Chiral conformal field theory

Course notes

André Henriques, spring 2018

Contents

Introduction

There are two things which go by the name "conformal field theory" (CFT), and which are quite distinct: Chiral conformal field theories and Full conformal field theories. A chiral conformal field theory is not a full conformal field theory, and a full conformal field theory is not a chiral conformal field theory. Instead, they are related by constructions

$$\begin{tikzcd} \left\{\begin{array}{c}\text{Chiral conformal}\\ \text{field theories}\end{array}\right\} \arrow[rr, "\text{diagonal}"] & & \left\{\begin{array}{c}\text{Full conformal}\\ \text{field theories}\end{array}\right\} \arrow[ll, bend right, "\text{anti-chiral part}"'] \arrow[ll, bend left, "\text{chiral part}"] \arrow[loop right, "\begin{array}{l}\text{complex}\\ \text{conjugation}\end{array}"] \end{tikzcd}$$

In these notes, we will be dealing with chiral conformal field theories only.

There exist three mathematical formalizations of the concept of chiral conformal field theory:

$$\chi\text{CFT} \begin{array}{l} \nearrow \bullet \text{ Vertex operator algebras,} \\ \rightarrow \bullet \text{ Conformal nets, and} \\ \searrow \bullet \text{ Segal CFTs.} \end{array}$$

Terminology warning: Whereas the term 'vertex operator algebra' (VOA) unambiguously refers to chiral CFTs, there exist variants of the notions of conformal net and of Segal CFT which model the notion of full CFT. In order to avoid any ambiguity, it is therefore preferable to use the terminology 'chiral conformal net' and 'chiral Segal CFT'. (There also exits a variant of the notion of vertex operator algebra which formalizes full CFTs, and which goes by the name 'full field algebra'.)

Note: Graeme Segal himself uses the term 'weakly conformal CFT' for what we call here a chiral Segal CFT.

The notions of vertex operator algebra, of chiral conformal net, and of chiral Segal CFT are expected/conjectured to be equivalent, provided appropriate qualifiers are added. Note that these notions cannot be completely equivalent because:

- *Unitarity* is built into conformal nets, but not into VOAs, nor Segal CFTs.
- *Rationality* is built into Segal CFTs, but not into VOAs, nor conformal nets (rationality is a certain finiteness condition that a chiral CFT might or might not satisfy).
- There exists a certain equivalence between Segal CFTs called *infinitesimal equivalence*. Infinitesimally equivalent Segal CFTs model the same physics and should therefore be treated as 'the same'.

We propose:

Conjecture 1 *(i) There is a bijection*

$$\textit{unitary VOAs} \quad \Leftrightarrow \quad \textit{chiral conformal nets.}$$

(ii) There is a bijection

$$\textit{rational VOAs} \quad \Leftrightarrow \quad \begin{array}{c}\textit{chiral Segal CFTs up to}\\ \textit{infinitesimal equivalence.}\end{array}$$

(iii) There is a bijection

$$\textit{rational conformal nets} \quad \Leftrightarrow \quad \textit{unitary chiral Segal CFTs.}$$

There exist a couple of constructions in the literature which connect VOAs, chiral conformal nets, and chiral Segal CFTs. But these constructions only work in special cases, and it is fair to say that the above conjecture is wide open.

These notes will be focusing mostly on chiral Segal CFTs, which is the *least developed* of the above three mathematical formalizations of chiral CFT (for example, the only chiral CFTs which have been constructed so far, or proven to exist as Segal CFTs are free field CFTs). But this is also, in some sense, the most powerful one of the above three formalisms, and we expect that it should be easy (in comparison to other constructions) to construct a VOA or a conformal net from a chiral Segal CFT.

Complex cobordisms

The definition of Segal CFT (from now on, 'Segal CFT' = 'chiral Segal CFT') is based on the notion of *complex cobordism*.

Before talking about complex cobordisms, let us first describe the notion of a *Riemann surface with boundary*. Let $\mathbb{H} := \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) \geq 0\}$ be the complex upper half plane, and let $\mathring{\mathbb{H}} := \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$ be its interior.

Definition: *A Riemann surface with boundary is a ringed space* $(\Sigma, \mathcal{O}_\Sigma)$ *which is locally isomorphic to* $(\mathbb{H}, \mathcal{O}_\mathbb{H})$*, where* $\mathcal{O}_\mathbb{H}$ *is the sheaf on* $\mathbb{H}$ *given by*

$$\mathcal{O}_\mathbb{H}(U) := \left\{ f : U \to \mathbb{C} \;\middle|\; \begin{array}{l} f|_{U\cap\mathring{\mathbb{H}}} \textit{ is holomorphic,} \\ \exists\, V \subset \mathbb{C} \textit{ open and } g \in C^\infty(V) \textit{ s.t. } f = g|_U \end{array} \right\}$$

for $U \subset \mathbb{H}$ *an open subset.*

By a classical result known as Borel's lemma, for an open subset $U \subset \mathbb{H}$, the condition that a function $f : U \to \mathbb{C}$ be the restriction a C^∞ function defined on some open $V \subset \mathbb{C}$ is equivalent to f being smooth all the way to the boundary. Here, 'smooth all the way to the boundary' is just the usual notion of smoothness, adapted to the case of manifolds with boundary (when writing down the limits which are used to define the derivative of a function, restrict the domain of the limit to just one side if necessary, so as to not fall outside of the manifold).

An equivalent definition of the sheaf $\mathcal{O}_\mathbb{H}$ is to declare $\mathcal{O}_\mathbb{H}(U)$ to be the set of continuous functions on U which are holomorphic when restricted to $U \cap \mathring{\mathbb{H}}$, and smooth when restricted to $U \cap \partial\mathbb{H}$:

$$\mathcal{O}_\mathbb{H}(U) = \left\{ f \in C^0(U, \mathbb{C}) \;\middle|\; \begin{array}{l} f|_{U\cap\mathring{\mathbb{H}}} \textit{ is holomorphic,} \\ f|_{U\cap\partial\mathbb{H}} \textit{ is smooth} \end{array} \right\}.$$

The equivalence between the above two definitions of $\mathcal{O}_{\mathbb{H}}$ will be proven below, in Lemma 2. We first state an important theorem:

Theorem. *(Riemann mapping theorem for simply connected domains with smooth boundary)* Let $D \subset \mathbb{C}$ be a compact simply connected domain with smooth boundary. Then there exists an isomorphism

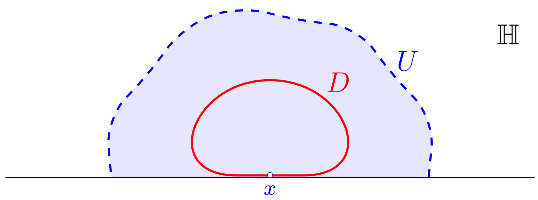
$$D \;\cong\; \mathbb{D} := \{z \in \mathbb{C} \,:\, |z| \leq 1\}$$

which is holomorphic in the interior, and smooth all the way to the boundary. Moreover, that isomorphism is unique up to an element of

$$\mathrm{Aut}(\mathbb{D}) = PSU(1,1) = \left\{z \mapsto \tfrac{az+b}{\bar{b}z+\bar{a}} : |a|^2 - |b|^2 = 1\right\}.$$

Lemma 2 *Let $U \subset \mathbb{H}$ be an open subset, and let $f : U \to \mathbb{C}$ be a continuous function such that $f|_{U\cap\mathring{\mathbb{H}}}$ is holomorphic and $f|_{U\cap\partial\mathbb{H}}$ is smooth. Then f is smooth all the way to the boundary.*

Proof. Let $x \in U \cap \partial\mathbb{H}$ be a point, and let $D \subset U$ be a neighbourhood of x which is compact, simply connected, and with smooth boundary:



Let $\psi : \mathbb{D} \to D$ be a uniformizing map. The function $g := \psi^* f$ is continuous, holomorphic in the interior of $\mathbb{D}$, and smooth on the boundary⚠. The Taylor coefficients a_n of $g(z) = \sum a_n z^n$ satisfy

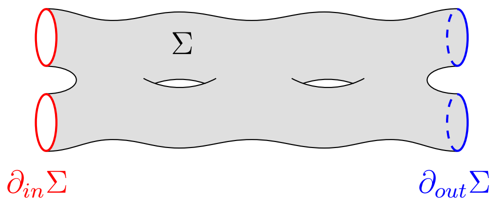
$$\begin{aligned} a_n &= \tfrac{1}{2\pi i}\oint_{|z|=r} g(z) z^{-(n+1)} dz && \text{for any } r<1 \\ &= \tfrac{1}{2\pi i}\oint_{|z|=1} g(z) z^{-(n+1)} dz && \text{since } g \text{ is continuous} \\ &= \tfrac{\pm 1}{n(n-1)\ldots(n-k+1)} \cdot \tfrac{1}{2\pi i}\oint_{|z|=1} \left(g|_{\partial\mathbb{D}}\right)^{(k)}(z)\cdot z^{-(n-k+1)} dz && \forall k \leq n. \end{aligned}$$

It follows that $|a_n| \leq \frac{1}{n(n-1)\ldots(n-k+1)} \cdot \|(g|_{\partial\mathbb{D}})^{(k)}\|_\infty$. The coefficients a_n decay faster that any power of n, so $g(z)$ is smooth all the way to the boundary. The same therefore holds for f around x.

⚠ There is a gap in the above proof, because we don't know that $f|_{\partial D}$ is smooth at the two boundary points of the interval $[a,b] := \partial D \cap \partial\mathbb{H}$. But we can fix that gap. Let $h \in \mathcal{O}_{\mathbb{H}}(\mathbb{H})$ be an auxiliary function with zeros of infinite order at a and b (for example, $h(z) = e^{-\frac{1+i}{\sqrt{z-a}} - \frac{1+i}{\sqrt{z-b}}}$). We run the same argument with $\tilde{f} := hf$ (the function $\tilde{f}|_{\partial D}$ is now smooth at a and b because it vanishes to infinite order), deduce that $\tilde{f}$ is smooth all the way to the boundary, and divide by h to get our result. $\square$

We'll be distinguishing two types of complex cobordisms. There's the ones which we'll be calling 'thick complex cobordisms', and there's the ones which we'll be calling 'complex cobordisms with thin parts'. Thick complex cobordisms are special cases of complex cobordisms with thin parts.

Definition: *A thick complex cobordism is a Riemann surface with boundary equipped with a decomposition of its boundary into a disjoint union $\partial\Sigma = \partial_{in}\Sigma \sqcup \partial_{out}\Sigma$:*



We equip $\partial_{out}\Sigma$ with the orientation induced by that of Σ, and we equip $\partial_{in}\Sigma$ with the opposite of that orientation.

More generally, given oriented 1-manifolds S_1 and S_2, a (thick) complex cobordism from S_1 to S_2 is a triple $(\Sigma, \varphi_{in}, \varphi_{out})$ where Σ is a complex cobordism as defined above, and $\varphi_{in} : S_1 \to \partial_{in}\Sigma$ and $\varphi_{out} : S_2 \to \partial_{out}\Sigma$ are diffeomorphisms.

The following result is quite non-trivial. We will not prove it in these notes.

Theorem. *(Conformal welding)* Let Σ_1 and Σ_2 be thick complex cobordisms, and let $\phi : \partial_{in}\Sigma_1 \to \partial_{out}\Sigma_2$ be an orientation preserving diffeomorphism. Then $(\Sigma_1 \cup_\phi \Sigma_2, \mathcal{O}_{\Sigma_1 \cup_\phi \Sigma_2})$ with

$$\partial_{in}(\Sigma_1 \cup_\phi \Sigma_2) = \partial_{in}\Sigma_2, \qquad \partial_{out}(\Sigma_1 \cup_\phi \Sigma_2) = \partial_{out}\Sigma_1,$$
$$\mathcal{O}_{\Sigma_1 \cup_\phi \Sigma_2}(U) := \big\{ f : U \to \mathbb{C} \,\big|\, f|_{U \cap \Sigma_i} \in \mathcal{O}_{\Sigma_i}(U \cap \Sigma_i), \text{ for } i = 1, 2 \big\}$$

is a thick complex cobordism. Moreover, the image of $\partial_{in}\Sigma_1$ inside $\Sigma_1 \cup_\phi \Sigma_2$ (equivalently, the image of $\partial_{out}\Sigma_2$) is a smooth curve.

