Affine algebras, loop algebras and central extensions

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Abstract

In these lecture notes we present how to construct a direct affine algebra, starting from a simple Lie algebra, by centrally extending its associated loop algebra. By exploring the root space associated with the affine algebra, the method is then compared to the axiomatic Cartan matrix method and the similarities are highlighted. The affine algebra $\mathfrak{sl}_2$ is given explicitly as an example. It is subsequently generalised to the quantum group $U_q(\mathfrak{sl}_2)$, which is also shown to have a Hopf algebra structure, via its universal enveloping algebra $U(\mathfrak{sl}_2)$. The final sections introduce the Witt algebra, its central extension, the Virasoro algebra, and explains their relevance to conformal field theories in physics. The notational conventions, definitions, figures and explanations can be found in [1].

Contents

1 Affine Lie algebras

We present here the construction of a general direct affine algebra. Starting with a finite dimensional simple Lie algebra, we provide a method to generalise it to its loop algebra and then to its affine algebra via a central extension.

1.1 Lie algebra

Let $\mathfrak{g}$ be a Lie algebra, that is a vector space defined here over $\mathbb{C}$, endowed with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket. It satisfies the following axioms:

$$\begin{aligned}
\text{Bilinearity:} \qquad & [ax+by, z] = a[x,z] + b[y,z], \\
& [z, ax+by] = a[z,x] + b[z,y]. && (1)
\end{aligned}$$

Alternating on $\mathfrak{g}$:

$$[x,x] = 0. \qquad (2)$$

The Jacobi identity:

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0, \qquad (3)$$

$$\forall a, b \in \mathbb{C} \text{ and } \forall x, y, z \in \mathfrak{g}.$$

A subspace $I \subseteq \mathfrak{g}$ satisfying $[\mathfrak{g}, I] \subseteq I$ is called an ideal in the Lie algebra $\mathfrak{g}$. A simple Lie algebra is a non-abelian Lie algebra whose only ideals are 0 and itself. As an example, let us consider the simple Lie algebra $\mathfrak{sl}_2$ which can be presented as the set of traceless 2×2 matrices:

$$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbb{C} \right\}$$

Then

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (4)$$

form a basis of $\mathfrak{sl}_2$. Their Lie algebra structure is generated by the commutators of these basis elements, $[h,e] = 2e$, $[h,f] = -2f$ and $[e,f] = h$.

We may fix on $\mathfrak{g}$, an invariant symmetric bilinear form, $(\cdot|\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, given by $(x|y) = \mathrm{tr}_{\mathbb{C}^2}(xy)$, satisfying

$$\begin{aligned}
&([x,y]|z) = (x|[y,z]), && (5) \\
&(x|y) = (y|x), && (6) \\
&\forall x, y, z \in \mathfrak{g}.
\end{aligned}$$

As an example we may consider again $\mathfrak{sl}_2$,

$$\begin{aligned}
(e|e) &= \mathrm{tr}(e^2) = 0 = (f|f) \\
(e|f) &= \mathrm{tr}(ef) = 1 = (f|e) \\
(h|h) &= \mathrm{tr}(h^2) = 2 \\
(h|e) &= \mathrm{tr}(he) = 0 = (h|f)
\end{aligned}$$

1.2 Loop algebra

The *loop* algebra of a simple Lie algebra $\mathfrak{g}$ is[1]

$$\widetilde{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}, \tag{7}$$

where $\mathbb{C}[t, t^{-1}]$ is the associative algebra of complex Laurent polynomials in t. A typical element of $\widetilde{\mathfrak{g}}$ is a sum of terms of the form: $f(t) \otimes x$, where $x \in \mathfrak{g}$ and $f(t) = \sum_n a_n t^n, a_n \in \mathbb{C}$. For example, let us consider the loop algebra of all 2×2 traceless matrices $\mathfrak{sl}_2(\mathbb{C})$,

$$\widetilde{\mathfrak{sl}}_2(\mathbb{C}) = \left\{ \begin{pmatrix} f_{11}(t) & f_{12}(t) \\ f_{21}(t) & f_{22}(t) \end{pmatrix} \mid f_{ij}(t) \in \mathbb{C}[t, t^{-1}], f_{11}(t) + f_{22}(t) = 0 \right\}$$

Each matrix is a loop in $\mathfrak{sl}_2$ with Lie bracket

$$[f(t) \otimes x, g(t) \otimes y] = f(t)g(t) \otimes [x, y]. \tag{8}$$

In general, if $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$, then $\widetilde{\mathfrak{g}}$ is a Lie algebra with the same generators $\{x, y, \ldots\}$ satisfying (8), but with coefficients taken from $\mathbb{C}[t, t^{-1}]$.

1.3 Affine algebra

The *direct affine* algebras are of the form[2]

$$\widehat{\mathfrak{g}} = \widetilde{\mathfrak{g}} \oplus \mathbb{C}\not{c}, \tag{9}$$

where $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$ and $\mathbb{C}\not{c}$ is a 1-dimensional space in the centre of $\widehat{\mathfrak{g}} : [\mathbb{C}\not{c}, x] = 0$ for all $x \in \mathfrak{g}$, with $\not{c}$ the *central element*. The Lie bracket is extended to $\widehat{\mathfrak{g}}$ by

$$[f(t) \otimes x, g(t) \otimes y] = f(t)g(t) \otimes [x, y] + m\delta_{m+n,0} k(x|y)\not{c}, \tag{10}$$

$$[\not{c}, f(t) \otimes x] = 0, \tag{11}$$

where the *Killing form* is defined up to a scalar multiplication by

$$k(x|y) = \mathrm{tr}[\mathrm{ad}(x)\mathrm{ad}(y)], \tag{12}$$

with $\mathrm{ad}(x) : y \to [x, y]$ the *adjoint action.*

[1]The Loop algebra $\widetilde{\mathfrak{g}}$ is sometimes denoted $L\mathfrak{g}$.

[2]The affine algebra $\widehat{\mathfrak{g}}$ is sometimes called afffine-$\mathfrak{g}$.

