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THE STRONG OPERATOR TOPOLOGY ON $\mathscr{B}(\mathbf{H})$ AND THE DOUBLE COMMUTANT THEOREM

ABSTRACT. These are the notes for a presentation on the strong and weak operator topologies on $\mathscr{B}(\mathbf{H})$ and on commutants of unital self-adjoint subalgebras of $\mathscr{B}(\mathbf{H})$ in the seminar on von Neumann algebras in Utrecht. The main goal for this talk was to prove the double commutant theorem of von Neumann. We will also give a proof of Vigiers theorem and we will work out several useful properties of the commutant.

Recall that a seminorm on a vector space $\mathbf{V}$ is a map $p : \mathbf{V} \to [0,\infty)$ with the properties that (i) $p(\lambda x) = |\lambda| p(x)$ for every vector $x \in \mathbf{V}$ and every scalar λ and (ii) $p(x+y) \leq p(x) + p(y)$ for every pair of vectors $x, y \in \mathbf{V}$. If $\mathscr{P}$ is a family of seminorms on $\mathbf{V}$ there is a topology generated by $\mathscr{P}$ of which the subbasis is defined by the sets

$$\{v \in \mathbf{V} : p(v-x) < \varepsilon\},$$

where $\varepsilon > 0$, $p \in \mathscr{P}$ and $x \in \mathbf{V}$. Hence a subset U of $\mathbf{V}$ is open if and only if for every $x \in U$ there exist $p_1, \ldots, p_n \in \mathscr{P}$, and $\varepsilon > 0$ with the property that

$$\bigcap_{i=1}^{n} \{v \in \mathbf{V} : p_i(v-x) < \varepsilon\} \subset U.$$

A family $\mathscr{P}$ of seminorms on $\mathbf{V}$ is called separating if, for every non-zero vector x, there exists a seminorm p in $\mathscr{P}$ such that $p(x) \neq 0$. The topology generated by a separating family of seminors is always Hausdorff.

Definition 1. Suppose that $\mathbf{H}$ is a Hilbert space. The weak operator topology on $\mathscr{B}(\mathbf{H})$ is the topology generated by collection $\{A \mapsto |\langle A(x), y\rangle| : x, y \in \mathbf{H}\}$ of seminorms.

Lemma 2. *For every net $\{A_i : i \in I\}$ in $\mathscr{B}(\mathbf{H})$ we have that $\{A_i\}$ converges in the weak operator topology to A if and only if $\langle A_i(x), y\rangle \to \langle A(x), y\rangle$ for all $x, y \in \mathbf{H}$.*

Proof. Suppose that the net $\{A_i : i \in I\}$ converges weakly to A. Then, for every open set U there exists $i \in I$ such that $A_j \in U$ whenever $j \geq i$ in I. In particular, for every $x, y \in \mathbf{H}$ and $\varepsilon > 0$ we can take $U_{x,y,\varepsilon} := \{B \in \mathscr{B}(\mathbf{H}) : |\langle (A-B)(x), y\rangle| < \varepsilon\}$. Since for every $\varepsilon > 0$ there exists an $i_\varepsilon \in I$ with the property that $A_j \in U_{x,y,\varepsilon}$ whenever $j \geq i_\varepsilon$, we see that the net $\{|\langle (A-A_i)(x), y\rangle| : i \in I\}$ converges to 0. And hence $\langle A_i(x), y\rangle \to \langle A(x), y\rangle$ for all $x, y \in \mathbf{H}$.

Suppose now that $\langle A_i(x), y\rangle \to \langle A(x), y\rangle$ for all $x, y \in \mathbf{H}$ and suppose that $U \subset \mathscr{B}(\mathbf{H})$ is weakly open with $A \in U$. Then there are $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbf{H}$ and $\varepsilon > 0$ with the property that

$$\bigcap_{k=1}^{n}\{B \in \mathscr{B}(\mathbf{H}) : |\langle (A-B)(x_k), y_k\rangle| < \varepsilon\} \subset U$$

By assumption there are $i_1, \ldots, i_n \in I$ with the property that $|\langle (A-A_j)(x_k), y_k\rangle| < \varepsilon$ for all $j \geq i_k$. This implies that $A_j \in U$ for all $j \geq \{i_1, \ldots, i_n\}$. and hence that $A_i \to A$ in the weak operator topology. $\square$

Definition 3. The strong operator topology on $\mathscr{B}(\mathbf{H})$ is the topology generated by the collection $\{A \mapsto \|A(x)\| : x \in \mathbf{H}\}$.