The non-trivial claim is that the ringed space $(\Sigma_1 \cup_\phi \Sigma_2, \mathcal{O}_{\Sigma_1 \cup_\phi \Sigma_2})$ is isomorphic to $(\mathbb{C}, \mathcal{O}_\mathbb{C})$ in a neighbourhood of the image of $\partial_{out}\Sigma_1$.

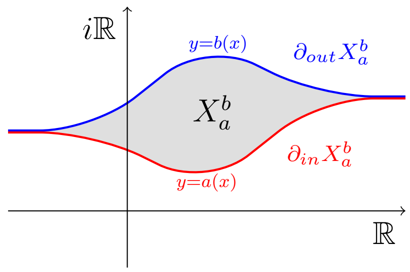
One big problem with the above definitions is that they do not allow for *identity cobordisms*. This is addressed by the following variant:

Definition: *A complex cobordism with thin parts is a ringed space $(\Sigma, \mathcal{O}_\Sigma)$ equipped with two subspaces $\partial_{in}\Sigma \subset \Sigma$ and $\partial_{out}\Sigma \subset \Sigma$ (typically not disjoint), which is locally isomorphic to $(X_a^b, \mathcal{O}_{X_a^b})$ for some $a \le b$ as below.*

Here, given two smooth functions $a, b : \mathbb{R} \to \mathbb{R}$ satisfying $a \leq b$, we set

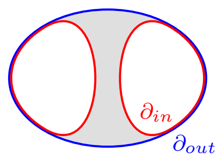
$$X_a^b := \{x + iy \in \mathbb{C} \,|\, a(x) \leq y \leq b(x)\},$$
$$\partial_{in}(X_a^b) = \{x + ia(x) \,|\, x \in \mathbb{R}\}, \qquad \partial_{out}(X_a^b) = \{x + ib(x) \,|\, x \in \mathbb{R}\},$$
$$\mathcal{O}_{X_a^b}(U) := \left\{ f : U \to \mathbb{C} \;\middle|\; \begin{array}{l} f|_{U \cap \mathring{X}_a^b} \textit{ is holomorphic,} \\ \exists V \subset \mathbb{C} \textit{ open and } g \in C^\infty(V) \textit{ s.t. } f = g|_U \end{array} \right\}, \tag{1}$$

where $\mathring{X}_a^b := \{x + iy \in \mathbb{C} \,|\, a(x) < y < b(x)\}$.



Once again, a complex cobordism from S_1 to S_2 is a triple $(\Sigma, \varphi_{in}, \varphi_{out})$ where Σ is as above, and $\varphi_{in} : S_1 \to \partial_{in}\Sigma$ and $\varphi_{out} : S_2 \to \partial_{out}\Sigma$ are diffeomorphisms.

The following is an example of a complex cobordism with thin parts:



An equivalent definition of the sheaf $\mathcal{O}_{X_a^b}$ is to declare

$$\mathcal{O}_{X_a^b}(U) := \left\{ f \in C^0(U, \mathbb{C}) \;\middle|\; \begin{array}{l} f|_{U \cap \mathring{X}_a^b} \textit{ is holomorphic,} \\ f|_{U \cap \partial_{in} X_a^b} \textit{ and } f|_{U \cap \partial_{out} X_a^b} \textit{ are smooth} \end{array} \right\} \tag{2}$$

Proposition. *(Conformal welding for complex cobordism with thin parts)* Let Σ_1 and Σ_2 be complex cobordisms with thin parts, and let $\phi : \partial_{in}\Sigma_1 \to \partial_{out}\Sigma_2$ be an orientation preserving diffeomorphism. Then $(\Sigma_1 \cup_\phi \Sigma_2, \mathcal{O}_{\Sigma_1 \cup_\phi \Sigma_2})$ with $\partial_{in}(\Sigma_1 \cup_\phi \Sigma_2) = \partial_{in}\Sigma_2$, $\partial_{out}(\Sigma_1 \cup_\phi \Sigma_2) = \partial_{out}\Sigma_1$, and

$$\mathcal{O}_{\Sigma_1 \cup_\phi \Sigma_2}(U) := \left\{ f : U \to \mathbb{C} \,\middle|\, f|_{U \cap \Sigma_i} \in \mathcal{O}_{\Sigma_i}(U \cap \Sigma_i), \text{ for } i = 1, 2 \right\}$$

is a complex cobordism with thin parts.

Sketch of proof. (modulo the proof of conformal welding). The problem being local, we may assume that $\Sigma_1 = \overline{X}_a^b := X_a^b \cup \{\infty\}$ and $\Sigma_2 = \overline{X}_c^d := X_c^d \cup \{\infty\}$, for some functions $a, b, c, d : \mathbb{R} \to \mathbb{R}$ which are compactly supported. Write

$$\overline{X}_a^b = \overline{X}_a^\infty \cap \overline{X}_{-\infty}^b \qquad \text{and} \qquad \overline{X}_c^d = \overline{X}_c^\infty \cap \overline{X}_{-\infty}^d.$$

The curve $\overline{X}_{-\infty}^b \cup_\phi \overline{X}_c^\infty$ has genus zero, and is therefore isomorphic to $\mathbb{CP}^1$. Pick an isomorphism $\psi : \overline{X}_{-\infty}^b \cup_\phi \overline{X}_c^\infty \to \mathbb{CP}^1$. The image of $\partial_{out}\overline{X}_{-\infty}^b$ under ψ (equivalently, the image of $\partial_{in}\overline{X}_c^\infty$ under ψ) is a smoothly embedded curve. By the Riemann mapping theorem for simply connected domains with smooth boundary, the map $\psi|_{\overline{X}_{-\infty}^b}$ is smooth all the way to the boundary (and holomorphic in the interior). The same holds for $\psi|_{\overline{X}_c^\infty}$. It follows that $\psi(\partial_{in}\overline{X}_a^b)$ and $\psi(\partial_{out}\overline{X}_c^d)$ are smooth curves in $\mathbb{CP}^1$ (being the image of a smooth curve under a smooth map). The space $\overline{X}_a^b \cup_\phi \overline{X}_c^d$ can be therefore identified with the subset of $\mathbb{CP}^1$ that lies between these two curves.

It remains to identify the sheaf $\mathcal{O}_{\overline{X}_a^b \cup_\phi \overline{X}_c^d}$ with the sheaf of functions on that subset which are holomorphic in the interior, and are the restriction of a smooth function on some open of $\mathbb{CP}^1$. This is difficult to do with the definition (1) alone, but it is easy to do once we know the equivalence between (1) and (2). $\square$

Full CFT versus chiral CFT

A *concrete linear category* is a pair $(\mathcal{C}, U)$ consisting of a linear category $\mathcal{C}$ together with a faithful functor U from $\mathcal{C}$ to the category of topological vector spaces [*Think:* $\mathcal{C}$ is the category of representations of a group or an algebra, and U the functor which sends a representation to its underlying vector space]. A *concrete functor* $(\mathcal{C}_1, U_1) \to (\mathcal{C}_2, U_2)$ between concrete linear categories is a pair consisting of a linear functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ and a linear natural transformation

$$\begin{tikzcd} \mathcal{C}_1 \arrow[rr, "F"] \arrow[dr, "U_1"'] & \arrow[d, phantom, "Z \Rrightarrow"] & \mathcal{C}_2 \arrow[dl, "U_2"] \\ & \mathsf{TopVec} & \end{tikzcd}$$

A full Segal CFT is a symmetric monoidal functor from the category of complex cobordisms (or rather, a certain central extension of that category by $\mathbb{R}_+$) into the category of topological vector spaces. By contrast, a chiral Segal CFT is a symmetric monoidal functor from the category of complex cobordisms (no central extension) into the category of concrete linear categories. It comes with an extra piece of structure which ensures that the functors (just the functors, not the concrete functors!) associated to annuli are trivial, and there is also a holomorphicity condition.

The word chiral in *'chiral CFT'* refers to that last holomorphicity condition.

Let us be precise with what we mean by an "annulus":

Definition 3 *Given a circle S (a manifold diffeomorphic to S^1) let*

$$\mathrm{Ann}(S) := \left\{ \begin{array}{c|c} \textit{partially thin complex cobordisms } A & \partial_{in}A \hookrightarrow A \textit{ and} \\ + \textit{ orientation preserving diffeomorphisms} & \partial_{out}A \hookrightarrow A \textit{ are} \\ \varphi_{in} : S \xrightarrow{\cong} \partial_{in}A,\ \ \varphi_{out} : S \xrightarrow{\cong} \partial_{out}A & \textit{htpy equivalences} \end{array} \right\} \Big/ \textit{iso.}$$

be the semigroup of annuli *with boundary components parametrized by S, with operation given by composition of cobordisms (conformal welding).*

The semigroup of annuli admits a certain central extension

$$0 \to \mathbb{C}^\times \times \mathbb{Z} \to \tilde{\mathrm{Ann}}(S) \to \mathrm{Ann}(S) \to 0$$

which depends on a number $c \in \mathbb{C}$ called the *central charge* (for rational chiral CFTs, the central charge is always an element of $\mathbb{Q}$).

Definition: *A chiral Segal CFT consists of:*

(1.a) For every closed 1-manifold S, a linear category $\mathcal{C}(S)$ isomorphic to $\mathsf{Vec}_{\mathrm{fd}}^{\oplus r}$ for some $r \in \mathbb{N}$. The assignment $S \mapsto \mathcal{C}(S)$ is symmetric monoidal with respect to disjoint union of 1-manifolds, and tensor product of linear categories.

(1.b) For every closed 1-manifold S, a faithful functor $U : \mathcal{C}(S) \to \mathsf{TopVec}$. The assignment $S \mapsto U$ is a symmetric monoidal transformation from $S \mapsto \mathcal{C}(S)$ to the constant 2-functor $S \mapsto \mathsf{TopVec}$.

(2.a) For every complex cobordism Σ, a linear functor $F_\Sigma : \mathcal{C}(\partial_{\mathrm{in}}\Sigma) \to \mathcal{C}(\partial_{\mathrm{out}}\Sigma)$. These functors are compatible with the operations of disjoint union, identity cobordisms, and composition of cobordisms.

(2.b) For every complex cobordism Σ, and every object $\lambda \in \mathcal{C}(\partial_{\mathrm{in}}\Sigma)$, a linear map $Z_\Sigma : U(\lambda) \to U(F_\Sigma(\lambda))$. The maps Z_Σ are compatible with the operations of disjoint union, identity cobordisms, and composition of cobordisms.

(3.a) For every $\tilde{A} \in \tilde{\mathrm{Ann}}(S)$, a trivialization $T_{\tilde{A}} : F_A \to \mathrm{id}_{\mathcal{C}(S)}$. The $T_{\tilde{A}}$ are compatible with identities and composition, and the central $\mathbb{C}^\times$ acts in a standard way.

(3.b) For every $\lambda \in \mathcal{C}(S)$, the map which sends $\tilde{\Sigma}$ to the composite $U(T_{\tilde{\Sigma}}) \circ Z_\Sigma : U(V) \to U(F_\Sigma(V)) \to U(V)$ is continuous on $\tilde{\mathrm{Ann}}(S)$ and holomorphic on its interior.

We summarise the above definition of chiral Segal CFT in Table 1, on the next page.

The items in the first column of that table [items (1a), (2a), (3a)] correspond to the notion of a *modular functor*.[1] In that column, everything is finite dimensional; everything is topological.

The items in the second column [items (1b), (2b), (3b)] correspond to the notion of a *twisted field theory* (the modular functor is the twist). If we were to remove the twist, then we would be left with a single topological vector space for every 1-manifold S, and a single linear map for every complex cobordism.

The two items in the first row [items (1a), (1b)] correspond to the idea that, for every 1-manifold S, there is an associated algebra $\mathcal{A}(S)$. That algebra is called the *algebra of*

[1]...at least conjecturally (I say 'conjecturally' because I don't think that this particular definition has ever been compared to the other definitions of modular functor.)

observables, and can be defined as the algebra of endomorphisms of the functor U. In more down-to-earth terms, the algebra of observables is given by

$$\mathcal{A}(S) = \bigoplus_{\substack{\lambda \in \mathcal{C}(S) \\ \lambda \text{ is simple}}} \mathrm{End}(U(\lambda))$$

where the sum ranges of a set of representatives of the isomorphism classes of simple objects of $\mathcal{C}(S)$. Provided we restrict appropriately the class of representations that we allow, we can recover $\mathcal{C}(S)$ as the category of representations of $\mathcal{A}(S)$:

$$\mathcal{C}(S) \;=\; \mathrm{Rep}(\mathcal{A}(S)).$$

Finally, the two items in the third row [items (3a), (3b)] correspond to the idea that H_Σ depends *topologically* on Σ (this means, in particular, that if Σ_1 and Σ_2 are diffeomorphic cobordisms, then H_{Σ_1} and H_{Σ_2} are isomorphic bimodules), while $\Omega_\Sigma \in H_\Sigma$ depends *holomorphically* on Σ.