1.4 Central extension

The second term in (10) is called a *central extension*. It appears to have been added in a somewhat ad hoc fashion; how do we know that this is the only term that we are permitted to adjoin to the algebra? Well it is indeed true that for simple Lie algebras there exists only one non-vanishing invariant, symmetric bilinear form, which is precisely the Killing form. Therefore due to the axioms (1), that the Lie bracket must satisfy, the second term in (10) is, up to a scalar multiplication, the unique term that we are allowed to adjoin to the Lie bracket.

As well as being unique, the central extension is rather important from the point of view of the representation theory. This is because the existence of a nontrivial highest weight representation requires a central extension, and so many of the interesting applications depend on the presence of the central term. For many simple models, the state of highest weight is often the ground state of a physical system, moreover it is always a vacuum state. Therefore from a physical point of view, the central extension is important, in fact, for the quantisation of field theories, it is quite necessary.

2 Root systems

2.1 Cartan subalgebra

Let us begin by defining a maximal abelian subalgebra (or *Cartan subalgebra*)[3] $\mathfrak{g}^0$, consisting of d diagonal elements of a *rank-d* simple Lie algebra $\mathfrak{g}$. In the *adjoint representation*, the Lie algebra itself serves as the representation space, so for notation convenience we will also denote this as $\mathfrak{g}$.[4] As the generators[5] of the Cartan subalgebra can be simultaneously diagonalised, a basis of simultaneous eigenvectors for $\mathfrak{g}$ can be found such that each Cartan generator acts on each basis element as a scalar. For example in the *Cartan-Weyl basis* we have $H^i|\alpha\rangle = \alpha^i|\alpha\rangle$, $(i = 1, \ldots, d)$[6] with H^i a Cartan

[3]The Cartan subalgebra of a Lie algebra $\mathfrak{g}$ is often denoted by $\mathfrak{h}$.

[4]To make it clear that $\mathfrak{g}$ is serving as the representation space, sometimes it is denoted as $V(\mathfrak{g})$ or $\mathrm{ad}(\mathfrak{g})$.

[5]Physicists and mathematicians use the word "generator" in different ways. Here the generators of a Lie algebra are basis elements.

[6]Note that α^i with an upper index indicates a scalar component of the root vector α.

generator and α a *root vector* or simply a *root*.

We can therefore generate a *root decomposition* of the representation space such that the basis elements are labelled by d-tuples of scalars called roots, where each component of the root is a scalar corresponding to a particular Cartan generator. In this respect the root space should be viewed as the *dual space*, that is, the space of linear functionals. Meaning that the roots are linear functionals on the Cartan subalgebra (via the Killing form) and are used to define $\mathfrak{g}$.

2.2 Simple roots and the Cartan matrix

A root α belongs to a set of roots of $\mathfrak{g}$, denoted by Δ.[7] In general the roots are linearly dependent. A *positive* root is a root where the first nonzero component is positive. A *simple* root is then a root that cannot be written as the sum of two positive roots. There are necessarily d simple roots, and their set, denoted at Π, provides the most convenient basis for the d-dimensional space of roots.[8] The root space has a natural scalar product $(\cdot|\cdot)$ that may be identified with a scalar product of the Lie algebra, namely the Killing form. In the Cartan-Weyl basis we may write this as

$$(\alpha|\beta) = k(H^{\alpha}, H^{\beta}). \tag{13}$$

The scalar product of the simple roots define the *Cartan matrix*,[9]

$$A_{ij} = \frac{2(\alpha_i|\alpha_j)}{(\alpha_j|\alpha_j)}, \qquad \alpha_i \in \Pi. \tag{14}$$

The elements A_{ij} are always integer valued and the diagonal elements $A_{ii} = 2$. It is not necessarily a symmetric matrix with respect to transposition, although the zeros are placed symmetrically.

[7] Although it is not customary to include 0 in the root system Δ for the finite dimensional theory, we do so here because it is convenient for the infinite dimensional case.

[8] Note that the choice of a set of simple roots Π is not unique.

[9] Note that α_i with a lower index indicates a specific simple root.

2.3 Coroots

Before we move on to the infinite dimensional algebras, it will be useful to first introduce the *coroot*, which if ($\alpha \neq 0$) is simply a rescaling of the roots,

$$\breve{\alpha} = \frac{2\alpha}{(\alpha|\alpha)}. \tag{15}$$

The set of coroots is

$$\breve{\Delta} = \{\breve{\alpha} \mid \alpha \in \Delta \backslash \{0\} \cup \{0\}. \tag{16}$$

By the isomorphism provided by the Killing form, we use $\breve{\Delta}$ to select a basis of the Cartan subalgebra called the *Chevalley basis*. In this basis the eigenvalue equation for generators in $\mathfrak{g}^0$ takes the following form

$$h_j|\alpha\rangle = \alpha_i(h_j)|\alpha\rangle = (\alpha_i|\breve{\alpha}_j)|\alpha\rangle = \frac{2(\alpha_i|\alpha_j)}{(\alpha_j|\alpha_j)}|\alpha\rangle = A_{ij}|\alpha\rangle. \tag{17}$$

The Killing form of the generators of the Cartan subalgebra is easily translated from the Cartan-Weyl to the Chevalley basis,

$$k(h^i, h^j) = (\breve{\alpha}_i|\breve{\alpha}_j). \tag{18}$$

2.4 Root space decomposition

Let the root space decomposition of the finite dimensional $\mathfrak{g}$ be

$$\mathfrak{g} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha, \tag{19}$$

the root space decomposition of the loop algebra $\tilde{\mathfrak{g}}$ is then

$$\tilde{\mathfrak{g}} = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{\alpha \in \Delta} t^n \otimes \mathfrak{g}^\alpha. \tag{20}$$

Thus a root β, of the loop algebra root system $\tilde{\Delta}$, consists of the pair

$$\beta = (n, \alpha), \qquad \beta \in \tilde{\Delta}, n \in \mathbb{Z}, \alpha \in \Delta. \tag{21}$$