The strong operator topology is the topology on $\mathscr{B}(\mathbf{H})$ in which convergence is equivalent to pointwise convergence:

Lemma 4. *A net $\{A_i : i \in I\}$ in $\mathscr{B}(\mathbf{H})$ converges to A in the strong operator topology if and only if $A_i(x) \to A(x)$ for all $x \in \mathbf{H}$.*

Proof. Suppose that $A_i \to A$ in the strong operator topology. Then, for every $x \in \mathbf{H}$ and for every $\varepsilon > 0$, there exists an $i \in I$ with the property that $A_j \in \{B \in \mathscr{B}(\mathbf{H}) : \|(A-A_j)(x)\| < \varepsilon\}$ for all $j \geq i$ in I, which shows that $\|(A-A_j)(x)\| \to 0$ and hence that $A_j(x) \to A(x)$.

On the other hand, suppose that $A_i(x) \to A(x)$ for all $x \in \mathbf{H}$ and let U be a strongly open subset of $\mathscr{B}(\mathbf{H})$ which contains A. Then there are $x_1, \ldots, x_n \in \mathbf{H}$ and $\varepsilon > 0$ such that

$$\bigcap_{k=1}^{n}\{B \in \mathscr{B}(\mathbf{H}) : \|(A-B)(x_k)\| < \varepsilon\} \subset U.$$

By assumption, there are $i_1, \ldots, i_k \in I$ with the property that $\|(A-A_j)(x_k)\| < \varepsilon$ whenever $j \geq i_k$. Hence for $j \geq \{i_1, \ldots, i_k\}$ it follows that $A_j \in U$ and we conclude that $A_i \to A$ in the strong operator topology. $\square$

Lemma 5. *The weak operator topology is weaker than the strong operator topology and the strong operator topology is weaker than the uniform topology on $\mathscr{B}(\mathbf{H})$.*

Proof. Suppose first that U weakly open. Then we can find for every operator A in U elements $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbf{H}$ and $\varepsilon > 0$ such that

$$\bigcap_{i=1}^{n}\{B \in \mathscr{B}(\mathbf{H}) : |\langle A(x_i) - B(x_i), y_i\rangle| < \varepsilon\} \subset U.$$

Without loss of generality we can assume that each y_i is non-zero. Since $|\langle A(x_i) - B(x_i)\rangle| \leq \|A(x_i) - B(x_i)\|\|y_i\|$ it follows that

$$\begin{aligned}\{B \in \mathscr{B}(\mathbf{H}) : \|A(x_i) - B(x_i)\| \leq \tfrac{\varepsilon}{\|y_i\|}\} &= \{B \in \mathscr{B}(\mathbf{H}) : \|A(x_i) - B(x_i)\|\|y_i\| \leq \varepsilon\} \\ &\subset \{B \in \mathscr{B}(\mathbf{H}) : |\langle A(x_i) - B(x_i), y_i\rangle| < \varepsilon\}\end{aligned}$$

Hence if we take $\delta := \min\{\frac{\varepsilon}{\|y_i\|} : 1 \leq i \leq n\}$ we see that

$$\bigcap_{i=1}^{n}\{B \in \mathscr{B}(\mathbf{H}) : \|A(x_i) - B(x_i)\| \leq \delta\} \subset U$$

and hence that U is open in the strong operator topology.

The same trick works to show that the strong operator topology is weaker than the uniform topology on $\mathscr{B}(\mathbf{H})$. Indeed, we have the inequality $\|A(x_i) - B(x_i)\| \leq \|A - B\|\|x_i\|$ and therefore we have the inclusion

$$\{B \in \mathscr{B}(\mathbf{H}) : \|A - B\|\|x\| < \varepsilon\} \subset \{B \in \mathscr{B}(\mathbf{H}) : \|A(x) - B(x)\| < \varepsilon\}$$

for all $x \in \mathbf{H}$ and $\varepsilon > 0$. If U is open in the strong operator topology we can find $x_1, \ldots, x_n \in \mathbf{H}$ and $\varepsilon > 0$ for each bounded operator A, with the property that

$$\bigcap_{i=1}^{n} \{B \in \mathscr{B}(\mathbf{H}) : \|A(x_i) - B(x_i)\| < \varepsilon\} \subset U.$$