The two items in the second row [items (2a), (2b)] correspond to the idea that, for every complex cobordism Σ, there is an associated $\mathcal{A}(\partial_{out}\Sigma)$-$\mathcal{A}(\partial_{in}\Sigma)$-bimodule H_Σ, equipped with a distinguished 'vacuum vector' $\Omega_\Sigma \in H_\Sigma$. One recovers F_Σ as the functor $H_\Sigma \otimes_{\mathcal{A}(\partial_{in}\Sigma)} -$, and Z_Σ as the operation of tensoring with Ω_Σ:

$$F_\Sigma = H_\Sigma \otimes - \qquad\qquad Z_\Sigma = \Omega_\Sigma \otimes - \,.$$

DEFINITION (SKETCH): CHIRAL SEGAL CFT

(1a) For every 1-manifold S, a category $\mathcal{C}(S)$.	(1b) A 'forgetful' functor $U : \mathcal{C}(S) \to \mathsf{TopVec}$.
(2a) For every cpx cobordism Σ, a functor $F_\Sigma : \mathcal{C}(\partial_{in}\Sigma) \to \mathcal{C}(\partial_{out}\Sigma)$.	(2b) For every $\lambda \in \mathcal{C}(\partial_{in}\Sigma)$, a linear map $Z_\Sigma : U(\lambda) \to U(F_\Sigma(\lambda))$.
(3a) For every $\tilde{A} \in \widetilde{\mathrm{Ann}}(S)$, a trivialization $T_{\tilde{A}} : F_A \to \mathrm{id}_{\mathcal{C}(S)}$.	(3b) For every $\lambda \in \mathcal{C}(S)$, the map $\widetilde{\mathrm{Ann}}(S) \to \mathrm{End}(U(\lambda))$ $\tilde{A} \mapsto U(T_{\tilde{A}}) \circ Z_A$ is holomorphic.

Table 1.

The definition of (rational) chiral Segal CFT

In the previous section, we provided a summary of the notion of chiral Segal CFT. Here, we spell out all the details for the convenience of the reader. Recall that rationality is built into the definition of chiral Segal CFT. As before, we organise the definition into six parts, labelled (1a), (1b), (2a), (2b), (3a), (3b).

<u>Main definition</u>

A (rational) chiral Segal CFT of central charge c consists of:

(1a) For every closed (compact, smooth, oriented) 1-manifold S, a category $\mathcal{C}(S)$, isomorphic to $\mathsf{Vec}_{\mathrm{f.d.}}^{\oplus r}$ for some $r \in \mathbb{N}$ which depends on S.

[Think: There is a certain group or algebra associated to S, and $\mathcal{C}(S)$ is the category of representations of that group or algebra (r = number of irreps.)]

For every pair of 1-manifolds S_1, S_2 there is a bilinear functor $\mathcal{C}(S_1) \times \mathcal{C}(S_2) \to \mathcal{C}(S_1 \sqcup S_2) : (\lambda, \mu) \mapsto \lambda \otimes \mu$ which induces an equivalence of categories

$$\mathcal{C}(S_1) \otimes \mathcal{C}(S_2) \xrightarrow{\cong} \mathcal{C}(S_1 \sqcup S_2).$$

Here, given two linear categories $\mathcal{C}$ and $\mathcal{D}$ isomorphic to $\mathsf{Vec}_{\mathrm{f.d.}}^{\oplus r}$, their tensor product $\mathcal{C} \otimes \mathcal{D}$ has objects of the form $\bigoplus c_i \otimes d_i$ for $c_i \in \mathcal{C}$ and $d_i \in \mathcal{D}$, and hom-spaces given by $\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}}(\bigoplus c_i \otimes d_i, \bigoplus c'_j \otimes d'_j) = \bigoplus_{ij} \mathrm{Hom}_{\mathcal{C}}(c_i, c'_j) \otimes \mathrm{Hom}_{\mathcal{D}}(d_i, d'_j)$.

We also have an equivalence $\mathsf{Vec}_{\mathrm{f.d.}} \xrightarrow{\cong} \mathcal{C}(\emptyset) : \mathbb{C} \mapsto 1$.

There is an associator $(\lambda \otimes \mu) \otimes \nu \overset{\cong}{\to} \lambda \otimes (\mu \otimes \nu)$, unitors $1 \otimes \lambda \overset{\cong}{\to} \lambda$ and $\lambda \otimes 1 \overset{\cong}{\to} \lambda$, and a braiding $\lambda \otimes \mu \overset{\cong}{\to} \mu \otimes \lambda$ *[we omit the isomorphisms $(S_1 \sqcup S_2) \sqcup S_3 \cong S_1 \sqcup (S_2 \sqcup S_3)$, $\emptyset \sqcup S \cong S$, $S \sqcup \emptyset \cong S$, and $S_1 \sqcup S_2 \cong S_2 \sqcup S_1$]* which are natural (i.e. for any morphisms $\lambda \to \lambda'$, $\mu \to \mu'$, $\nu \to \nu'$ the following diagrams commute

$$\begin{tikzcd} (\lambda\otimes\mu)\otimes\nu \arrow[r] \arrow[d] & (\lambda'\otimes\mu')\otimes\nu' \arrow[d] \\ \lambda\otimes(\mu\otimes\nu) \arrow[r] & \lambda'\otimes(\mu'\otimes\nu') \end{tikzcd} \quad \begin{tikzcd} 1\otimes\lambda \arrow[r] \arrow[d] & 1\otimes\lambda' \arrow[d] \\ \lambda \arrow[r] & \lambda' \end{tikzcd} \quad \begin{tikzcd} \lambda\otimes 1 \arrow[r] \arrow[d] & \lambda'\otimes 1 \arrow[d] \\ \lambda \arrow[r] & \lambda' \end{tikzcd} \quad \begin{tikzcd} \lambda\otimes\mu \arrow[r] \arrow[d] & \lambda'\otimes\mu' \arrow[d] \\ \mu\otimes\lambda \arrow[r] & \mu'\otimes\lambda' \end{tikzcd} \quad)$$

and subject to the well-known pentagon, triangle, hexagon, and symmetry axioms (the same axioms which appear in the definition of a symmetric monoidal category):

$$\begin{tikzcd} & (\lambda\otimes\mu)\otimes(\nu\otimes\rho) \arrow[dr] & \\ ((\lambda\otimes\mu)\otimes\nu)\otimes\rho \arrow[ur] \arrow[d] & & \lambda\otimes(\mu\otimes(\nu\otimes\rho)) \\ (\lambda\otimes(\mu\otimes\nu))\otimes\rho \arrow[r] & \lambda\otimes((\mu\otimes\nu)\otimes\rho) \arrow[ur] & \end{tikzcd} \quad \begin{tikzcd} & \lambda\otimes\mu \arrow[dr] & \\ (\lambda\otimes 1)\otimes\mu \arrow[ur] \arrow[rr] & & \lambda\otimes(1\otimes\mu) \end{tikzcd}$$

$$\begin{tikzcd} & (\mu\otimes\lambda)\otimes\nu \arrow[r] & \mu\otimes(\lambda\otimes\nu) \arrow[dr] & \\ (\lambda\otimes\mu)\otimes\nu \arrow[ur] \arrow[dr] & & & \mu\otimes(\nu\otimes\lambda) \\ & \lambda\otimes(\mu\otimes\nu) \arrow[r] & (\mu\otimes\nu)\otimes\lambda \arrow[ur] & \end{tikzcd} \quad \begin{tikzcd} & \mu\otimes\lambda \arrow[dr] & \\ \lambda\otimes\mu \arrow[ur] \arrow[rr, "="] & & \lambda\otimes\mu \end{tikzcd}$$

(1b) For every closed 1-manifold S, a faithful functor $U : \mathcal{C}(S) \to \mathsf{TopVec}$ which equips $\mathcal{C}(S)$ with the structure of a concrete category.[2]

[2]Depending on the type of topological vector spaces one works with, one might want to modify the notion of tensor product accordingly.

If $\mathcal{C}(S) \cong \mathsf{Vec}_{\text{f.d.}}^{\oplus r}$, so that an object can be written as an r-tuple of finite dimensional vector spaces, then the functor U is always of the form $(V_1, \ldots, V_r) \mapsto \bigoplus V_i \otimes W_i$, where the W_i are typically infinite dimensional.

The forgetful functor satisfies $U(\lambda \otimes \mu) = U(\lambda) \otimes U(\mu)$ and $U(1) = \mathbb{C}$, naturally in λ and μ, and compatibly with the associator, unitors, and braiding:

$$\begin{tikzcd} U((\lambda\otimes\mu)\otimes\nu) \arrow[r, equal] \arrow[d] & (U(\lambda)\otimes U(\mu))\otimes U(\nu) \arrow[d] \\ U(\lambda\otimes(\mu\otimes\nu)) \arrow[r, equal] & U(\lambda)\otimes(U(\mu)\otimes U(\nu)) \end{tikzcd} \qquad \begin{tikzcd} U(1)\otimes U(\lambda) \arrow[r, equal] \arrow[d, equal] & U(1\otimes\lambda) \arrow[d] \\ \mathbb{C}\otimes U(\lambda) \arrow[r] & U(\lambda) \end{tikzcd} \qquad \begin{tikzcd} U(\lambda)\otimes U(1) \arrow[r, equal] \arrow[d, equal] & U(\lambda\otimes 1) \arrow[d] \\ U(\lambda)\otimes\mathbb{C} \arrow[r] & U(\lambda) \end{tikzcd} \qquad \begin{tikzcd} U(\lambda\otimes\mu) \arrow[r, equal] \arrow[d] & U(\lambda)\otimes U(\mu) \arrow[d] \\ U(\mu\otimes\lambda) \arrow[r, equal] & U(\mu)\otimes U(\lambda) \end{tikzcd}$$

(2a) For every complex cobordism with thin parts Σ from S_1 to S_2, a linear functor $F_\Sigma : \mathcal{C}(S_1) \to \mathcal{C}(S_2)$.

If Σ and Σ' are complex cobordisms from S_1 to S_2, then for every biholomorphic map $\phi : \Sigma \overset{\cong}{\to} \Sigma'$ such that $\phi|_{S_1} = \mathrm{id}$ and $\phi|_{S_2} = \mathrm{id}$, we have an invertible natural transformation $F_\Sigma \cong F_{\Sigma'}$, compatible with composition of maps.

We also have invertible natural transformations $F_{\mathbf{1}_S} \cong \mathrm{id}_{\mathcal{C}(S)}$, $F_{\Sigma_1 \cup \Sigma_2} \cong F_{\Sigma_1} \circ F_{\Sigma_2}$, and $F_{\Sigma_1 \sqcup \Sigma_2} \cong F_{\Sigma_1} \otimes F_{\Sigma_2}$. They are natural with respect to biholomorphic maps of complex cobordisms, and make the following diagrams commute:

$$\begin{tikzcd} F_{\mathbf{1}_S\cup\Sigma} \arrow[r] \arrow[d] & F_{\mathbf{1}_S}\circ F_\Sigma \arrow[d] \\ F_\Sigma \arrow[r, equal] & \mathrm{id}_{\mathcal{C}(S)}\circ F_\Sigma \end{tikzcd} \qquad \begin{tikzcd} F_{\Sigma\cup\mathbf{1}_S} \arrow[r] \arrow[d] & F_\Sigma\circ F_{\mathbf{1}_S} \arrow[d] \\ F_\Sigma \arrow[r, equal] & F_\Sigma\circ\mathrm{id}_{\mathcal{C}(S)} \end{tikzcd} \qquad \begin{tikzcd} F_{\mathbf{1}_{S_1}\sqcup\mathbf{1}_{S_2}} \arrow[r] \arrow[d] & F_{\mathbf{1}_{S_1}}\otimes F_{\mathbf{1}_{S_2}} \arrow[d] \\ \mathrm{id}_{\mathcal{C}(S_1\sqcup S_2)} \arrow[r] & \mathrm{id}_{\mathcal{C}(S_1)}\otimes\mathrm{id}_{\mathcal{C}(S_2)} \end{tikzcd} \qquad \begin{tikzcd} F_{\Sigma_1\cup\Sigma_2\cup\Sigma_3} \arrow[r] \arrow[d] & F_{\Sigma_1}\circ F_{\Sigma_2\cup\Sigma_3} \arrow[d] \\ F_{\Sigma_1\cup\Sigma_2}\circ F_{\Sigma_3} \arrow[r] & F_{\Sigma_1}\circ F_{\Sigma_2}\circ F_{\Sigma_3} \end{tikzcd}$$