When discussing loop and affine algebras, a root α of the finite algebra is given by $(0,\, \alpha)$, $\alpha \in \Delta$. We may extend the scalar product $(\cdot|\cdot)$ to the root space of the loop algebra by defining the *null root* as $\delta = (1, 0)$. We require that $(\delta|\delta) = 0$ and $(\delta|\alpha) = 0$ for all $\alpha \in \widetilde{\Delta}$. Thus the scalar product is positive semidefinite and $(\alpha + n\delta|\alpha + n\delta) = (\alpha|\alpha)$ for all n. Although the extension $(\cdot|\cdot)$ may seem arbitrary, it is the only one that is useful in the subsequent theory. The set of roots of $\widetilde{\mathfrak{g}}$ is then

$$\widetilde{\Delta} = \bigcup_{n\in\mathbb{Z}}(\Delta + n\delta) = \Delta + \mathbb{Z}\delta. \tag{22}$$

The root system of the loop algebra is a union of real roots and imaginary roots,[10] $\widetilde{\Delta} = \widetilde{\Delta}^{\mathrm{re}} \cup \widetilde{\Delta}^{\mathrm{im}}$, with

$$\widetilde{\Delta}^{\mathrm{re}} = \{\alpha + n\delta|\ \alpha \neq 0,\ \ n \in \mathbb{Z}\}, \tag{23}$$

$$\widetilde{\Delta}^{\mathrm{im}} = \{(n\delta|\ n \in \mathbb{Z}\} = \mathbb{Z}\delta. \tag{24}$$

The root space decomposition of $\widehat{\mathfrak{g}}$ is essentially the same as for $\widetilde{\mathfrak{g}}$, the only difference being the addition of the central element to the Cartan subalgebra $\widetilde{\mathfrak{g}}^0$.

$$\begin{aligned} \widehat{\mathfrak{g}} &= \bigoplus_{\alpha\in\widehat{\Delta}} \widehat{\mathfrak{g}}^{\alpha}, \\ \widehat{\mathfrak{g}}^{\alpha} &= \widetilde{\mathfrak{g}}^{\alpha}, \qquad \text{if } \alpha \neq 0, \alpha \in \widehat{\Delta}, \\ \widehat{\mathfrak{g}}^{0} &= \widetilde{\mathfrak{g}}^{0} + \mathbb{C}\not{c}. \end{aligned} \tag{25}$$

For each $\alpha + n\delta = (n, \alpha) \in \widetilde{\Delta}^{\mathrm{re}}$, there is a subalgebra $\mathfrak{sl}_2^{\alpha+n\delta}$ defined as

$$t^{-n} \otimes \mathfrak{g}^{-\alpha} \oplus [t^{-n} \otimes \mathfrak{g}^{-\alpha}, t^{n} \otimes \mathfrak{g}^{\alpha}] \oplus t^{n} \otimes \mathfrak{g}^{\alpha}, \tag{26}$$

and the (linear) generators from these root spaces are isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

The loop algebra construction of the direct affine algebras yields a concise summary of the root system in terms of the roots of the finite dimensional Lie algebra $\mathfrak{g}$: one adjoint representation of $\mathfrak{g}$ occurs at each integer multiple of the null root δ. As an example consider the root system $\widetilde{\Delta}$ of $\widetilde{\mathfrak{su}}_3$ or ($\widehat{\Delta}$

[10]The null root δ is often called an imaginary root because it has zero norm. This however is a misnomer, as it has nothing to do with the imaginary numbers. It simply means that for the imaginary roots the scalar product is no longer positive definite, it is in fact zero.

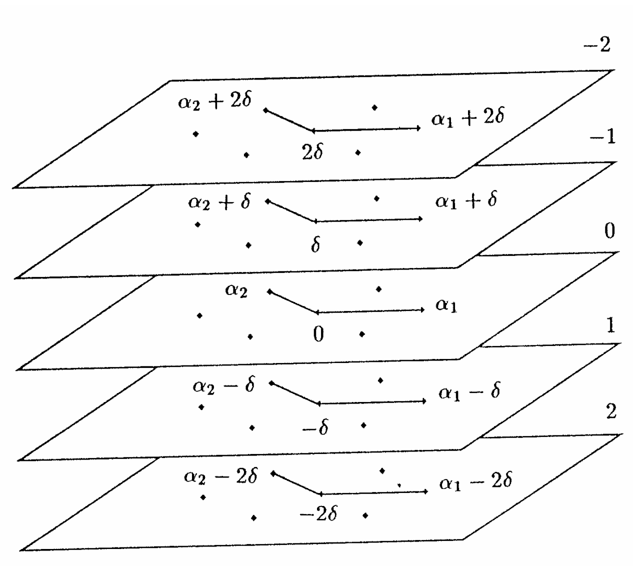

Figure 1: Root system of $\widetilde{\mathfrak{su}}_3$ or $\widehat{\mathfrak{su}}_3$ algebra. The stack of sheets continues indefinitely both upwards and downwards.

of $\widehat{\mathfrak{su}}_3$)[11], we can imagine an infinite stack of sheets with an $\mathfrak{su}_3$ root system drawn on each page, as shown in Figure 1. The n^{th} page in the stack contains the set of roots $\Delta(\mathfrak{su}_3) + \mathbb{Z}\delta$. The root $n\delta$ has multiplicity 2 on each sheet of the root system of $\widetilde{\mathfrak{su}}_3$, corresponding to the two Cartan generators. The same is true for $\widehat{\mathfrak{su}}_3$, with the exception that on sheet 0 the zero root has multiplicity 3, corresponding to the addition of the central element $\not{c}$, with $\alpha(\not{c}) = 0$, to the Cartan subalgebra of $\mathfrak{su}_3$.

2.5 Affine simple roots and the affine Cartan matrix

The next step is the identification of a basis of simple roots for the affine algebra. This basis must contain $d+1$ elements, d of which are necessarily the finite simple roots, whereas the remaining simple root must be a linear combination involving the null root δ. The proper choice for this extra simple

[11]The classical algebra A_d of *rank* d is often denoted as $\mathfrak{su}_{d+1}$ (special unitary) when referring to the compact real form of A_d . The name $\mathfrak{sl}$ (special linear) refers to the complexified $\mathfrak{su}$.

