{\otimes}N$. Then, if A is a topological braided algebra with universal R-matrix R, for any topological A-module M the K-linear automorphism $c^R_{M,M}$ defined by

$$c^R_{M,M}(m_1\widetilde{\otimes}m_2) = \left(R\left(m_1\widetilde{\otimes}m_2\right)\right)_{21}$$

is a solution of the Yang-Baxter equation in $M\widetilde{\otimes}M\widetilde{\otimes}M$.

DEF: A **quantum enveloping algebra** for the complex Lie algebra $\mathfrak{g}$ is a topological braided quasi-bialgebra $A = (A, \mu, \eta, \Delta, \epsilon, \Phi, R)$ s.t.

(i)A is a topologically free module
(ii) The induced braided quasi-bialgebra $\bar{A} = (A/hA, \bar{\mu}, \bar{\eta}, \bar{\Delta}, \bar{\epsilon}, \bar{\Phi}, \bar{R})$ coincides with the trivial braided quasi-bialgebra structure of $U(\mathfrak{g})$
(iii) The map η is trivially extended from $\bar{\eta}$.

Let's make the definition explicit. First, note that a QUE is topologically free. Thus $A = (A/hA)[[h]]$ as a left K-module. By hypothesis, we also have $A/hA = U(\mathfrak{g})$, and thus $A = U(\mathfrak{g})[[h]]$ as a K-module. We also have

$$A^{\widetilde{\otimes}n} = (U(\mathfrak{g})^{\otimes n})[[h]]$$

for all $n > 0$. Thus, using the proof of that earlier proposition I said we'd need when we got to QUE's, the maps $\mu, \eta, \Delta, \epsilon$ are determined by their restrictions to $U(\mathfrak{g}) \otimes U(\mathfrak{g})$, $\mathbb{C}$, $U(\mathfrak{g})$, and $U(\mathfrak{g})$, respectively. For instance, for $a, a' \in U(\mathfrak{g})$,

$$\mu(a \otimes a') = \sum_{n\geq 0} \mu_n(a \otimes a')h^n$$

where $(\mu_n)_{n\geq 0}$ is a family of linear maps from $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ to $U(\mathfrak{g})$ s.t. μ_0 is the product in the enveloping algebra. Likewise for Δ and ϵ; for η, we have $\eta(f) = f1$ for all $f \in K$. Lastly,

$$\Phi = \sum_{n\geq 0} \Phi_n h^n, \qquad R = \sum_{n\geq 0} R_n h^n$$

where $(\Phi_n)_{n\geq 0}$ and $(R_n)_{n\geq 0}$ are families of elements of $U(\mathfrak{g})^{\otimes 3}$ and $U(\mathfrak{g})^{\otimes 2}$, respectively, s.t.

$$\Phi_0 = 1 \otimes 1 \otimes 1, \qquad R_0 = 1 \otimes 1$$

Note that these identities ensure the invertibility of Φ and R (recall the first lemma), a necessary condition for quasi-, respectively braided, bialgebras.

We can recover $\mathfrak{g}$ from A by

$$\mathfrak{g} = \{x \in A/hA \mid \Delta_0(x) = 1 \otimes x + x \otimes 1\}$$

recalling that the subspace of primitive elements in $U(\mathfrak{g})$ is $\mathfrak{g}$ provided that the ground field is of characteristic zero.

Semisimple Lie Algebras

We first review the data used by Élie Cartan to completely characterize the semisimple (and hence finite-dimensional) compex Lie algebras. This will allow us to deform the semisimple Lie algebras in a natural way. (Here, $n = N/3$, where N is the number of generators of the semisimple Lie algebra and is always a multiple of three.)

DEF: A **Cartan matrix** is a square matrix $A = (a_{ij})_{1\leq i,j\leq n}$ with the following properties:
(i) its coefficients a_{ij} are non-positive integers when $i \neq j$, and $a_{ii} = 2$
(ii) there exists a diagonal matrix $D = \mathrm{Diag}(d_1, ..., d_n)$ with entries in the set $\{1, 2, 3\}$ s.t. the matrix DA is symmetric positive-definite

Jean-Pierre Serre then showed that the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is isomorphic to the algebra generated by $\{X_i, Y_i, H_i\}_{1\leq i\leq n}$ and the relations

$$[H_i, H_j] = 0, \qquad [X_i, Y_j] = \delta_{ij} H_i, \qquad [H_i, X_j] = a_{ij} X_j, \qquad [H_i, Y_j] = -a_{ij} Y_j$$

and, if $i \neq j$,

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1 - a_{ij} \\ k \end{pmatrix} X_i^k X_j X_i^{1-a_{ij}-k} = 0$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1 - a_{ij} \\ k \end{pmatrix} Y_i^k Y_j Y_i^{1-a_{ij}-k} = 0$$

EX: In the case of $\mathfrak{sl}(2)$, $A = (2)$, $D = (1)$, and $U(\mathfrak{sl}(2))$ is isomorphic to the algebra generated by $\{X, Y, H\}$ subject to the relations $[X, Y] = H$, $[H, X] = 2X$, $[H, Y] = -2Y$.