$$\begin{tikzcd} F_{\Sigma\sqcup\emptyset} \arrow[rr] \arrow[dr] & & F_\Sigma\otimes\mathrm{id}_{\mathsf{Vec}} \arrow[dl] \\ & F_\Sigma & \end{tikzcd} \qquad \begin{tikzcd} F_{\Sigma_1\sqcup\Sigma_2\sqcup\Sigma_3} \arrow[r] \arrow[d] & F_{\Sigma_1}\otimes F_{\Sigma_2\sqcup\Sigma_3} \arrow[d] \\ F_{\Sigma_1\sqcup\Sigma_2}\otimes F_{\Sigma_3} \arrow[r] & F_{\Sigma_1}\otimes F_{\Sigma_2}\otimes F_{\Sigma_3} \end{tikzcd} \qquad \begin{tikzcd} F_{(\Sigma_1\cup\Sigma_2)\sqcup(\Sigma_1'\cup\Sigma_2')} \arrow[r] \arrow[d] & F_{\Sigma_1\cup\Sigma_2}\otimes F_{\Sigma_1'\cup\Sigma_2'} \arrow[d] \\ F_{\Sigma_1\sqcup\Sigma_1'}\circ F_{\Sigma_2\sqcup\Sigma_2'} \arrow[r] & (F_{\Sigma_1}\circ F_{\Sigma_2})\otimes(F_{\Sigma_1'}\circ F_{\Sigma_2'}) \end{tikzcd} \qquad \begin{tikzcd} F_{\Sigma_1\sqcup\Sigma_2} \arrow[r] \arrow[d] & F_{\Sigma_2}\otimes F_{\Sigma_1} \arrow[d] \\ F_{\Sigma_2\sqcup\Sigma_1} \arrow[r] & F_{\Sigma_1}\otimes F_{\Sigma_2} \end{tikzcd}$$

(The astute reader will have noticed that the above diagrams are a bit sloppy: the functors being compared don't always have the same domain/codomain. Fixing them is not difficult, but would make them very bulky.)

(2b) For every complex cobordism with thin parts Σ from S_1 to S_2 and every object $\lambda \in \mathcal{C}(S_1)$, a continuous linear map $Z_\Sigma : U(\lambda) \to U(F_\Sigma(\lambda))$.

The maps Z_Σ are natural in λ. They're also natural in Σ, meaning that for every biholomorphic map $\phi : \Sigma' \to \Sigma$ fixing S_1 and S_2, and every $\lambda \in \mathcal{C}(S_1)$, we have a commutative diagram

$$\begin{tikzcd} U(\lambda) \arrow[r, "Z_\Sigma"] \arrow[d, equal] & U(F_\Sigma(\lambda)) \arrow[d, "\cong" sloped] \\ U(\lambda) \arrow[r, "Z_{\Sigma'}"] & U(F_{\Sigma'}(\lambda)) \end{tikzcd}$$

We also have $Z_{\mathbf{1_S}} = \mathrm{id}_{U(\lambda)}$, $Z_{\Sigma_1 \cup \Sigma_2} = Z_{\Sigma_1} \circ Z_{\Sigma_2}$, and $Z_{\Sigma_1 \sqcup \Sigma_2} = Z_{\Sigma_1} \otimes Z_{\Sigma_2}$. (Some isomorphisms have been omitted for better readability. For example, the last equality should say that the following diagram is commutative:

$$\begin{tikzcd} U(\lambda\otimes\mu) \arrow[r, "Z_{\Sigma_1\sqcup\Sigma_2}"] \arrow[dd, equal] & U(F_{\Sigma_1\sqcup\Sigma_2}(\lambda\otimes\mu)) \arrow[d, "\cong" sloped] \\ & U(F_{\Sigma_1}(\lambda)\otimes F_{\Sigma_2}(\mu)) \arrow[d, equal] \\ U(\lambda)\otimes U(\mu) \arrow[r, "Z_{\Sigma_1}\otimes Z_{\Sigma_2}"] & U(F_{\Sigma_1}(\lambda))\otimes U(F_{\Sigma_2}(\mu)) \end{tikzcd}$$
)

Before describing items (3a) and (3b) of the definition of chiral Segal CFT, we need a couple of facts about $\mathrm{Diff}(S)$ and its "complexification", the semigroup of annuli $\mathrm{Ann}(S)$.

Let S be a circle (a manifold diffeomorphic to S^1) and let $\mathrm{Diff}(S)$ be its group of orientation preserving diffeomorphisms, and let $\mathrm{Ann}(S)$ be the semigroup of annuli with boundary components parametrized by S. There is an obvious embedding

$$\mathrm{Diff}(S) \hookrightarrow \mathrm{Ann}(S)$$

which sends a diffeomorphism φ to the completely thin annulus $(A{=}S, \varphi_{in}{=}\varphi, \varphi_{out}{=}\mathrm{id})$.

We will postpone the proof of the following proposition until after the definition of chiral Segal CFT is completed:

Proposition. *The group* $\mathrm{Diff}(S)$ *admits a universal central extension in the category of Fréchet Lie groups, and the center of that universal central extension is canonically isomorphic to* $i\mathbb{R} \oplus \mathbb{Z}$*.*

The semigroup $\mathrm{Ann}(S)$ *admits a universal central extension in the category of complex Fréchet semigroups*[3]*, and the center of that universal central extension is canonically isomorphic to* $\mathbb{C} \oplus \mathbb{Z}$*.*

Writing ${}^{i\mathbb{R}\oplus\mathbb{Z}}\mathrm{Diff}(S)$ for the universal central extension of $\mathrm{Diff}(S)$, and ${}^{\mathbb{C}\oplus\mathbb{Z}}\mathrm{Ann}(S)$ for the universal central extension of $\mathrm{Ann}(S)$, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & i\mathbb{R}\oplus\mathbb{Z} \arrow[r] \arrow[d] & {}^{i\mathbb{R}\oplus\mathbb{Z}}\mathrm{Diff}(S) \arrow[r] \arrow[d] & \mathrm{Diff}(S) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C}\oplus\mathbb{Z} \arrow[r] & {}^{\mathbb{C}\oplus\mathbb{Z}}\mathrm{Ann}(S) \arrow[r] & \mathrm{Ann}(S) \arrow[r] & 0
\end{tikzcd}$$

where each vertical arrow is the inclusion of a group into its "complexification".

Given a complex number $c \in \mathbb{C}$ (this will later be the *central charge* of the CFT), we can form the associated central extension

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}\oplus\mathbb{Z} \arrow[r] \arrow[d, "{(z,n)\mapsto(e^{cz},n)}"'] \arrow[dr, phantom, "\ulcorner"] & {}^{\mathbb{C}\oplus\mathbb{Z}}\mathrm{Ann}(S) \arrow[r] \arrow[d] & \mathrm{Ann}(S) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{C}^\times\oplus\mathbb{Z} \arrow[r] & {}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S) \arrow[r] & \mathrm{Ann}(S) \arrow[r] & 0
\end{tikzcd}$$

where ${}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S)$ is defined as the pushout. Assuming $c \in \mathbb{R}$ (the central charge of a rational CFTs is always a rational number), we can also form the central extension

$$\begin{tikzcd}
0 \arrow[r] & i\mathbb{R}\oplus\mathbb{Z} \arrow[r] \arrow[d, "{(z,n)\mapsto(e^{cz},n)}"'] \arrow[dr, phantom, "\ulcorner"] & {}^{i\mathbb{R}\oplus\mathbb{Z}}\mathrm{Diff}(S^1) \arrow[r] \arrow[d] & \mathrm{Diff}(S^1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U(1)\oplus\mathbb{Z} \arrow[r] & {}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S^1) \arrow[r] & \mathrm{Diff}(S^1) \arrow[r] & 0
\end{tikzcd}$$

[3] A semigroup in the category whose objects are closed subspaces of complex Fréchet manifolds *[e.g., a Fréchet manifold with boundary, or with corners]*, and whose morphisms are restrictions of smooth maps. *(⚠ I hope that my description of the morphisms is right.)*

which sits as a subgroup ${}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S^1) \subset {}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S)$.

We can now finish the definition of chiral Segal CFT of central charge c (the items (1a), (1b), (2a), (2b) didn't depend on c, which is why we only mention it now):

(3a) For every circle S, every annulus $A \in \mathrm{Ann}(S)$, and every lift $\tilde{A} \in {}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S)$, a trivialization $T_{\tilde{A}} : F_A(\lambda) \xrightarrow{\cong} \lambda$.

[Think: 'the map $\Sigma \mapsto F_\Sigma$ is topological.']

These should satisfy $T_{\mathbf{1}_S} = \mathrm{id}$ and $T_{\tilde{A}_1\cup\tilde{A}_2} = T_{\tilde{A}_1} \circ T_{\tilde{A}_2}$ (omitting the isomorphism $F_{A_1\cup A_2} \cong F_{A_1} \circ F_{A_2}$ for better readability). Moreover, the central $\mathbb{C}^\times$ should act in the standard way: $T_{z\tilde{A}} = z \cdot T_{\tilde{A}}$ for every $z \in \mathbb{C}^\times$.

(3b) For every circle S and every object $\lambda \in \mathcal{C}(S)$, the map

$$\begin{array}{ccc} {}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S) & \longrightarrow & \mathrm{End}(U(\lambda)) \\ \Psi & & \Psi \\ \tilde{A} & \mapsto & \Big(U(\lambda) \xrightarrow{Z_A} U(F_A(\lambda)) \xrightarrow{U(T_{\tilde{A}})} U(\lambda)\Big) \end{array} \tag{3}$$

is holomorphic (holomorphic in the interior, and smooth all the way to the boundary[4]).

[Think: 'the map $A \mapsto Z_A$ is holomorphic.']

The vacuum sector and its symmetries

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$, and $S^1 := \partial\mathbb{D}$. Given a chiral Segal CFT, let us define the *unit object*

$$\mathbf{1} \in \mathcal{C}(S^1)$$

to be the image of $1_\emptyset := \mathbb{C} \in \mathsf{Vec}_{\mathrm{f.d.}} = \mathcal{C}(\emptyset)$ under the functor $F_\mathbb{D} : \mathcal{C}(\partial_{in}\mathbb{D}) = \mathcal{C}(\emptyset) \to \mathcal{C}(\partial_{out}\mathbb{D}) = \mathcal{C}(S^1)$. We define the *vacuum sector* H_0 of the CFT to be the underlying vector space of $\mathbf{1}$:

$$H_0 := U(\mathbf{1}).$$

The vacuum sector comes with a *vacuum vector*

$$\Omega := Z_\mathbb{D}(1) \in H_0$$

defined as the image of $1 \in \mathbb{C}$ under the map $Z_\mathbb{D} : \mathbb{C} = U(1_\emptyset) \to U(F_\mathbb{D}(1_\emptyset)) = U(\mathbf{1})$. More generally, given a complex cobordism Σ with empty incoming boundary, we get a vector space $H_\Sigma := U(F_\Sigma(1_\emptyset))$, and a vacuum vector

$$\Omega_\Sigma := Z_\Sigma(1) \in H_\Sigma.$$

[4]Depending on the type of topological vector spaces that one uses, it might be appropriate to only require the map $\tilde{A} \mapsto U(T_{\tilde{A}}) \circ Z_A$ to be continuous for the topology of pointwise convergence, as opposed to smooth all the way to the boundary.

Remark. In examples of interest, the unit object $\mathbf{1} \in \mathcal{C}(S^1)$ is always a simple, equivalently, the vacuum sector H_0 is an irreducible $\mathcal{A}(S^1)$-module, but this property is not guaranteed by the axioms. A chiral Segal CFT with that property is called *irreducible*.

Given a finite collection $(\mathcal{C}_i, U_i, F_i, Z_i, T_i)$ of irreducible Segal CFTs of same central charge, their direct sum is defined on connected manifolds by $S \mapsto (\bigoplus \mathcal{C}_i(S), \bigoplus U_i)$, $\Sigma \mapsto (\bigoplus F_{i,\Sigma}, Z_{i,\Sigma})$, $\tilde{A} \mapsto \bigoplus T_{i,\tilde{A}}$, and is defined on disconnected manifolds to be the tensor product rule of what the theory assigns to each connected component. Every chiral Segal CFT is a direct sum of irreducible ones, and that the direct sum decomposition is canonical. In that sense, the study of chiral Segal CFTs completely reduces to the study of irreducible ones.