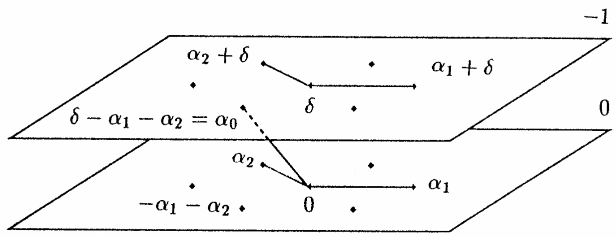

Figure 2: Simple roots of $\widehat{\mathfrak{su}}_3$ algebra are α_1, α_2 and $\alpha_0 = \delta - \alpha_1 - \alpha_2$.

root is $\alpha_0 = \delta - \theta$ with θ defined as the *highest root* of the finite root system Δ,[12]

$$\theta = \sum_{i=1}^{d} c_i \alpha_i, \tag{27}$$

for which $\sum_{i=1}^{d} c_i$ is maximum, where c_i are the *marks*. To give an example of what a root system looks like with the additional simple root α_0, let us again consider $\widehat{\mathfrak{su}}_3$ with the set of simple roots $\widehat{\Pi} = \{\alpha_0, \alpha_1, \alpha_2\}$ and highest root $\theta = \alpha_1 + \alpha_2$, see Figure 2.
According to (27) the null root may be written as

$$\delta = \sum_{i=0}^{d} c_i \alpha_i, \tag{28}$$

with c_0=1. Given a set of affine simple roots $\widehat{\Pi}$ and a scalar product, we can define the *direct affine Cartan matrix* as[13]

$$\widehat{A}_{ij} = \frac{2(\alpha_i|\alpha_j)}{(\alpha_j|\alpha_j)}, \qquad \alpha_i \in \widehat{\Pi}. \tag{29}$$

Note that the equation $(\delta|\alpha) = 0$ implies that for $\alpha \neq 0$,

$$\frac{2(\delta|\alpha_j)}{(\alpha_j|\alpha_j)} = \sum_{i=0}^{d} c_i \widehat{A}_{ij} = 0. \tag{30}$$

[12]Notice that in the affine case there is no highest root, (i.e. the adjoint representation is not a highest weight representation).
[13]This matrix is also often referred to as an *extended Cartan matrix*.

Since $\sum_{i=0}^{d} c_i$ is non zero, the matrix $\widehat{A}$ is singular and hence its determinant vanishes, $\det\widehat{A} = 0$.

In conclusion, we have successfully built the affine algebra $\widehat{\mathfrak{g}}$, starting from the finite Lie algebra $\mathfrak{g}$, via the Loop algebra construction. There exists however, another complementary method which begins with the Cartan matrix and a list of axioms which they must satisfy. By relaxing one of these axioms, namely that $\det\widehat{A} = 0$ instead of being strictly positive, the corresponding Lie algebras derived are precisely that of the affine algebras. Such that the two methods are in accord with each other.

2.6 Extended affine algebra

We now come to an important problem. In the finite dimensional case the roots are linear functionals on the Cartan subalgebra (via the Killing form) and are used to define $\mathfrak{g}$. We would like the same to be true for the affine case, such that a root of the affine algebra $\alpha + n\delta$ acts as a linear functional on the Cartan subalgebra $\widehat{\mathfrak{g}}^0$. However if we let $h \in \widehat{\mathfrak{g}}^0$ be a Cartan generator in the Chevalley basis, it follows that $\delta(h_i) = 0$ since $\delta(h_j) = \sum_{i=0}^{d} c_i\alpha_i(h_j) = \sum_{i=0}^{d} c_i\widehat{A}_{ij} = 0$ by (17), (28) and (30). Thus, $\delta(h) = 0$, and $(n\delta + \alpha)(h) = \alpha(h)$ for all n. This agrees with the previous use of roots, except that $(n\delta + \alpha)$ define the *same* functional no matter what n is. The $\delta(h) = 0$ problem is that there is *no operator* in $\widehat{\mathfrak{g}}^0$ that measures the value of n for the affine root $(n\delta + \alpha)$.

The $\delta(h) = 0$ problem can be solved by augmenting the Cartan subalgebra to include a so called *grading* operator L_0, for which $\delta(L_0) = -1$. The immediate purpose of L_0 is to measure the n value of a root $(n\delta + \alpha)$. We therefore define L_0 on $\widetilde{\mathfrak{g}}$ to act as $-t\frac{d}{dt}$ such that the Lie bracket is

$$[L_0, t^n \otimes x] = -nt^n \otimes x \qquad \forall x \in \widetilde{\mathfrak{g}}. \tag{31}$$

L_0 is a derivation on $\widetilde{\mathfrak{g}}$ and extends uniquely to a derivation of $\widehat{\mathfrak{g}}$ if we impose $L_0 \not{c} = 0$. Since $[L_0, \mathfrak{g}^0 + \mathbb{C}\not{c}] = 0$, we may define an *extended Cartan subalgebra*,

$$\widehat{\mathfrak{g}}^{e,0} = \widetilde{\mathfrak{g}}^0 \oplus \mathbb{C}\not{c} \oplus \mathbb{C}L_0. \tag{32}$$

The extended affine algebra is then

$$\widehat{\mathfrak{g}}^e = \widetilde{\mathfrak{g}} \oplus \mathbb{C}\not{c} \oplus \mathbb{C}L_0 = \widehat{\mathfrak{g}} \oplus \mathbb{C}L_0. \tag{33}$$

The Lie bracket of the elements of $\widehat{\mathfrak{g}}^e$ is

$$\begin{aligned}
&[t^m \otimes x + a\not{c} + bL_0, t^n \otimes y + c\not{c} + dL_0] \\
&= t^{m+n} \otimes [x,y] + m\delta_{m+n,0}k(x|y)\not{c} + bL_0(t^n \otimes y + c\not{c}) - dL_0(t^m \otimes x + a\not{c}), \\
&= t^{m+n} \otimes [x,y] + m\delta_{m+n,0}k(x|y)\not{c} - nbt^n \otimes y + mdt^m \otimes x, \qquad (34)
\end{aligned}$$

with $a, b, c, d \in \mathbb{C}$.
Let $x \in \widetilde{\mathfrak{g}}^{n\delta+\alpha}$, $(n, \alpha) \neq 0$. Then

$$\begin{aligned}
[h, x] &= \alpha(h)x, \qquad h \in \widehat{\mathfrak{g}}^{e,0},\ x \in \widehat{\mathfrak{g}}^{n\delta+\alpha},\ n\delta + \alpha \in \widehat{\Delta}, \\
[L_0, x] &= -nx, \qquad L_0 \in \widehat{\mathfrak{g}}^{e,0},\ x \in \widehat{\mathfrak{g}}^{n\delta+\alpha}. \qquad (35)
\end{aligned}$$