This Doesn't Fit Anywhere Else

DEF: The **topologically free algebra generated by *X*** is the algebra of formal series over the free complex algebra generated by the set X:

$$K(X) = (\mathbb{C}\langle X\rangle)[[h]]$$

equipped with the h-adic topology.

DEF: Let X be a set and R be a subset of the topologically free algebra $K(X)$ generated by X. A K-algebra A is said to be the ***K*-algebra topologically generated by the set *X* of generators and the set *R* of relations** if A is isomorphic to the quotient of $K(X)$ by the closure (for the h-adic topology) of the two-sided ideal generated by R.

Drinfeld-Jimbo Algebras
Let $\mathfrak{g}$ be a complex semisimple Lie algebra and A, D, and n as before. We create a 1-parameter deformation $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$:

DEF: The **Drinfeld-Jimbo algebra** $U_h(\mathfrak{g})$ is the K-algebra topologically generated by the set of generators $\{X_i, Y_i, H_i\}_{1\leq i\leq n}$ and the relations

$$[H_i, H_j] = 0, \qquad [X_i, Y_j] = \delta_{ij}\frac{\sinh(hd_iH_i/2)}{\sinh(hd_i/2)}, \qquad [H_i, X_j] = a_{ij}X_j, \qquad [H_i, Y_j] = -a_{ij}Y_j$$

and if $i \neq j$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} X_i^k X_j X_i^{1-a_{ij}-k} = 0$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} Y_i^k Y_j Y_i^{1-a_{ij}-k} = 0$$

where $q_i = e^{hd_i/2}$ and where sinh is the usual formal series

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = \sum_{n\geq 0}\frac{x^{2n+1}}{(2n+1)!}$$

Note that, while $\sinh(hd_i/2)$ is not invertible (since the zeroth-order coefficient equals zero), it is the product of h with a unique invertible element, so that $\sinh(hd_iH_i/2)/\sinh(hd_i/2)$ is a well-defined element of $K\langle\{X_i, Y_i, H_i\}_{1\leq i\leq n}\rangle$. Also, note that the above relations closely resemble the Serre relations for semisimple Lie algebras. In particular,

$$\frac{\sinh(hd_iH_i/2)}{\sinh(hd_i/2)} \equiv H_i \bmod h$$

That is to say, to zeroth-order in h, this relation reduces to Serre's relation for $U(\mathfrak{g})$. The same goes for the two summation identities. We'll see in a minute that if we set $h = 0$ we'll recover the enveloping algebra of $\mathfrak{g}$ in Serre's presentation.

THM: The topological algebra $U_h(\mathfrak{g})$ is a quantum enveloping algebra

$$(U_h(\mathfrak{g}), \mu_h, \eta_h, \Delta_h, \epsilon_h, \Phi_h, R_h)$$

for the Lie algebra $\mathfrak{g}$, with $\Phi_h = 1\tilde{\otimes}1\tilde{\otimes}1$ and comultiplication and counit determined by

$$\Delta_h(H_i) = H_i \otimes 1 + 1 \otimes H_i$$

$$\Delta_h(X_i) = X_i \otimes e^{hd_iH_i/4} + e^{-hd_iH_i/4} \otimes X_i$$

$$\Delta_h(Y_i) = Y_i \otimes e^{hd_iH_i/4} + e^{-hd_iH_i/4} \otimes Y_i$$

$$\epsilon_h(H_i) = \epsilon_h(X_i) = \epsilon_h(Y_i) = 0$$

In particular, $U_h(\mathfrak{g})$ is a topological braided quasi-bialgebra. (No proof provided.) $\square$

A few comments:
(i) Setting $h = 0$ gives us back the enveloping algebra of $\mathfrak{g}$ in Serre's presentation, i.e.