Given an object $\lambda \in \mathcal{C}(S^1)$, we sometimes write $H_\lambda := U(\lambda)$. If λ is irreducible and $\lambda \ncong \mathbf{1}$, we call this a *charged sector* of the CFT.

The vacuum sector depends functorially on the disc $\mathbb{D}$. The automorphism group $\mathrm{Aut}(\mathbb{D}) = PSU(1,1)$ therefore acts of H_0. And since the construction of $\Omega \in H_0$ also depends functorially on $\mathbb{D}$, it is invariant under the action of that group:

$$\Omega \in H_0^{PSU(1,1)}.$$

But there is a much bigger (semi)group which acts on H_0 *[we've set up things in such a way that all these actions are right actions]*. First of all, by the construction described in (3), the semigroup ${}^{\mathbb{C}^\times \oplus \mathbb{Z}}\mathrm{Ann}_c(S)$ acts on the right on H_0. But there is another semigroup which also acts. Let us define the semigroup of *univalent maps* of the disc by:

$$\mathrm{Univ}(\mathbb{D}) := \{\psi : \mathbb{D} \to \mathbb{D} \,|\, \psi \text{ is an embeddings}\}.$$

There is an obvious embedding $\mathrm{Univ}(\mathbb{D}) \hookrightarrow \mathrm{Ann}(S^1)$ which sends an element $\psi \in \mathrm{Univ}(\mathbb{D})$ to the annulus $A_\psi := \big(\mathbb{D} \setminus \psi(\mathring{\mathbb{D}}), \varphi_{in} = \psi|_{\partial\mathbb{D}}, \varphi_{out} = \mathrm{id}\big)$. It satisfies $A_{\psi_1 \circ \psi_2} = A_{\psi_1} \cup A_{\psi_2}$. The action of ψ on H_0 is then given by

$$H_0 = U(F_{\mathbb{D}}(1_\emptyset)) \xrightarrow{Z_{A_\psi}} U(F_{A_\psi} F_{\mathbb{D}}(1_\emptyset)) \cong U(F_{A_\psi \cup \mathbb{D}}(1_\emptyset)) \cong U(F_{\mathbb{D}}(1_\emptyset)) = H_0. \tag{4}$$

We prove the next lemma under the assumption that the Segal CFT is irreducible (the statement also holds true without that assumption):

Lemma 4 *Let $\psi : \mathbb{D} \to \mathbb{D}$ be a univalent map, let A be its image in $\mathrm{Ann}(S)$, and let $\tilde{A} \in {}^{\mathbb{C}^\times \times \mathbb{Z}}\mathrm{Ann}_c(S)$ be an arbitrary lift. Then the actions of $\tilde{A}$ and of ψ on H_0 given by* (3) *and* (4)*, respectively, agree up to scalar.*

Proof. Write S_ψ for the isomorphism $F_{A_\psi} F_{\mathbb{D}}(1_\emptyset) \xrightarrow{\cong} F_{A_\psi \cup \mathbb{D}}(1_\emptyset) \xrightarrow{\cong} F_{\mathbb{D}}(1_\emptyset)$. By definition, the actions of $\tilde{A}$ and ψ are given by $U(T_{\tilde{A}}) \circ Z_A$ and $U(S_\psi) \circ Z_A$, respectively. Since $\mathbf{1} = F_{\mathbb{D}}(1_\emptyset) \in \mathcal{C}(S^1)$ is a simple object, there exists a constant $a \in \mathbb{C}^\times$ such that $T_{\tilde{A}} = a \cdot S_\psi$. It follows that $U(T_{\tilde{A}}) \circ Z_A = a \cdot U(S_\psi) \circ Z_A$. $\square$

Summarizing, we have the following four (semi)groups which all act compatibly on the vacuum sector of a CFT. The ones in the top row act honestly (i.e., without central extension), whereas the ones in the bottom row only act projectively:

$$\begin{tikzcd} PSU(1,1) \arrow[r, hook] \arrow[d, hook] & \mathrm{Univ}(\mathbb{D}) \arrow[d, hook] & \text{act honestly on } H_0 \arrow[l] \\ \mathrm{Diff}(S^1) \arrow[r, hook] & \mathrm{Ann}(S^1) & \text{only act projectively} \arrow[l] \end{tikzcd}$$

Let ${}^{\mathbb{Z}}PSU(1,1)$ denote the universal cover of $PSU(1,1)$, which is also its universal central extension. Similarly, let us write ${}^{\mathbb{Z}}\mathrm{Univ}(\mathbb{D})$ for the universal cover of $\mathrm{Univ}(\mathbb{D})$. The inclusion $\mathrm{Univ}(\mathbb{D}) \hookrightarrow \mathrm{Ann}(S)$ induces a map of central extensions

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "{n \mapsto (0,n)}"'] & {}^{\mathbb{Z}}\mathrm{Univ}(\mathbb{D}) \arrow[r] \arrow[d] & \mathrm{Univ}(\mathbb{D}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{C}^\times \oplus \mathbb{Z} \arrow[r] & {}^{\mathbb{C}^\times \oplus \mathbb{Z}}\mathrm{Ann}_c(S) \arrow[r] & \mathrm{Ann}(S) \arrow[r] & 0 \end{tikzcd}$$

By Lemma 4, the restriction of the action of ${}^{\mathbb{C}^\times \oplus \mathbb{Z}}\mathrm{Ann}_c(S)$ on H_0 to the subsemigroup ${}^{\mathbb{Z}}\mathrm{Univ}(\mathbb{D})$ descends to the quotient $\mathrm{Univ}(\mathbb{D})$. So the action of ${}^{\mathbb{C}^\times \oplus \mathbb{Z}}\mathrm{Ann}_c(S)$ on H_0 descends to

$${}^{\mathbb{C}^\times}\mathrm{Ann}_c(S) := {}^{\mathbb{C}^\times \oplus \mathbb{Z}}\mathrm{Ann}_c(S)/\mathbb{Z}.$$

Let also:

$${}^{U(1)}\mathrm{Diff}_c(S) := {}^{U(1) \oplus \mathbb{Z}}\mathrm{Diff}_c(S)/\mathbb{Z}.$$

All in all, *the following (semi)groups act on the vacuum sector of any chiral CFT:*

$$\boxed{\begin{tikzcd} PSU(1,1) \arrow[r, hook] \arrow[d, hook] & \mathrm{Univ}(\mathbb{D}) \arrow[d, hook] \\ {}^{U(1)}\mathrm{Diff}_c(S^1) \arrow[r, hook] & {}^{\mathbb{C}^\times}\mathrm{Ann}_c(S^1) \end{tikzcd}}$$

This should be contrasted with the case of charged sectors, where it's only the following (semi)groups which act:

$$\begin{tikzcd} {}^{\mathbb{Z}}PSU(1,1) \arrow[r, hook] \arrow[d, hook] & {}^{\mathbb{Z}}\mathrm{Univ}(\mathbb{D}) \arrow[d, hook] \\ {}^{U(1) \oplus \mathbb{Z}}\mathrm{Diff}_c(S^1) \arrow[r, hook] & {}^{\mathbb{C}^\times \oplus \mathbb{Z}}\mathrm{Ann}_c(S^1) \end{tikzcd}$$

Let $\lambda \in \mathcal{C}(S^1)$ be a simple object, and let $H_\lambda = U(\lambda)$ be the corresponding charged sector. For $\tilde{A}$ in the kernel of the map ${}^{\mathbb{C} \oplus \mathbb{Z}}\mathrm{Ann}(S) \to \mathrm{Ann}(S)$,

$$\tilde{A} \in \ker\left({}^{\mathbb{C} \oplus \mathbb{Z}}\mathrm{Ann}(S) \to \mathrm{Ann}(S)\right) \;\cong\; \mathbb{C} \oplus \mathbb{Z},$$

since F_A and Z_A are trivial, the action (3) of $\tilde{A}$ on H_λ simplifies to $\tilde{A} \mapsto U(T_{\tilde{A}})$, where moreover $T_{\tilde{A}} : \lambda \to \lambda$ is just a scalar. To recapitulate, by (3), for any simple object $\lambda \in \mathcal{C}(S^1)$, the semigroup

$$^{\mathbb{C}\oplus\mathbb{Z}}\mathrm{Ann}(S)$$

acts on the corresponding charged sector H_λ. The central $\mathbb{C}$ acts via the character $z \mapsto e^{cz}$, where c is the central charge. This is an invariant of the chiral CFT and does not depend on the choice of sector. The central $\mathbb{Z}$ acts via some character $n \mapsto (\theta_\lambda)^n$, where θ_λ is called the conformal spin of the sector (in a rational CFT, the conformal spins are always roots of unity). This number does depend on the sector (for example, the conformal spin of the vacuum sector is always trivial). Let L_0 be the infinitesimal generator or rotations (we'll be more specific about this later). This operator always has spectrum bounded from below, its smallest eigenvalue is denoted h_λ and called the minimal energy. It satisfies $e^{2\pi i h_\lambda} = \theta_\lambda$.

The semigroup of annuli as a complexification of $\mathrm{Diff}(S^1)$

$\mathrm{Diff}(S^1)$ is an infinite dimensional Lie group whose Lie algebra can be identified with

$$\mathfrak{X}(S^1) := \big\{ f(z)\tfrac{\partial}{\partial z} \,\big|\, \tfrac{f(z)}{z} \in i\mathbb{R} \big\},$$

vector fields on S^1. We recall the well-known formula for the Lie bracket of vector fields:

$$\big[f(z)\tfrac{\partial}{\partial z}, g(z)\tfrac{\partial}{\partial z}\big] = (fg' - gf')\tfrac{\partial}{\partial z}$$

Remark. *It is a great annoyance in differential geometry that, for a manifold M, the Lie algebra of $\mathrm{Diff}(M)$ is not $\mathfrak{X}(M)$ but instead $\mathfrak{X}(M)^{\mathrm{op}}$, the Lie algebra of vector field on M equipped with the opposite of the usual Lie bracket of vector fields.*

The complexification $\mathfrak{X}_{\mathbb{C}}(S^1)$ of the Lie algebra $\mathfrak{X}(S^1)$ admits a topological basis given by the vector fields

$$\ell_n := -z^{n+1}\tfrac{\partial}{\partial z}. \tag{5}$$

These satisfy an algebra known as the *Witt algebra*:

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n}.$$

Remark. *When acting on the vacuum sector of a CFT, the ℓ_n with $n < 0$ will be acting as creation operators, whereas ℓ_n with $n > 0$ will be acting as annihilation operators. The operator associated to ℓ_0 will have an interpretation as "the energy", and will always have positive spectrum. This forces us to include the minus sign in the definition* (5)*.*

The subalgebra spanned by ℓ_{-1}, ℓ_0, and ℓ_0 is isomorphic to $\mathfrak{su}(1,1)_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$ under the isomorphism

$$\ell_{-1} \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \ell_0 \leftrightarrow \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \ell_1 \leftrightarrow \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

It corresponds to the subgroup $PSU(1,1) \subset \mathrm{Diff}(S^1)$.

The Witt algebra $\mathbb{W} = \mathrm{Span}\{\ell_n\}_{n\in\mathbb{Z}}$ is famously known for being a Lie algebra that does not have an associated Lie group. Said otherwise, the Lie group $\mathrm{Diff}(S^1)$ does not have a complexification. To see that, note that the subalgebra $\mathrm{Span}\{\ell_{-n}, \ell_0, \ell_n\} \subset \mathbb{W}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ via the map

$$\tfrac{1}{n}\ell_{-n} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \tfrac{1}{n}\ell_0 \mapsto \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \tfrac{1}{n}\ell_n \mapsto \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

If $\mathbb{W}$ or $\mathfrak{X}_{\mathbb{C}}(S^1)$ were to integrate to a Lie group G, then the above map $\mathfrak{sl}(2,\mathbb{C}) \to \mathbb{W}$ would integrate to a homomorphism $SL(2,\mathbb{C}) \to G$ (since $SL(2,\mathbb{C})$ is simply connected). But the relation $\exp\left(4\pi i \cdot \frac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right) = 1$ holds in $SL(2,\mathbb{C})$. Therefore, for every n, the relation $\exp(4\pi i \cdot \frac{1}{n}\ell_0) = 1$ would have to hold in G. Clearly impossible. $\lightning$

One of the wonderful ideas of Graeme Segal (independently also due to Y. Neretin) is that, even though there is no Lie group which is the complexification of $\mathrm{Diff}(S^1)$, there is a complex semigroup which plays that role: the semigroup of annuli.