Thus, if we define $n\delta + \alpha$ as a linear functional on $\widehat{\mathfrak{g}}^{e,0}$,

$$(n\delta + \alpha)(h) = \alpha(h), \qquad \text{if } h \in \widehat{\mathfrak{g}}^{e,0}, \tag{36}$$

$$(n\delta + \alpha)(L_0) = -n, \tag{37}$$

then we summarise (35) by

$$[h, x] = (n\delta + \alpha)(h)x, \qquad x \in \widetilde{\mathfrak{g}}^{n\delta+\alpha}, h \in \widehat{\mathfrak{g}}^{e,0}. \tag{38}$$

Moreover, $(n\delta + \alpha)$ and α are distinguishable as functions on $\widehat{\mathfrak{g}}^{e,0}$, since

$$\delta(L_0) = -1, \qquad \delta(h) = 0 \qquad \text{for all } h \in \widehat{\mathfrak{g}}^0. \tag{39}$$

3 The $\mathfrak{sl}_2$ algebra

3.1 The loop algebra $\widetilde{\mathfrak{sl}}_2$

The *loop* algebra of $\mathfrak{sl}_2$ is

$$\widetilde{\mathfrak{sl}}_2 = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{sl}_2. \tag{40}$$

It is the Lie algebra of 2×2 traceless matrices with matrix elements in $\mathbb{C}[t, t^{-1}]$, the algebra of polynomials in t and t^{-1}.

For $x \in \mathfrak{sl}_2$ we may define $x_n = t^n \otimes x$, $n \in \mathbb{Z}$, and choose the following basis

$$e_n = \begin{pmatrix} 0 & t^n \\ 0 & 0 \end{pmatrix}, \qquad h_n = \begin{pmatrix} t^n & 0 \\ 0 & -t^n \end{pmatrix}, \qquad f_n = \begin{pmatrix} 0 & 0 \\ t^n & 0 \end{pmatrix}. \tag{41}$$

As n varies over $\mathbb{Z}$, these matrices span $\widetilde{\mathfrak{sl}}_2$.

3.2 The affine algebra $\widehat{\mathfrak{sl}}_2$

$\widetilde{\mathfrak{sl}}_2$ has a covering algebra $\widehat{\mathfrak{sl}}_2$.

- It has a one dimensional centre $\mathbb{C}\not{c}$.
- $\widetilde{\mathfrak{sl}}_2 \simeq \widehat{\mathfrak{sl}}_2/\mathbb{C}\not{c}$.

Similar to how the unitary groups cover the orthogonal groups, $\widehat{\mathfrak{sl}}_2$ is a central extension of $\widetilde{\mathfrak{sl}}_2$. Note however that $\widetilde{\mathfrak{sl}}_2$ is not a subalgebra of $\widehat{\mathfrak{sl}}_2$, since it is not closed under the Lie bracket of $\widehat{\mathfrak{sl}}_2$. Furthermore the Lie algebra $\widehat{\mathfrak{sl}}_2$ is not simple. The Lie bracket of $\widehat{\mathfrak{sl}}_2$ is defined by

$$[x_m, y_n] = [x, y]_{(m+n)} + m\delta_{m+n,0}k(x|y)\not{c}, \tag{42}$$

where $x_m \in \widetilde{\mathfrak{sl}}_2$ and $\mathrm{k}(\cdot|\cdot)$ is the Killing form on $\mathfrak{sl}_2$ (see (12)). In the root space decomposition of $\mathfrak{sl}_2$ we have $e \in \mathfrak{sl}_2^{\alpha_1}$, $f \in \mathfrak{sl}_2^{-\alpha_1}$, and $\alpha_1(h) = 2$ since the set of simple roots of $\mathfrak{sl}_2$ is $\Pi = \{\alpha_1\}$.

3.3 The root space decomposition of $\widehat{\mathfrak{sl}}_2$

The root space decomposition of $\widehat{\mathfrak{sl}}_2$ is

$$\widehat{\mathfrak{sl}}_2 = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{k=-1}^{1} \widehat{\mathfrak{sl}}_2^{k\alpha+n\delta}, \tag{43}$$

with

$$\begin{aligned} \widehat{\mathfrak{sl}}_2^{\alpha+n\delta} &= \mathbb{C}e_n, \\ \widehat{\mathfrak{sl}}_2^{-\alpha+n\delta} &= \mathbb{C}f_n, \end{aligned} \tag{44}$$

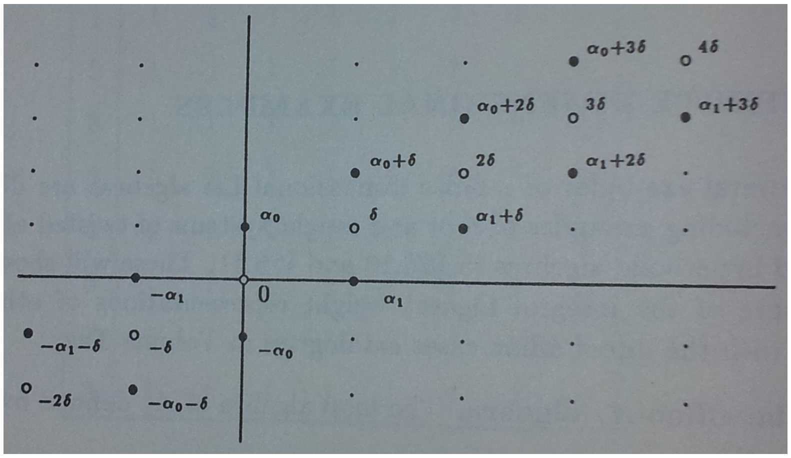

Figure 3: Roots of the $\widehat{\mathfrak{sl}}_2$ algebra. All roots have multiplicity 1 except for 0, which has multiplicity 2. Roots indicated by open circles are imaginary.