We can safely assume that each x_i is non-zero. Taking $\delta = \min\{\frac{\varepsilon}{\|x_i\|} : 1 \leq i \leq n\}$ it follows from the mentioned inclusion that $\bigcap_{i=1}^{n} \{B \in \mathscr{B}(\mathbf{H}) : \|A - B\| < \delta\} \subset U$. □

The following theorem basically says that an increasing net of positive operators has a least upper bound and converges to it whenever it has an upper bound:

Theorem 6 (Vigiers theorem)**.** *Suppose that $\{A_\lambda : \lambda \in I\}$ is a net of self-adjoint operators on a Hilbert space $\mathbf{H}$. If $A_\kappa - A_\lambda$ is positive for all $\lambda \leq \kappa$ and if there is an $M \in \mathbb{R}$ such that $\|A_\lambda\| \leq M$ for all $\lambda \in I$, then $\{A_\lambda\}$ is strongly convergent.*

Proof. Note that we can always pick $\lambda_0 \in I$ and look at the net $\{A_\lambda - A_{\lambda_0} : \lambda \geq \lambda_0 \in I\}$, so without loss of generality we can assume that the net $\{A_\lambda\}$ consists of positive operators. Hence the net $\{\langle A_\lambda(x), x\rangle : \lambda \in I\}$ is increasing and bounded above by $M\|x\|^2$ and therefore the net $\{\langle A_\lambda(x), x\rangle\}$ is convergent for each $x \in \mathbf{H}$. For any $x, y \in \mathbf{H}$ the polarisation identity

$$\langle A_\lambda(x), y\rangle = \sum_{k=0}^{3} i^k \langle A_\lambda(x + i^k y), x + i^k y\rangle$$

gives us that the net $\{\langle A_\lambda(x), y\rangle\}$ is also convergent. Denote its limit by $\sigma(x, y)$; one can verify that $\sigma : \mathbf{H} \times \mathbf{H} \to \mathbf{H}$ defines a sesquilinear form on $\mathscr{H}$ which is bounded by $|\sigma(x, y)| \leq M\|x\|\|y\|$. Hence there is a bounded operator A for which $\sigma(x, y) = \langle A(x), y\rangle$ and $\|A\| = \|\sigma\|$. A is a positive operator, larger than any A_λ, since for every $\lambda \in I$ and for every $\varepsilon > 0$ there is a $\lambda_0 \geq \lambda \in I$ with the property that $|\langle (A - A_\kappa)(x), x\rangle| < \varepsilon$ whenever $\kappa \geq \lambda_0$ and

$$\langle A(x), x\rangle - \langle A_\lambda(x), x\rangle \geq \langle A_\kappa(x), x\rangle - \langle A_\lambda(x), x\rangle - \varepsilon.$$

Since $\langle A_\kappa(x), x\rangle \geq \langle A_\lambda(x), x\rangle \geq 0$ it follows, by taking ε smaller and smaller, that A is positive and $A_\lambda \leq A$ for all λ. In particular we can take the positive square root of $A - A_\lambda$ for all $\lambda \in I$. So we have

$$\begin{aligned}\|A(x) - A_\lambda(x)\|^2 &= \|(A - A_\lambda)^{\frac{1}{2}}(A - A_\lambda)^{\frac{1}{2}}(x)\|^2 \\ &\leq \|A - A_\lambda\|\|(A - A_\lambda)^{\frac{1}{2}}(x)\|^2 \\ &\leq 2M\langle (A - A_\lambda)(x), x\rangle \to 0\end{aligned}$$

We conclude that A_i converges pointwise (hence strongly) to A. □

Definition 7. The commutant $\mathcal{S}'$ of a subset $\mathcal{S}$ of an algebra $\mathscr{A}$ is the set

$$\{a \in \mathscr{A} : as - sa = 0 \text{ for all } s \in \mathcal{S}\}.$$

Since every element of $\mathcal{S}$ commutes with every element of $\mathcal{S}'$ we have the inclusion $\mathcal{S} \subset \mathcal{S}''$ (and also $\mathcal{S}' \subset \mathcal{S}'''$). Also, it is easy to see that if $\mathcal{S} \subset T$ then $T' \subset \mathcal{S}'$ and hence we have $\mathcal{S}''' \subset \mathcal{S}'$. Therefore we have the identity

$$\mathcal{S}' = \mathcal{S}''' \tag{1}$$

for every subset $\mathcal{S}$ of an algebra $\mathscr{A}$.