To convince you that $\mathrm{Ann}(S^1)$ indeed behaves like $\mathrm{Diff}_{\mathbb{C}}(S^1)$, let us compute its tangent space at the identity, and check that we get $\mathfrak{X}_{\mathbb{C}}(S^1)$. There is of course a problem, because $\mathrm{Ann}(S^1)$ is not a manifold: it is some kind of infinite dimensional manifold with boundary (or rather with corners) and its identity element definitely sits at the boundary. But things are not too bad. We'll see that $\mathrm{Ann}(S^1)$ can be identified with a closed subspace of an honest manifold M, and that its interior $\mathring{\mathrm{Ann}}(S^1) := \{A \in \mathrm{Ann}(S^1) \,|\, A \text{ is thick}\}$ is an open subset of M.

Let M be the quotient of the space of pairs of embeddings $\psi_- : \mathbb{D} \hookrightarrow \mathbb{CP}^1$, $\psi_+ : \mathbb{D} \hookrightarrow \mathbb{CP}^1$ by the action of $PSL(2,\mathbb{C})$. Equivalently,

$$M := \left\{ \begin{matrix} \psi_- : \mathbb{D} \hookrightarrow \mathbb{CP}^1 \\ \psi_+ : \mathbb{D} \hookrightarrow \mathbb{CP}^1 \end{matrix} \,\middle|\, \psi_-(z) = 1/z + O(z) \right\} \tag{6}$$

(where we use the identification $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ to make sense of the condition $\psi_-(z) = 1/z + O(z)$).

Given an annulus $A \in \mathrm{Ann}(S^1)$, let $\mathbb{P}_A := \mathbb{D} \cup A \cup \mathbb{D}$ (the left $\mathbb{D}$ has its boundary parametrized by $z \mapsto \bar{z}$), and let us write $\iota_- : \mathbb{D} \to \mathbb{P}_A$ and $\iota_+ : \mathbb{D} \to \mathbb{P}_A$ for the two inclusions. By the classification of genus zero Riemann surfaces (there is only one up to isomorphism), $\mathbb{P}_A \cong \mathbb{CP}^1$ and there is a unique isomorphism $\psi : \mathbb{P}_A \to \mathbb{CP}^1$ that satisfies $\psi(\iota_-(z)) = 1/z + O(z)$ *[there is a three dimensional space of isomorphisms $\mathbb{P}_A \cong \mathbb{CP}^1$; the condition $\psi(\iota_-(z)) = 1\cdot z^{-1} + 0\cdot 1 + 0\cdot z + \ldots$ cuts down that three dimensional space to something zero dimensional]*. So we get a map

$$\begin{array}{rcl} \mathrm{Ann}(S^1) & \to & M \\ A & \mapsto & (\psi\iota_-, \psi\iota_+) \end{array}$$

whose image is the set of $(\psi_-, \psi_+) \in M$ such that $\psi_-(\mathring{\mathbb{D}}) \cap \psi_+(\mathring{\mathbb{D}}) = \emptyset$.

The unit element $\mathbf{1}_{S^1} \in \mathrm{Ann}(S^1)$ maps to the pair $(\psi_-^0, \psi_+^0) := (z \mapsto z^{-1}, z \mapsto z)$. The tangent space of M at that point can be identified with the set of pairs of vector fields

(v_-, v_+), $v_\pm \in \Gamma(\mathbb{D}, (\psi^0_\pm)^*T\mathbb{CP}^1)$, with v_- vanishing to third order at the origin. Here,

$$\Gamma\big(\mathbb{D}, (\psi^0_-)^*T\mathbb{CP}^1\big) = \mathrm{Span}\{\ell_n\}_{n\leq 1} \qquad \Gamma\big(\mathbb{D}, (\psi^0_+)^*T\mathbb{CP}^1\big) = \mathrm{Span}\{\ell_n\}_{n\geq -1}$$

with the coefficients of ℓ_n decaying faster than any power as $n \to \infty$. The condition that v_- vanishes to third order at the origin says that the first three coefficients (the coefficients of ℓ_{-1}, ℓ_0, ℓ_1) are zero. So we get $T_{(\psi^0_-,\psi^0_+)}M = \mathrm{Span}\{\ell_n\}_{n<-1} \oplus \mathrm{Span}\{\ell_n\}_{n\geq -1} = \mathfrak{X}_\mathbb{C}(S^1)$, as desired.

We can also describe the tangent space of $\mathrm{Ann}(S)$ at an arbitrary point $A \in \mathrm{Ann}(S)$.

Lemma.

$$T_A\mathrm{Ann}(S) = \frac{\mathfrak{X}_\mathbb{C}(\partial_{out}A) \oplus \mathfrak{X}_\mathbb{C}(\partial_{in}A)}{\mathfrak{X}_{\mathrm{hol}}(A)} = \frac{\mathfrak{X}_\mathbb{C}(S) \oplus \mathfrak{X}_\mathbb{C}(S)}{\mathfrak{X}_{\mathrm{hol}}(A)},$$

where $\mathfrak{X}_{\mathrm{hol}}(A)$ denotes the set of vector field on A which are holomorphic in the interior, and smooth all the way to the boundary (as in (1)*).*

Proof. Wlog $S = S^1$ (the standard circle). Let M be as in (6), and let

$$M_0 = \big\{(\psi_-, \psi_+) \in M \,\big|\, \psi_-(\mathring{\mathbb{D}}) \cap \psi_+(\mathring{\mathbb{D}}) = \emptyset\big\}.$$

Then $M_0 \cong \mathrm{Ann}(S^1)$ as explained above. Let

$$N = \big\{(\gamma_{out}, \gamma_{in}) \,\big|\, \gamma_{out}, \gamma_{in} : S^1 \hookrightarrow \mathbb{CP}^1\big\}$$

and let $s : M \to N$ be the map given by $s : (\psi_-, \psi_+) \mapsto (\gamma_{out} := \psi_-|_{\partial\mathbb{D}}, \gamma_{in} := \psi_+|_{\partial\mathbb{D}})$. Let $N_0 \subset N$ be the saturation of $s(M_0) \subset N$ under the action of $\mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$. In more down to earth terms, N_0 is the set of pairs $(\gamma_{out}, \gamma_{in})$ such that the curves $\gamma_{out}(S^1)$ and $\gamma_{in}(S^1)$ 'bound an annulus', and $\gamma_{out}(S^1)$ 'encircles ∞'.

The map s admits a retraction⚠ $r : N_0 \to M_0$ which sends a pair $(\gamma_{out}, \gamma_{in})$ to the annulus bound by $\gamma_{out}(S^1)$ and $\gamma_{in}(S^1)$, and then identifies that annulus with an element of M_0. By construction, this map satisfies $s \circ r = \mathrm{id}$. The fiber $r^{-1}(A)$ over a point $A \in \mathrm{Ann}(S^1) \cong M_0$ is the set of embeddings $\sigma : A \hookrightarrow \mathbb{CP}^1$ such that $\partial_{out}A$ 'encircles ∞'.

At the level of tangent spaces, the diagram

$$\begin{tikzcd} r^{-1}(A) \arrow[r] & N_0 \arrow[r, "r"] & M_0 \arrow[l, bend right, "s"'] \end{tikzcd}$$

induces a split short exact sequence

$$0 \to \Gamma_{\mathrm{hol}}\big(A, \sigma^*T\mathbb{CP}^1\big) \to \Gamma\big(S^1 \sqcup S^1, (\gamma_{out} \sqcup \gamma_{in})^*T\mathbb{CP}^1\big) \to T_A\mathrm{Ann}(S) \to 0.$$

The result follows since $\Gamma_{\mathrm{hol}}(A, \sigma^*T\mathbb{CP}^1) = \Gamma_{\mathrm{hol}}(A, TA) = \mathfrak{X}_{\mathrm{hol}}(A)$, and $\Gamma(S^1 \sqcup S^1, (\gamma_{in} \sqcup \gamma_{out})^*T\mathbb{CP}^1) = \Gamma(S^1, T_\mathbb{C}S^1)^{\oplus 2} = \mathfrak{X}_\mathbb{C}(S^1) \oplus \mathfrak{X}_\mathbb{C}(S^1)$.
⚠Unfortunately, there is a gap in the argument: I haven't showed that r is smooth (the restriction of a smooth map $N \to M$). I don't know how to prove that fact (and hence I don't know whether it's true — I only think that it is). $\square$

Note that, for $A = \mathbf{1}_S$, the above lemma recovers our earlier result:

$$T_{\mathbf{1}}\mathrm{Ann}(S) \;=\; \frac{\mathfrak{X}_{\mathbb{C}}(S) \oplus \mathfrak{X}_{\mathbb{C}}(S)}{\mathfrak{X}_{\mathbb{C}}(S)} \;=\; \mathfrak{X}_{\mathbb{C}}(S).$$

Remark. *Unlike in the case of groups, the map $T_{\mathbf{1}}\mathrm{Ann}(S) \to T_A\mathrm{Ann}(S)$ induced by left multiplication by A is not an isomorphism (except when A is completely thin). Instead, it is an embedding with dense image.*

Proposition. *The multiplication map $\mathrm{Ann}(S) \times \mathrm{Ann}(S) \to \mathrm{Ann}(S)$ is holomorphic. I.e., the map*

$$T_{A_1}\mathrm{Ann}(S) \oplus T_{A_2}\mathrm{Ann}(S) \;\to\; T_{A_1 \cup A_2}\mathrm{Ann}(S)$$

induced by the composition of annuli is complex linear.

Proof. Given $A_1, A_2 \in \mathrm{Ann}(S)$, let $\mathbb{P}_{A_1,A_2} := \mathbb{D} \cup A_1 \cup A_2 \cup \mathbb{D}$, and let $\iota_- : \mathbb{D} \to \mathbb{P}_{A_1,A_2}$ and $\iota_+ : \mathbb{D} \to \mathbb{P}_{A_1,A_2}$ be the two inclusions. Letting $\psi : \mathbb{P}_{A_1,A_2} \to \mathbb{CP}^1$ be the unique isomorphism such that $\psi(\iota_-(z)) = 1/z + O(z)$, one gets a map

$$\begin{aligned} \mathrm{Ann}(S) \times \mathrm{Ann}(S) &\;\to\; \big\{(\gamma_1, \gamma_2, \gamma_3) \,\big|\, \gamma_i : S \hookrightarrow \mathbb{CP}^1\big\} \\ (A_1, A_2) &\;\mapsto\; \big(\psi|_{\partial_{out} A_1}, \psi|_{\partial_{in} A_1} = \psi|_{\partial_{out} A_2}, \psi|_{\partial_{in} A_2}\big). \end{aligned}$$

At the level of tangent spaces, the existence of that map means that the vector space

$$\big\{((v_1^{out}, v_1^{in}), (v_2^{out}, v_2^{in})) \in \big(\mathfrak{X}_{\mathbb{C}}S \oplus \mathfrak{X}_{\mathbb{C}}S\big) \oplus \big(\mathfrak{X}_{\mathbb{C}}S \oplus \mathfrak{X}_{\mathbb{C}}S\big) \,\big|\, v_1^{out} = v_2^{in}\big\}$$

surjects onto

$$T_{A_1}\mathrm{Ann}(S) \oplus T_{A_2}\mathrm{Ann}(S) \;=\; \frac{\mathfrak{X}_{\mathbb{C}}S \oplus \mathfrak{X}_{\mathbb{C}}S}{\mathfrak{X}_{\mathrm{hol}}(A_1)} \oplus \frac{\mathfrak{X}_{\mathbb{C}}S \oplus \mathfrak{X}_{\mathbb{C}}S}{\mathfrak{X}_{\mathrm{hol}}(A_2)}\,.$$

So we get a commutative diagram

$$\begin{tikzcd} \big\{((v_1^{out}, v_1^{in}), (v_2^{out}, v_2^{in})) \in \big(\mathfrak{X}_{\mathbb{C}}S \oplus \mathfrak{X}_{\mathbb{C}}S\big) \oplus \big(\mathfrak{X}_{\mathbb{C}}S \oplus \mathfrak{X}_{\mathbb{C}}S\big) \,\big|\, v_1^{in} = v_2^{out}\big\} \arrow[r, "{((v_1^{out}, v_1^{in}), (v_2^{out}, v_2^{in})) \,\mapsto\, (v_2^{out}, v_1^{in})}"'] \arrow[d, two heads] & \mathfrak{X}_{\mathbb{C}}S \oplus \mathfrak{X}_{\mathbb{C}}S \arrow[d, two heads] \\ T_{A_1}\mathrm{Ann}(S) \oplus T_{A_2}\mathrm{Ann}(S) \arrow[r] & T_{A_1 \cup A_2}\mathrm{Ann}(S) \end{tikzcd}$$

The top horizontal map and the two vertical maps are visibly complex linear. Therefore so is the bottom map. $\square$

We finish this section by explaining why the Lie algebra of $\mathrm{Ann}(S)$ is $(\mathfrak{X}_{\mathbb{C}}(S), [\ ,\ ]^{\mathrm{op}})$, the Lie algebra of complexified vector fields on S equipped with the opposite of the usual bracket of vector fields. We already saw that the Lie algebra $\mathrm{Ann}(S)$ is isomorphic to $\mathfrak{X}_{\mathbb{C}}(S)$ as a vector space. Let us write $[\ ,\ ]_{\mathrm{Ann}}$ for the Lie bracket on $\mathfrak{X}_{\mathbb{C}}(S)$ induced by the fact that it is the Lie algebra of $\mathrm{Ann}(S)$. The inclusion $\mathrm{Diff}(S) \hookrightarrow \mathrm{Ann}(S)$ induces an inclusion of Lie algberas $(\mathfrak{X}(S), [\ ,\ ]^{\mathrm{op}}) \hookrightarrow (\mathfrak{X}_{\mathbb{C}}(S), [\ ,\ ]_{\mathrm{Ann}})$. The latter being a complex Lie algebra, its bracket is completely determined by what happens on the real subspace $\mathfrak{X}(S)$. Therefore $[\ ,\ ]_{\mathrm{Ann}} = [\ ,\ ]^{\mathrm{op}}$.