$$\widehat{\mathfrak{sl}}_2^{n\delta} = \begin{cases} \mathbb{C}h_n & \text{if } n \neq 0, \\ \mathbb{C}h + \mathbb{C}\not{c} & \text{if } n = 0. \end{cases} \tag{45}$$

The set of roots $\widehat{\Delta}$ of $\widehat{\mathfrak{sl}}_2$ is

$$\widehat{\Delta} = \{(n\delta - \alpha), (n\delta), (n\delta + \alpha) |\ n \in \mathbb{Z}\},$$

defining the extra simple root as $\alpha_0 := \delta - \alpha_1 = (1, -\alpha_1)$, the entire root system becomes

$$\{k\alpha_0 + n\alpha_1 |\ k, n \in \mathbb{Z}, |k - n| \leqslant 1\}, \tag{46}$$

and is shown in Figure 3. The roots $k\alpha_0 + n\alpha_1$, $|k - n| = 1$, are called real roots and the imaginary roots are the multiples of δ. The set of roots may be divided up as $\widehat{\Delta}_- \cup \{0\} \cup \widehat{\Delta}_+$, where

$$\widehat{\Delta}_+ = \{n\delta - \alpha_1 |\ n > 0\} \cup \{n\delta |\ n > 0\} \cup \{n\delta + \alpha_1 |\ n \geqslant 0\},$$

and $\widehat{\Delta}_- = -\widehat{\Delta}_+$. The roots in the upper right-hand quadrant of Figure 3 are those in $\widehat{\Delta}_+$. The decomposition (43) is said to be *triangular*, and can be written as

$$\widehat{\mathfrak{sl}}_2 = \widehat{\mathfrak{sl}}_{2+} \oplus \widehat{\mathfrak{sl}}_2^0 \oplus \widehat{\mathfrak{sl}}_{2-}, \qquad \widehat{\mathfrak{sl}}_{2\pm} = \bigoplus_{\alpha\in\hat{\Delta}_\pm} \widehat{\mathfrak{sl}}_2^\alpha, \tag{47}$$

where $\widehat{\mathfrak{sl}}_2^0$ is the Cartan subalgebra. The subalgebra $\widehat{\mathfrak{sl}}_{2+}$ is generated by two elements (as a Lie algebra), namely $e_0 = e$ and $f_1 = t \otimes f$. One has a similar situation for $\widehat{\mathfrak{sl}}_{2-}$, with the two elements given as $e_{-1} = t^{-1} \otimes e$ and $f_0 = f$. Finally we set $h_1 = h$, $h_0 = \not{c} - h$ and define the direct affine Cartan matrix as[14]

$$\widehat{A}_{ij} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{48}$$

The Lie algebra $\widetilde{\mathfrak{sl}}_2$, can now be defined in terms of the Cartan matrix by the following presentation on $\{e_i, f_i, h_i\}_{i=0,1}$

$$\begin{aligned} [h_i, h_j] &= 0, \\ [h_i, e_j] &= \alpha_j(h_i)e_j = \widehat{A}_{ji}e_j, \\ [h_i, f_j] &= -\alpha_j(h_i)f_j = -\widehat{A}_{ji}f_j, \\ [e_i, f_j] &= \delta_{ij}h_i, \end{aligned} \tag{49}$$

and is completed by the constraints (called the *Serre* relations),

$$\begin{aligned} \mathrm{ad}(e_i)^{1-\hat{A}_{ji}}e_j &= \mathrm{ad}(e_i)^3 e_j = 0, \qquad \text{if } i \neq j \\ \mathrm{ad}(f_i)^{1-\hat{A}_{ji}}f_j &= \mathrm{ad}(f_i)^3 f_j = 0, \qquad \text{if } i \neq j \end{aligned} \tag{50}$$

where, for example if $\widehat{A}_{ji} = -1$, then $\mathrm{ad}(e_i)^2 e_j = [e_i, [e_i, e_j]]$.

3.4 The extended $\widehat{\mathfrak{sl}}_2$ algebra

The Lie algebra presented above has only a two dimensional Cartan subalgebra. We must therefore extend the Cartan subalgebra by including the grading operator L_0. The Lie algebra $\widehat{\mathfrak{sl}}_2$ is then presented as above, but with $\{e_i, f_i, h_i, L_0\}_{i=0,1}$ and the additional relations

[14] Compared to the finite Cartan matrix, (which for $\mathfrak{sl}_2$ is just $A_{11} = 2$), $\widehat{A}_{ij}$ contains an extra row and column, corresponding to the extra simple root α_0.

$$
\begin{aligned}
[L_0, h_i] &= 0, \\
[L_0, e_0] &= \alpha_0(L_0)e_0 = e_0, \\
[L_0, f_0] &= \alpha_0(L_0)f_0 = -f_0, \\
[L_0, e_1] &= \alpha_1(L_0)e_1 = 0, \\
[L_0, f_1] &= \alpha_1(L_0)f_1 = 0,
\end{aligned} \tag{51}
$$

where $\alpha_i(L_0)$ is constrained by the convention in (39) that $\delta(L_0) = -1$.