Now suppose that $\mathscr{A}$ is a unital algebra and consider the map $\varphi : \mathscr{A} \to M_n(\mathscr{A})$, from $\mathscr{A}$ to the $n \times n$ matrices with coefficients in $\mathscr{A}$, which is defined by $\varphi(a) = a\mathbf{1}$ (the matrix with

a on every diagonal entry and zero everywhere else). This defines a unital homomorphism and we have the following lemma concerning commutants:

Lemma 8. *For $\varphi : \mathscr{A} \to M_n(\mathscr{A})$ as above and $\mathscr{B}$ a unital subalgebra of $\mathscr{A}$ we have*

i. $\varphi(\mathscr{B})' = M_n(\mathscr{B}')$

ii. $\varphi(\mathscr{B})'' = \varphi(\mathscr{B}'')$.

Proof. For the first assertion, suppose that $x \in \mathscr{B}$ and that $M \in M_n(\mathscr{A})$, then

$$(\varphi(X)M)_{ij} = \sum_k \varphi(X)_{ik} M_{kj} = XM_{ij} \quad \text{and} \quad (M\varphi(X))_{ij} = \sum_k M_{ik}\varphi(X)_{kj} = M_{ij}X$$

and therefore we see that M is in the commutant of $\varphi(\mathscr{B})$ if and only if every coefficient M_{ij} is an operator in the commutant $\mathscr{B}'$ of $\mathscr{B}$.

For the second assertion, let E_{ij} be the matrix-unit with the unit 1 of $\mathscr{A}$ at the ij-th coefficient and zero everywhere else. Suppose that $A_{ij} \neq 0$ for some i and j with $i \neq j$ and some $A \in M_n(\mathscr{A})$. Then, taking $M = E_{ji}$ — pay attention to the order of the indices and note that $M \in \varphi(\mathscr{B})'$ — we see that

$$(AM)_{ii} = \sum_k A_{ik} M_{ki} = A_{ij} \quad \text{while} \quad (MA)_{ii} = \sum_k M_{ik} A_{ki} = 0$$

and hence that such A cannot be in the commutant of $\varphi(\mathscr{A})'$ whenever any of its off-diagonal coefficients are non-zero. If we choose $M = E_{ij}$, then for all diagonal matrices A we have

$$(AM)_{ij} = \sum_k A_{ik} M_{kj} = A_{ii} \quad \text{while} \quad (MA)_{ij} = \sum_k M_{ik} A_{kj} = A_{jj}.$$

It follows that A is of the form $\varphi(x)$ for some $x \in \mathscr{A}$ whenever $A \in \varphi(\mathscr{A})''$. Suppose now that $M = bE_{ii}$ for some $b \in \mathscr{B}'$. Then $(M\varphi(x))_{pq} = M_{pq}X$, which is zero except when $p = q = i$. So $M\varphi(x) = (bx)E_{ii}$; similarly we see that $\varphi(x)M = (xb)E_{ii}$. We see that $\varphi(x) \in \varphi(\mathscr{B})''$ if and only if $x \in \mathscr{B}''$, which concludes the proof. $\square$

Lemma 9. *For a Hilbert space* $\mathbf{H}$ *and a subset* $\mathcal{S}$ *of* $\mathscr{B}(\mathbf{H})$ *the commutant* $\mathcal{S}'$ *is always weakly closed.*

Proof. Suppose that $\{A_i : i \in I\}$ is a net in $\mathcal{S}'$ which converges to A in the weak operator topology. We will show that $A \in \mathcal{S}'$. Let $X \in \mathcal{S}$ and note that $XA = AX$ if and only if $\langle XA(x), y\rangle = \langle AX(x), y\rangle$ for all $x, y \in \mathbf{H}$. By assumtion we have $\langle A(X(x)), y\rangle = \lim_i \langle A_i(X(x)), y\rangle$ and also we have

$$\langle XA(x), y\rangle = \langle A(x), X^*(y)\rangle = \lim_i \langle A_i(x), X^*(y)\rangle = \lim_i \langle XA_i(x), y\rangle$$