The Virasoro algebra

We now turn to the question of describing the universal central extensions of $\mathrm{Diff}(S)$ and of $\mathrm{Ann}(S)$. We claim that there exist central extensions

$$\begin{gathered} 0 \to i\mathbb{R} \oplus \mathbb{Z} \to {}^{i\mathbb{R}\oplus\mathbb{Z}}\mathrm{Diff}(S) \to \mathrm{Diff}(S) \to 0 \\ 0 \to \mathbb{C} \oplus \mathbb{Z} \to {}^{\mathbb{C}\oplus\mathbb{Z}}\mathrm{Ann}(S) \to \mathrm{Ann}(S) \to 0 \end{gathered} \tag{7}$$

and that these are universal central extensions.

At the level of Lie algebras (things are always easier at he level of Lie algebras), the corresponding claim is that there exist central extensions

$$\begin{gathered} 0 \to i\mathbb{R} \to {}^{i\mathbb{R}}X(S) \to \mathfrak{X}(S) \to 0 \\ 0 \to \mathbb{C} \to {}^{\mathbb{C}}X_{\mathbb{C}}(S) \to \mathfrak{X}_{\mathbb{C}}(S) \to 0 \end{gathered}$$

and that these are universal central extensions. If one takes S to be the standard circle S^1, one can be more specific, and write down the cocycle that describes these central extensions. This is the *Virasoro cocycle*:

$$\omega_{Vir}\left(f(z)\tfrac{\partial}{\partial z}, g(z)\tfrac{\partial}{\partial z}\right) = \tfrac{1}{12}\int_{S^1} \tfrac{\partial^3 f}{\partial z^3}(z)\, g(z)\, \tfrac{dz}{2\pi i} \tag{8}$$

equivalently:
$$\omega_{Vir}(\ell_m, \ell_n) = \tfrac{1}{12}(m^3 - m)\delta_{m+n,0}.$$

The Witt algebra $\mathbb{W}$ is the algebraic span of the ℓ_n's (the Lie algebra of algebraic vector fields on $\mathbb{C}^\times$), and its universal central extension is called the *Virasoro algebra*. It is standard convention to denote the basis vectors of the Witt algebra ℓ_n, and the corresponding basis vectors of the Virasoro by upper case letters L_n:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & Vir & \longrightarrow & \mathbb{W} & \longrightarrow & 0 \\ & & & & \cup\!\!\!\!\!\,\vert & & \cup\!\!\!\!\!\,\vert & & \\ & & & & L_n & \mapsto & \ell_n & & \end{array}$$

The commutation relations of the Virasoro algebra are given by

$$\boxed{[L_m, L_n] \;=\; (m-n)L_{m+n} + \tfrac{1}{12}(m^3 - m)\delta_{m+n,0}.}$$

Remark. The Virasoro commutation relations are usually written in a way that includes the central charge:

$$[L_m, L_n] \;=\; (m-n)L_{m+n} + \tfrac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{9}$$

Let Vir_c be the Lie algebra defined by (9). Provided c and c' are non-zero, there is an isomorphism $Vir_c \cong Vir_{c'}$ that fits in a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, "z \mapsto z\cdot c'/c"'] & Vir_c \arrow[r] \arrow[d, "\wr"'] & \mathbb{W} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{C} \arrow[r] & Vir_{c'} \arrow[r] & \mathbb{W} \arrow[r] & 0 \end{tikzcd}$$

The Lie algebras Vir_c and $Vir_{c'}$ are however distinct *as central extensions of* $\mathbb{W}$ *by* $\mathbb{C}$ (i.e, there's no way to arrange for the left vertical map to be the identity map $\mathrm{id}_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$).

Also, the notion of a *representation* of Vir_c is distinct from that of a representation of $Vir_{c'}$, because one always includes the requirement that the central element $1 \in \mathbb{C}$ acts by the identity operator.

Before proceeding, let us review some rudiments of Lie algebra cohomology. Let $\mathfrak{g}$ be a Lie algebra, and let A be a vector space.

A 2-cocycle is a bilinear map $\omega : \mathfrak{g} \times \mathfrak{g} \to A$ which is antisymmetric, and satisfies

$$\sum^{3} \omega([X,Y],Z) = 0.$$

Given a 2-cocycle, one can form a central extension $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus A$, with Lie bracket

$$\big[(X,a),(Y,b)\big]_{\tilde{\mathfrak{g}}} := \big([X,Y]_{\mathfrak{g}}, \omega(X,Y)\big).$$

which fits into a central extension $0 \to A \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$ (an extension such that $A \subset Z(\tilde{\mathfrak{g}})$). If the cocycle can written in the form

$$\omega(X,Y) = \mu([X,Y])$$

for some linear map $\mu : \mathfrak{g} \to \mathbb{C}$ (typically not a Lie algebra homomorphism), then we say that ω is a trivial 2-cocycle, and write $\omega = d\mu$.

Theorem. *The second Lie algebra cohomology group*

$$H^2(\mathfrak{g}; A) := \frac{\{\text{2-cocycles}\}}{\{\text{trivial 2-cocycles}\}}$$

is canonically isomorphic to the set of isomorphism classes of central extensions of $\mathfrak{g}$ by A, where two central extensions $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}'$ are called isomorphic if there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}_A"'] & \tilde{\mathfrak{g}} \arrow[r] \arrow[d, "\cong"'] & \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{id}_{\mathfrak{g}}"'] & 0 \\
0 \arrow[r] & A \arrow[r] & \tilde{\mathfrak{g}}' \arrow[r] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

where the two outer vertical maps are identity maps.

Proof outline. We already saw how to construct a central extension from a 2-cocycle. Suppose now that $\omega_2 - \omega_1 = d\mu$. Then

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & \tilde{\mathfrak{g}}_1 = \mathfrak{g} \oplus A \arrow[r] \arrow[d, "\left(\begin{smallmatrix}1 & 0\\ \mu & 1\end{smallmatrix}\right)"] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & \tilde{\mathfrak{g}}_2 = \mathfrak{g} \oplus A \arrow[r] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

is an isomorphism. So the map $\{\text{2-cocycles}\} \to \{\text{central extensions}\}$ descends to a map $H^2(\mathfrak{g}; A) \to \{\text{iso classes of central extensions}\}$.

$\leftarrow$ Given a central extension of $\mathfrak{g}$ by A, pick a splitting

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & \tilde{\mathfrak{g}} \arrow[r, "r"] & \mathfrak{g} \arrow[l, bend right, "s"'] \arrow[r] & 0 \end{tikzcd}$$

(usually not a Lie algebra homomorphism) and let $\omega(X,Y) := [s(X), s(Y)] - s([X,Y])$. Given another splitting, we can write it as $s' = s + \mu$ for some $\mu : \mathfrak{g} \to A$. The corresponding cocycles satisfy $\omega' = \omega - d\mu$. So they're equal in $H^2(\mathfrak{g}; A)$. $\square$

We also have:

Proposition 5 *Let $0 \to A_i \to \tilde{\mathfrak{g}}_i \to \mathfrak{g} \to 0$ be central extensions that fit into a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d, "f"'] & \tilde{\mathfrak{g}}_1 \arrow[r] \arrow[d] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A_2 \arrow[r] & \tilde{\mathfrak{g}}_2' \arrow[r] & \mathfrak{g} \arrow[r] & 0 \end{tikzcd}$$

and let $[\omega_i] \in H^2(\mathfrak{g}_i, A_i)$ be the corresponding cohomology classes. Then $[\omega_2]$ is the image of $[\omega_1]$ under the map $H^2(\mathfrak{g}, A_1) \to H^2(\mathfrak{g}, A_2)$ induced by $f : A_1 \to A_2$.

Theorem 6 *The second cohomology of the Witt algebra is one dimensional $H^2(\mathbb{W}, \mathbb{C}) \cong \mathbb{C}$, and the Virasoro cocycle $[\omega_{Vir}]$ is a generator.*

Let us check that ω_{Vir} is indeed a cocycle. For the purpose of this computation, we rewrite (8) in the following abbreviated (and less precise) form:

$$\omega_{Vir}(f,g) = \oint f'''g = \oint f'g''$$

We then easily compute: $\sum^3 \oint (fg' - f'g)'h'' = \sum^3 \oint (\cancel{f'g'} + fg'' - f''g - \cancel{f'g'})h'' = 0.$

The following lemma will be surprisingly useful:

Lemma 7 *Let $\mathfrak{g}$ be a Lie algebra, and let $X \in \mathfrak{g}$ be such that* $\mathrm{ad}(X)$ *exponentiates to a 1-parameter family of automorphisms of* $\mathfrak{g}$ [<u>For us</u>: $\mathfrak{g} = \mathbb{W}$, $X = i\ell_0$, and $\mathrm{ad}(i\ell_0)$ exponentiates to an action of S^1 on $\mathbb{W}$]. *For $\xi \in \mathfrak{g}$, let $\xi_t := \exp\big(t \cdot \mathrm{ad}(X)\big)(\xi)$, so that $\frac{d}{dt}\xi_t = [X, \xi_t]$. Then, for any 2-cocycle ω, we have*

$$[\omega] = [\omega_t] \in H^2(\mathfrak{g}),$$

where $\omega_t(\xi,\eta) := \omega(\xi_t, \eta_t)$.