$$U_h(\mathfrak{g})/hU_h(\mathfrak{g}) \cong U(\mathfrak{g})$$

(ii) The topological bialgebra $U_h(\mathfrak{g})$ has antipode S_h determined by

$$S_h(H_i) = -H_i, \qquad S_h(X_i) = -e^{hd_i/2}X_i, \qquad S_h(Y_i) = -e^{-hd_i/2}Y_i$$

(iii) Commultiplication is not cocommutative and the antipode is not involutive.
(iv) The above theorem tells us only that $R_h \in U_h(\mathfrak{g})$ exists. Further work can show that it has the form

$$R_h = \sum_{\ell\in\mathbb{N}^n} e^{h\left(\frac{t_0}{2}+\frac{1}{4}(H_\ell\otimes 1 - 1\otimes H_\ell)\right)}P_\ell$$

where $H_\ell = \sum_{1\le i\le n} \ell_i H_i$ for $\ell = (\ell_1, ..., \ell_n)$, t_0 the element

$$t_0 = \sum_{1\le i,j\le n} (DA)^{-1}_{ij} H_i \otimes H_j$$

in $\mathfrak{g}\otimes\mathfrak{g}$, and P_ℓ is a polynomial in the variables $X_1\otimes 1$,...,$X_n\otimes 1$ and in $1\otimes Y_1$,...,$1\otimes Y_n$ (homogeneous of degree ℓ_i in $X_i\otimes 1$ and $1\otimes Y_i$) that can be calculated via induction on ℓ. Note the exponentiated generators; it is essentially for just this reason alone that we introduced the notion of a topological algebra.
(v) We have $P_0 = 1\otimes 1$ and $R_h \equiv 1\otimes 1 \mod h$, and hence R_h is invertible, as desired.

The Case of $\mathfrak{sl}(2)$
$U_h = U_h(\mathfrak{sl}(2))$ is the K-algebra topologically generated by the three variable X, Y, H and the relations

$$[H,X] = 2X, \qquad [H,Y] = -2Y, \qquad [X,Y] = \frac{\sinh(hH/2}{\sinh(h/2)} = \frac{e^{hH/2} - e^{-hH/2}}{e^{h/2} - e^{-h/2}}$$

The U_q discussed earlier in the semester is not a quantum group. But it can be embedded in U_h:

PROP: There exists a map of Hopf algebras $i : U_q \to U_h$ s.t.

$$i(E) = Xe^{hH/4}, \qquad i(F) = e^{-hH/4}Y, \qquad i(K) = e^{hH/2}, \qquad i(K^{-1}) = e^{-hH/2}, \qquad i(q) = e^{h/2}$$

The above map is an injection, which allows us to identify U_q with the subalgebra of U_h generated by

$$E = Xe^{hH/4}, \qquad F = e^{-hH/4}Y, \qquad K = e^{hH/2}, \qquad K^{-1} = e^{-hH/2}, \qquad q = e^{h/2}$$

THM: The element $R_h \in U_h\widetilde{\otimes}U_h$ defined below is a univeral R-matrix for $U_h(\mathfrak{sl}(2))$

$$\begin{aligned} R_h &= e^{\frac{h(H\otimes H)}{4}} \left(\sum_{\ell\ge 0} \frac{(q-q^{-1})^\ell}{[\ell]_q!} q^{\ell(\ell-1)/2} (E^\ell \otimes F^\ell) \right) \\ &= \sum_{\ell\ge 0} \frac{(q-q^{-1})^\ell}{[\ell]_q!} q^{-\ell(\ell-1)/2} e^{\frac{h}{2}\left(\frac{H\otimes H}{2} + \frac{1}{2}(\ell H\otimes 1 - 1\otimes \ell H)\right)} (X^\ell \otimes Y^\ell) \end{aligned}$$

RMK: Clearly, this is not a useful working definition of R_h. However, it is enough to demonstrate to us that, because of the exponentiated generators, a universal R-matrix is impossible to define in the U_q (read: non-topological) paradigm.

EXERCISE: Ch. XVI, number 7. A quick note: although it is not explicitly about DJA's or QUE's, this exercise is assigned to help us get a better feel for the notion of -adic topology and the inverse limit. In particular, note that, in contrast to the case of (K_n, p_n), we do not have that $\mathbb{Z} \cong \mathbb{Z}_p$ (which we can think of as the completion of $\mathbb{Z}$); in fact, the latter is much, much bigger than the former. You'll need to read a bit about ultrametric distance and the idea of density, but a review of the proof of Prop XVI.2.3a and of Coro XVI.1.4 (which was skipped during the talk) should suffice.