Since $XA_i = A_iX$ for all $i \in \mathscr{I}$ we have $\langle AX(x), y\rangle = \lim_i \langle A_iX(x), y\rangle = \lim_i \langle XA_i(x), y\rangle = \langle XA(x), y\rangle$, and hence that $A \in \mathcal{S}'$. $\square$

Since the weak topology is weaker than the strong topology we have:

Corollary 10. *For a Hilbert space* $\mathbf{H}$ *and a subset* $\mathcal{S}$ *of* $\mathscr{B}(\mathbf{H})$ *the commutant* $\mathcal{S}'$ *of* $\mathcal{S}$ *is always strongly closed.*

Before stating the bicommutant theorem let us verify a useful property of unital self-adjoint subalgebras of $\mathscr{B}(\mathbf{H})$:

Lemma 11. *Suppose* $\mathscr{A}$ *is a self-adjoint algebra of linear operators on* $\mathbf{H}$ *and let* $\mathbf{K}$ *be a closed subspace of* $\mathbf{H}$*. The following are equivalent:*

i. $\mathscr{A}(\mathbf{K}) \subset \mathbf{K}$

ii. $\mathscr{A}(\mathbf{K}^{\perp}) \subset \mathbf{K}^{\perp}$
iii. $[\mathscr{A}, P_{\mathbf{K}}] = 0$.

A subspace $\mathbf{K}$ *of* $\mathbf{H}$ *with either of these properties is called reducing (with respect to* $\mathscr{A}$*).*

Proof. Suppose that $\mathscr{A}(\mathbf{K}) \subset \mathbf{K}$, i.e. that $A(y) \in \mathbf{K}$ for all $A \in \mathscr{A}$ and $y \in \mathbf{K}$, let $A \in \mathscr{A}$ and let $x \in \mathbf{K}^{\perp}$, $y \in \mathbf{K}$. Then $\langle y, A(x)\rangle = \langle A^*(y), x\rangle = 0$. Since $A^* \in \mathscr{A}$ we see that $A(x) \in \mathbf{K}^{\perp}$, so $\mathscr{A}(\mathbf{K}^{\perp}) \subset \mathbf{K}^{\perp}$. The assertion that $\mathscr{A}(\mathbf{K}^{\perp}) \subset \mathbf{K}^{\perp}$ implies that $\mathscr{A}(\mathbf{K}) \subset \mathbf{K}$ follows from the fact that $\mathbf{K} = \mathbf{K}^{\perp\perp}$.

Suppose again that $\mathscr{A}(\mathbf{K}) \subset \mathbf{K}$, let $A \in \mathscr{A}$ and let $x \in \mathbf{H}$. Then

$$\begin{aligned} A(P_{\mathbf{K}}(x)) - P_{\mathbf{K}}(A(x)) &= A(P_{\mathbf{K}}(x)) - P_{\mathbf{K}}(A(P_{\mathbf{K}}(x) + P_{\mathbf{K}^{\perp}}(x))) \\ &= A(P_{\mathbf{K}}(x)) - P_{\mathbf{K}}(A(P_{\mathbf{K}}(x))), \end{aligned}$$

which is zero and therefore $[A, P_{\mathbf{K}}] = 0$. As this is true for all $A \in \mathscr{A}$ we see that $[\mathscr{A}, P_{\mathbf{K}}] = 0$. For the last part, suppose that $[\mathscr{A}, P_{\mathbf{K}}] = 0$, let $x \in \mathbf{K}$ and let $y \in \mathbf{K}^{\perp}$. Then

$$\langle A(y), x\rangle = \langle A(y), P_{\mathbf{K}}(x)\rangle = \langle P_{\mathbf{K}}(A(y)), x\rangle = \langle A(P_{\mathbf{K}}(y)), x\rangle = 0$$

for all $A \in \mathscr{A}$ and hence we see that $A(\mathbf{K}^{\perp}) \subset \mathbf{K}^{\perp}$ for all $A \in \mathscr{A}$, so $\mathscr{A}(\mathbf{K}^{\perp}) \subset \mathbf{K}^{\perp}$. □

Theorem 12 (The double commutant theorem of von Neumann)**.** *Suppose that* $\mathbf{H}$ *is a Hilbert space. Then* $\mathscr{A}''$ *is the strong closure of* $\mathscr{A}$ *for every unital self-adjoint subalgebra* $\mathscr{A}$ *of* $\mathscr{B}(\mathbf{H})$.