4 Enveloping algebra $U(\mathfrak{g})$

The enveloping algebra $U(\mathfrak{g})$ is defined to be the unique solution to the following universal problem: $U(\mathfrak{g})$ is an associative unital algebra with a linear map $\rho : \mathfrak{g} \to U(\mathfrak{g})$ such that $\rho([x, y]) = [\rho(x), \rho(y)]$, and if U is another such algebra there exists a unique unital algebra map $\pi : U(\mathfrak{g}) \to U$ such that the following commutative diagram is satisfied.

$$
\begin{tikzcd}
\mathfrak{g} \arrow[r, "\rho"] \arrow[dr, "\rho'"'] & U\mathfrak{g} \arrow[d, dotted, "\pi"] \\
& U
\end{tikzcd}
$$

We define the enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ as follows. Let $T(\mathfrak{g})$ be a tensor algebra defined as [2]

$$
T(\mathfrak{g}) = \bigoplus_k \mathfrak{g}^{\otimes k}, \qquad k \in \mathbb{Z}. \tag{52}
$$

Let $I(\mathfrak{g})$ be the two-sided ideal of the tensor algebra $T(\mathfrak{g})$ generated by all elements of the form $x \otimes y - y \otimes x - [x, y]$ where $x, y \in \mathfrak{g}$. We define

$$
U(\mathfrak{g}) = T(\mathfrak{g})/I(\mathfrak{g}), \tag{53}
$$

such that we mod out $T(\mathfrak{g})$ by the algebra generated by the Lie bracket. So we can think of the universal enveloping algebra $U(\mathfrak{sl}_2)$ as all the formal powers and linear combinations of $e, f, h \in \mathfrak{sl}_2$ modulo the standard Lie algebra relations.

5 The Quantum group $U_q(\mathfrak{sl}_2)$

5.1 The $U_q(\mathfrak{sl}_2)$ algebra

$U_q(\mathfrak{sl}_2)$ is a one-parameter deformation of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$, which as we will see shortly, is also a Hopf algebra. When the parameter $q \in k$, with k the field, is not a root of unity, the algebra $U_q(\mathfrak{sl}_2)$ has properties that parallel those of the enveloping algebra of $\mathfrak{sl}_2$, $U(\mathfrak{sl}_2)$.

Let us fix an invertible element $q \in k$ different from 1 and -1, so that the fraction $1/(q - q^{-1})$ is well-defined. We introduce some notation that will come in handy in a later section. For any integer n, we define

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + q^{-n+3} + q^{-n+1}, \tag{54}$$

which satisfies

$$[-n] = -[n], \qquad [m+n] = q^n[m] + q^{-m}[n].$$

Observe that if q is not a root of unity, then $[n] \neq 0$ for any non-zero integer. We define $U_q(\mathfrak{sl}_2)$ to be the unital associative algebra over a field k (which in most cases will be $\mathbb{C}$), generated by four variables E, F, K, K^{-1} subject to the following relations: [3]

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1, \\ KEK^{-1} &= q^2 E, \\ KFK^{-1} &= q^{-2} F, \\ [E, F] &= \frac{K - K^{-1}}{q - q^{-1}}, \end{aligned} \tag{55}$$

where $q \in k \backslash \{0, 1, -1\}$.
Although impossible with the definition given above, with another presentation we can take the limit $q \to 1$. This is the so called classical limit, in the sense that the "quantum" group $U_q(\mathfrak{sl}_2)$ becomes the "classical" universal enveloping algebra $U(\mathfrak{sl}_2)$.

5.2 Hopf Algebra structure

To facilitate the following section we need to define some of the structure of a Hopf algebra H. A Hopf algebra H is a bialgebra $H, \Delta, \epsilon, m, \eta$ with a map, $S : H \to H$ (an algebra anti-homomorphism, i.e. $S(x \otimes y) = s(y)S(x)$ with $x, y \in H$), called the antipode. The axioms that make a simultaneous algebra and coalgebra into a Hopf algebra can be found in, for example [4], as can the maps that form part of the bialgebra (algebra and coalgebra). For our purposes however, we only need to define Δ, the coassociative map in the coalgebra C, (an algebra homomorphism) $\Delta : C \to C \otimes C$, called the coproduct. The coproduct and the antipode will be relevant for the following section.

5.3 Quantum affine $\mathfrak{sl}_2$: $U_q(\widehat{\mathfrak{sl}}_2)$

$U_q(\widehat{\mathfrak{sl}}_2)$ is the associated algebra over k with generators $X_i^{\pm}, K_i^{\pm}$ for $i = 0, 1$ subject to the following relations [5]

$$\begin{aligned}
K_i K_i^{-1} &= K_i^{-1} K_i = 1, \\
K_0 K_1 &= K_1 K_0, \\
K_i X_j^{\pm} &= q^{\pm \hat{A}_{ij}} X_j^{\pm} K_i, \\
[X_i^{+}, X_j^{-}] &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\
(X_i^{\pm})^3 X_j^{\pm} - [3](X_i^{\pm})^2 X_j^{\pm} X_i^{\pm} &+ [3] X_i^{\pm} X_j^{\pm} (X_i^{\pm})^2 - X_j^{\pm} (X_i^{\pm})^3 = 0,
\end{aligned} \tag{56}$$

see (54) for the $[n]$ notation. The above algebra is a Hopf Algebra with coproduct

$$\begin{aligned}
\Delta(K_i) &= K_i \otimes K_i, \\
\Delta(X_i^{+}) &= X_i^{+} \otimes K_i + 1 \otimes X_i^{+}, \\
\Delta(X_i^{-}) &= X_i^{-} \otimes 1 + K_i^{-1} \otimes X_i^{-}.
\end{aligned} \tag{57}$$

and antipode

$$\begin{aligned}
S(K_i) &= K_i^{-1}, \\
S(X_i^{+}) &= -X_i^{+} K_i^{-1}, \\
S(X_i^{-}) &= K_i X_i^{-}.
\end{aligned} \tag{58}$$

6 The Witt algebra and the Virasoro algebra

The Virasoro algebra $\mathfrak{V}$ is an infinite dimensional Lie algebra over $\mathbb{C}$. The algebra $\mathfrak{V}$ naturally arises in the theory of two dimensional conformal invariance. This is the reason why it is particularly interesting to theoretical physicists.

Let $\mathfrak{W}$ be an infinite dimensional vector space over $\mathbb{C}$ with a basis $\{L_m\}$, $m \in \mathbb{Z}$. The Witt algebra is the Lie algebra obtained from $\mathfrak{W}$ by defining[15]

$$[L_m, L_n] = (m-n)L_{m+n}, \qquad \text{for all } m, n \in \mathbb{Z}. \tag{59}$$

The Witt algebra arises in several areas of mathematics. For example if one considers the Lie algebra $\mathfrak{L}$ of vector fields on the unit circle $U = \{e^{i\theta} | \theta \in \mathbb{R}\}$. Then the subalgebra $\mathfrak{L}_{\text{fin}}$, of vector fields $f(\theta)d/d\theta$, for which f has a finite Fourier expansion, has a basis $L_n := ie^{in\theta}d/d\theta$, $n \in \mathbb{Z}$ which satisfies (59), such that $\mathfrak{L}_{\text{fin}} \sim \mathfrak{W}$.

The Virasoro algebra $\mathfrak{V}$ is a 1-dimensional central extension of $\mathfrak{W}$:

$$0 \longrightarrow \mathbb{C}\cent \longrightarrow \mathfrak{V} \longrightarrow \mathfrak{W} \longrightarrow 0. \tag{60}$$

This is an example of an exact sequence. It is a sequence of objects (here algebras and vector spaces) and morphisms, such that the image of one morphism equals the kernel of the next. The zero represents the zero-dimensional vector space. It forces the morphism between $\mathbb{C}\cent$ and $\mathfrak{V}$ to be an injective homomorphism and the morpshim between $\mathfrak{V}$ and $\mathfrak{W}$ to be a surjective homomorphism with kernel $\mathbb{C}\cent$.
As a vector space the Virasoro algebra is written as

$$\mathfrak{V} = \mathfrak{W} \oplus \mathbb{C}\cent. \tag{61}$$

The Lie bracket is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}m(m^2-1)\delta_{m+n,0}(x|y)\cent, \tag{62}$$

$$[\cent, L_m] = 0, \qquad m, n \in \mathbb{Z}. \tag{63}$$

[15]We may recall from section 2.6 that the definition of the grading operator was $L_0 = -t\frac{d}{dt}$, generalising this to $L_m = -t^{m+1}\frac{d}{dt}$ gives us the full Witt algebra.

Remark: The 1/12 factor seems to be somewhat arbitrary, since it may be absorbed into the definition of $\not{c}$. It is however, chosen in accordance with the regularisation of the Riemann-zeta function: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, $\zeta(-1) = -1/12$.
From (63) we see that $\mathbb{C}\not{c}$ lies in (actually equals) the centre of $\mathfrak{V}$ and $\mathfrak{V}/\mathbb{C}\not{c} \sim \mathfrak{W}$. As was the case for the affine algebras, $\mathfrak{V}$ is a universal central extension of $\mathfrak{W}$. Evidently $\mathfrak{V}$ is graded by $\mathbb{Z}$, since from (62), $[L_0, L_n] = -nL_n$:

$$\mathfrak{V} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{V}^n,$$

where

$$\begin{aligned} \mathfrak{V}^n &= \mathbb{C}L_{-n}, \qquad \text{if } n \neq 0, \\ \mathfrak{V}^0 &= \mathbb{C}L_0 + \mathbb{C}\not{c}. \end{aligned} \tag{64}$$

From $[L_0, L_n] = -nL_n$ it follows that $\mathfrak{V}^n$ is the n-eigenspace of adL_0. Moreover, $\mathfrak{V}$ carries an anti-linear anti-involution[16] σ:

$$\begin{aligned} \sigma L_n = L_n, n \in \mathbb{Z}, \\ \sigma\not{c} = \not{c}. \end{aligned} \tag{65}$$

We can decompose $\mathfrak{V}$ into three subalgebras

$$\mathfrak{V} = \mathfrak{V}_- \oplus \mathfrak{V}^0 \oplus \mathfrak{V}_+, \tag{66}$$

where

$$\mathfrak{V}_+ = \bigoplus_{n>0} \mathfrak{V}^n, \qquad \mathfrak{V}_- = \bigoplus_{n`,0} \mathfrak{V}^n,$$

thereby obtaining a triangular decomposition of $\mathfrak{V}$, with $\sigma\mathfrak{V}_\pm = \mathfrak{V}_\mp$.

The triangular decomposition is a common feature of the Heisenberg algebra, finite simple Lie algebras (e.g. A_1), Kac-Moody algebras, and as shown above the Virasoro algebra. As a consquence these algebras have many important properties in common, one such feature is the existence of the highest weight representation. As touched upon before, from a physical point of view this is very appealing, because it ensures that the system posses a vacuum state.

[16] Anti-linearity means $\sigma(cx) = \bar{c}\sigma(x)$, for all $x \in \mathfrak{V}$, $c \in \mathbb{C}$. Anti-involution means $\sigma([x, y]) = -[\sigma x, \sigma y]$ for x, $y \in \mathfrak{V}$ with $\sigma^2 x = x$.

6.1 Conformal field theory

The mathematical elegance and sophistication of loop and affine algebras is evident. Moreover, the explicit realisation of their representations in models of statistical systems and elementary particles, makes the study of these algebraic structures all the more interesting. To be more specific, if we were to investigate the properties of a conformally invariant two dimensional quantum field theory. Then we shall discover that, quite remarkably, the solutions are indeed representations of Kac-Moody algebras, of which the direct affine algebras are a specific example.

The formalism of a conformal field theory [6] becomes quite elegant when the two dimensional space-time manifold is coordinatized by a single complex variable. The solutions to the equations of motion are then typically holomorphic or antiholomorphic. A conformally invariant field theory is defined by a Lagrangian that is invariant under conformal mappings of the complex plane. In two dimensions, the Lie algebra of conformal transformations is infinite dimensional; it is the Virasoro algebra. Furthermore, if the Lagrangian has an additional finite dimensional Lie symmetry, then the quantum solution will be a representation of the affine Lie algebra.

The archetypal example of a conformal field theory is string theory, [7] where a vibrating string sweeps out a two dimensional world-sheet in space-time. In fact, any two dimensional theory of gravity that enjoys both diffeomorphism and Weyl invariance will reduce to a conformally invariant theory, when the background metric is assumed not to be dynamical, but fixed. Conformal field theory however, has many applications outside of string theory. Most notably in statistical physics, where it offers a description of critical phenomena. This is because at the critical point, the system has scale invariance, and in certain theories, conformal invariance is a consequence of scale and Poincaré invariance.

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