The proof requires the following observation:

Lemma 13. *Suppose* $\mathscr{A}$ *is a unital self-adjoint subalgebra of* $\mathscr{B}(\mathbf{H})$*. For every* $A \in \mathscr{A}''$ *and every* $x \in \mathbf{H}$ *there is a net* $\{A_i : i \in I\} \subset \mathscr{A}$ *such that* $A_i(x) \to A(x)$.

Proof. Suppose that $A \in \mathscr{A}''$, that $x \in \mathbf{H}$ and that $\mathbf{K} := cl\{X(x) : X \in \mathscr{A}\} \leq \mathbf{H}$. Then $\mathscr{A}(\mathbf{K}) \subset \mathbf{K}$ and hence by lemma 11 it follows that every operator in $\mathscr{A}$ commutes with the projection $P_{\mathbf{K}}$ and hence that $[A, P_{\mathbf{K}}] = 0$. Since the identity $\mathrm{id}_{\mathbf{H}}$ is an element of $\mathscr{A}$ we see that $x \in \mathbf{K}$ and therefore that $A(x) = A(P_{\mathbf{K}}(x)) = P_{\mathbf{K}}(A(x)) \in \mathbf{K}$ as well. Since $\mathbf{K}$ is closed it follows that there is a net $\{X_i : i \in I\}$ of operators in $\mathscr{A}$ with the property that $X_i(x) \to A(x)$ as $i \to \infty$. □

Proof of theorem 12. Suppose that $A \in \mathscr{A}''$ and let $\mathcal{W}$ be a strong neighborhood of A. If we can show that $\mathcal{W} \cap \mathscr{A} \neq \varnothing$ it follows that $\mathscr{A}''$ is the strong closure of $\mathscr{A}$. Since $\mathcal{W}$ is a strong neighborhood of A there exist $x_1, \ldots, x_n \in \mathbf{H}$ and $\varepsilon > 0$ such that

$$\bigcap_{i=1}^{n} \{X \in \mathscr{B}(\mathbf{H}) : \|(A - X)(x_i)\| < \varepsilon\} \subset \mathcal{W}.$$

Now consider the map $\varphi : \mathscr{B}(\mathbf{H}) \to M_n(\mathscr{B}(\mathbf{H})) = \mathscr{B}(\mathbf{H}^n)$ as in lemma 8. Then φ is a unital $*$-homomorphism and $\varphi(A)$ commutes with all the $n \times n$ matrices with coefficients in $\mathscr{A}'$. Also, we can apply lemma 13 in the situation where we take the Hilbert space $\mathbf{H}^n$ in place of $\mathbf{H}$ to see that there is a net $\{A_i : i \in I\} \subset \mathscr{A}$ with the property that $\varphi(A_i)(x_1, \ldots, x_n) \to \varphi(A)(x_1, \ldots, x_n)$, which is equivalent with the assertion that $A_i(x_j) \to A(x_j)$ for $1 \leq j \leq n$. In particular, there is an $i_0 \in I$ with the property that $\|(A - A_i)(x_j)\| < \varepsilon$ for all $1 \leq j \leq n$ and $i \geq i_0$. Hence we see that $A_i \in \mathcal{W}$ for $i \geq i_0$, and the theorem is proven. □

To summarize, the double commutant of a unital self-adjoint subalgebra of $\mathscr{B}(\mathbf{H})$ is always weakly closed, this is immediate from lemma 9. Weakly closed sets are also strongly closed by lemma 5 and finally the double commutant theorem, theorem 12, revealed that the strongly closed unital self-adjoint subalgebras of $\mathscr{B}(\mathbf{H})$ are always their own double commutant. So we have

Corollary 14. *For a unital self-adjoint subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathbf{H})$ the following are equivalent:*

i. $\mathscr{A}$ *is weakly closed,*
ii. $\mathscr{A}$ *is strongly closed,*
iii. $\mathscr{A} = \mathscr{A}''$.

If $\mathscr{A}$ satisfies either of these conditions we say that $\mathscr{A}$ is a von Neumann algebra on $\mathbf{H}$.

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