Proof.

$$\begin{aligned}
\omega(\xi_T,\eta_T)-\omega(\xi,\eta) &= \int_0^T \left(\tfrac{d}{dt}\omega(\xi_t,\eta_t)\right)dt\\
&\overset{\frac{d}{dt}\xi_t=[X,\xi_t]}{=} \int_0^T \big(\omega([X,\xi_t],\eta_t)+\omega(\xi_t,[X,\eta_t])\big)dt\\
&\overset{\text{cocycle identity}}{=} \int_0^T \omega(X,[\xi_t,\eta_t])dt\\
&\overset{\xi\mapsto\xi_t \text{ is an automorphism}}{=} \int_0^T \omega(X,[\xi,\eta]_t)dt=\mu([\xi,\eta])
\end{aligned}$$

where $\mu(\xi) := \int_0^T \omega(X,\xi_t)dt$. $\square$

Suppose (as is the case in our example of interest), that $\mathrm{ad}(X)$ exponentiates to an action of S^1 on $\mathfrak{g}$ by Lie algebra automorphisms. Then letting $\mathrm{avg}_{S^1}(\omega) := \int_{S^1}\omega_t dt$, we have

$$\big[\mathrm{avg}_{S^1}(\omega)\big] = \big[\textstyle\int_{S^1}\omega_t dt\big] = \int_{S^1}[\omega_t]dt = \int_{S^1}[\omega]dt = [\omega] \qquad \text{in } \; H^2(\mathfrak{g},A)$$

for any 2-cocycle ω. Given a linear map $\mu : \mathfrak{g}\to A$, let $\mu_t(\xi) := \mu(\xi_t)$, and let us define $\mathrm{avg}_{S^1}(\mu) := \int_{S^1}\mu_t dt$. If a 2-cocycle ω is trivial, i.e., if there exists μ such that $\omega = d\mu$, then there also exists an S^1-invariant μ with that same property: indeed, letting $\mu' := \mathrm{avg}_{S^1}(\mu)$ we have

$$d\mu' = d\big(\mathrm{avg}_{S^1}(\mu)\big) = \mathrm{avg}_{S^1}(d(\mu)) = \mathrm{avg}_{S^1}(\omega) = \omega.$$

From the above discussion, we deduce that

$$H^2(\mathfrak{g};A) \;=\; \frac{\big\{\, S^1\text{-invariant 2-cocycles} \,\big\}}{\big\{d\mu \,\big|\, \mu:\mathfrak{g}\to A,\;\; \mu \text{ is } S^1\text{-invariant}\big\}}$$

Remark. *The same argument works with any compact Lie group H in place of S^1. Let $\mathfrak{g}$ be a (typically infinite dimensional) Lie algebra, and let $\mathfrak{h}\subset\mathfrak{g}$ be a finite dimensional subalgebra such that the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ exponentiates to the action of a compact Lie group H on $\mathfrak{g}$. Then $H^2(\mathfrak{g};A)=\{H\text{-invariant 2-cocycles}\}/\{d\mu\,|\,\mu \text{ is } H\text{-invariant}\}$.*

Armed with the above description of $H^2(\mathfrak{g};A)$ we can prove the theorem:

Proof of Theorem 6. We'll show that the space of S^1-invariant 2-cocycles $\omega:\mathbb{W}\times\mathbb{W}\to\mathbb{C}$ is two dimensional, spanned by the cocycles

$$\omega_1(\ell_m,\ell_n) := m^3{\cdot}\delta_{m+n,0} \qquad \text{and} \qquad \omega_2(\ell_m,\ell_n) := m{\cdot}\delta_{m+n,0},$$

and that the space of 2-cocycles which are of the form $d\mu$ for some S^1-invariant $\mu:\mathbb{W}\to\mathbb{C}$ is one dimensional, spanned by ω_2. It will follow that $\dim(H^2(\mathbb{W},\mathbb{C})) = 2-1 = 1$.

First of all, the space of S^1-invariant linear functionals $\mathbb{W}\to\mathbb{C}$ is one dimensional, spanned by $\mu:\ell_n\mapsto\delta_{n,0}$. An easy computation yields $d\mu = 2\omega_2$.

Let now ω be an S^1-invariant 2-cocycle. Let $c_{m,n} = \omega(\ell_m,\ell_n)$. S^1-invariance implies that $c_{m,n} = 0$ when $m+n\neq 0$. So let's write $c_n = \omega(\ell_n,\ell_{-n})$. We have $c_{-n} = -c_n$

by antisymmetry. The cocycle identity $\sum^3 \omega([\ell_m, \ell_n], \ell_p) = 0$ is only interesting when $m + n + p = 0$ (otherwise it's trivially satisfied). At the level of the c_n's, it reads

$$(m - n)c_{m+n} + (n - p)c_{n+p} + (p - m)c_{p+m} = 0.$$

Plugging in $p = -m - n$, we get

$$(m - n)c_{m+n} + (2n + m)c_{-m} - (2m + n)c_{-n} = 0.$$

Equivalently,

$$(m - n)c_{m+n} = (2n + m)c_m - (2m + n)c_n.$$

The case $n = 1$ of the above equation reads:

$$(m - 1)c_{m+1} = (2 + m)c_m - (2m + 1)c_1$$

It is a recurrence relation that expresses c_{m+1} in terms of c_m and c_1, provided $m + 1 \geq 3$ (otherwise $m - 1$ might be zero).

The sequence $\{c_m\}_{m\geq 1}$ is therefore entirely determined by the values of c_1 and of c_2. In particular, the space of S^1-invariant 2-cocycles is at most two dimensional. We already know that that space is at least two dimensional. So it's two dimensional. $\square$

Theorem 6 means that the Virasoro algebra is a universal central extension of the Witt algebra (I should say *the* universal central extension, because universal central extensions are unique up to unique isomorphism). Namely, by the same computation as above, one checks that $H^2(\mathbb{W}, A) = A$ for any vector space A. More precisely, any 2-cocycle is equivalent to $a \cdot \omega_{Vir}$ for some $a \in A$. Effectively, what this produces is, for every central extension $0 \to A \to \tilde{\mathfrak{g}} \to \mathbb{W} \to 0$, a homomorphism of central extensions

$$\begin{tikzcd} & 0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, "\exists!"'] & Vir \arrow[r] \arrow[d] & \mathbb{W} \arrow[r] \arrow[d, equal] & 0 \\ \forall & 0 \arrow[r] & A \arrow[r] & \tilde{\mathfrak{g}} \arrow[r] & \mathbb{W} \arrow[r] & 0 \end{tikzcd}$$

The left vertical map $\mathbb{C} \to A$ is uniquely characterized (by Proposition 5) by the fact that it sends $1 \in H^2(\mathbb{W}, \mathbb{C}) = \mathbb{C}$ to the element $a \in H^2(\mathbb{W}, A) = A$ that classifies the central extension $\tilde{\mathfrak{g}}$. The middle vertical map is also unique, because Vir is spanned by cummutators of lifts of elements of $\mathbb{W}$. That's exactly what it means, by definition, that the Virasoro algebra is the universal central extension of the Witt algebra.

Remark. *The same argument shows that the continuous cohomology $H^2_{cts}(\mathfrak{X}_{\mathbb{C}}(S^1), \mathbb{C})$ (defined in the same way as usual Lie algebra cohomology, except that we now also require the cocycles to be continuous) is one dimensional, generated by ω_{Vir}.*

The same proof can also be adapted to show that $H^2_{cts}(\mathfrak{X}(S^1), i\mathbb{R}) = \mathbb{R}$, generated by ω_{Vir}. (That last statement also follows from the fact that cohomology commutes with complexification: $H^2(\mathfrak{g}, A) \otimes_{\mathbb{R}} \mathbb{C} = H^2(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}, A \otimes_{\mathbb{R}} \mathbb{C})$.)

Little Fact. Inside the two dimensional space of S^1-invariant 2-cocycles, there is a one dimensional space of $PSU(1,1)$-invariant ones, spanned by ω_{Vir}. That's a good reason to prefer ω_{Vir} as opposed to, say, $(\ell_m, \ell_n) \mapsto m^3 \cdot \delta_{m+n,0}$.

Unfortunately, *the Virasoro cocycle is not coordinate independent*. What this means in practice is that, given a circle S (a manifold diffeomorphic to S^1), there is *no canonical* 2-cocycle on $\mathfrak{X}(S)$ (or on $\mathfrak{X}_{\mathbb{C}}(S)$). But the concept of universal central extension sill makes sense. And the good thing is that, if it exists, a universal central extension is unique up to unique isomorphism. So, even though we don't have a cocycle, we can still talk about *__the__* universal central extension of $\mathfrak{X}(S)$ (or of $\mathfrak{X}_{\mathbb{C}}(S)$) for any circle S.

Central extensions of (semi-)groups

We now address the question of, given a Lie (semi-)group G with Lie algebra $\mathfrak{g}$, and a central extension $0 \to A \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$, how to build a corresponding central extension of G?

If G is a Lie group, the Lie algebra $\mathfrak{g}$ can be naturally identified with the Lie algebra of left-invariant vector fields on G, equipped with the usual Lie bracket of vector fields. The chain complex which computes Lie algebra cohomology can then be naturally identified with the complex of left-invariant differential forms on G, equipped with the usual de Rham differental. Recall that, given a manifold M and a 2-form $\alpha \in \Omega^2(M)$, its de Rham differential is given by

$$d\alpha(X,Y,Z) = \sum^3 X \cdot \alpha(Y,Z) - \sum^3 \alpha([X,Y],Z)$$

where $[X,Y]$ is the Lie bracket of vector fields.

Given an antisymmetric bilinear form $\omega : \mathfrak{g} \times \mathfrak{g} \to A$, let us write $\underline{\omega} \in \Omega^2(G)$ for the corresponding left-invariant form on G. The 2-form $\underline{\omega}$ is closed if and only if ω is a 2-cocycle in the sense introduced before:

$$\begin{aligned} d\underline{\omega} = 0 \quad &\Leftrightarrow \quad d\underline{\omega}(X,Y,Z) = 0, \qquad \forall \text{ left invariant } X,Y,Z \in \mathfrak{X}(G), \\ &\Leftrightarrow \quad \cancel{\sum^3 X \cdot \underbrace{\underline{\omega}(Y,Z)}_{\textit{constant}}} - \sum^3 \underline{\omega}([X,Y],Z) = 0 \\ &\Leftrightarrow \quad \sum^3 \omega([X,Y],Z) = 0, \qquad \forall\, X,Y,Z \in \mathfrak{g}. \end{aligned}$$

Here, we've used the fact that, in order to check whether $\underline{\omega}$ is closed, it is enough to evaluate it against left-invariant vector fields.

Proposition 8 *1. Given a simply connected Lie group G with Lie algebra $\mathfrak{g}$, and a 2-cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \to A$, let*

$$\tilde{G}_\omega := \left\{ (\gamma, a) \,\middle|\, \begin{matrix} \gamma : [0,1] \to G, \\ \gamma(0) = e,\ a \in A \end{matrix} \right\} \Big/ \begin{matrix} (\gamma, a) \sim \left(\gamma', a + \int_h \underline{\omega}\right) \text{ when} \\ \gamma'(1) = \gamma(1) \text{ and } h \text{ is a homotopy from } \gamma \text{ to } \gamma', \end{matrix}$$

with group operation given by $(\gamma_1, a_1)(\gamma_2, a_2) = (\gamma_1 \cdot \gamma_1(1)\gamma_2, a_1 + a_2)$. *Then*

$$\tilde{G}_\omega \to G : (\gamma, a) \mapsto \gamma(1)$$

is a central extension of G *by* $\underline{A} := A/\{\text{periods of } \underline{\omega}\}$.

2. *If* $\omega' = \omega + d\mu$, *then* $\tilde{G}_\omega \cong \tilde{G}_{\omega'}$, *with isomorphism given by* $(\gamma, a) \mapsto (\gamma, a + \int_\gamma \underline{\mu})$.

3. *If* G *is merely connected then, provided* $\mathfrak{g}$ *has trivial abelianization,* $\tilde{G}_\omega$ *is a central extension of* G *by* $\underline{A} \times \pi_1(G)$. *(If* $\mathfrak{g}_{ab} \neq 0$, *the kernel of* $\tilde{G}_\omega \to G$ *might fail to be abelian.)*

4. *If* $\mathfrak{g}_{ab} = 0$ *and the central extension associated to* ω *is universal, then, provided the set of periods of* ω *is discrete inside* A, *the central extension*

$$1 \longrightarrow \underline{A} \times \pi_1(G) \longrightarrow \tilde{G}_\omega \longrightarrow G \longrightarrow 1$$

is a universal central extension in the category of Lie groups.

Proof. *1.* Since G is simply connected, any element of $K := \ker(\tilde{G}_\omega \to G)$ can be represented by a pair $(*, a)$, where $*$ denotes the constant path. By definition, we then have $(*, a) \sim (*, a + \int_h \underline{\omega})$ for every homotopy from the constant path to itself (also known as a based map $h : S^2 \to G$). The elements $\int_h \underline{\omega} \in A$ are, by definition, the periods of $\underline{\omega}$.

2. The map $(\gamma, a) \mapsto (\gamma, a + \int_\gamma \underline{\mu})$ is well-defined by an application of Stokes' theorem, and is visibly an isomorphism.

3. The projection map $K \to \pi_1(G) : [(\gamma, a)] \mapsto [\gamma]$ fits into a diagram

$$\begin{tikzcd}
\underline{A} \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d] & \pi_1(G) \arrow[d] \\
\underline{A} \arrow[r] & \tilde{G}_\omega \arrow[r] \arrow[d] & \tilde{G} \arrow[d] \\
 & G \arrow[r, equal] & G
\end{tikzcd}$$

where all rows and columns are group extensions. The conjugation action $\tilde{G} \circlearrowright \pi_1(G)$ is trivial (since $\tilde{G}$ is connected), so $\pi_1(G)$ is central in $\tilde{G}$. So, given $k \in K$ and $g \in \tilde{G}_\omega$, the commutator $[k, g] \in G_\omega$ maps to $1 \in \tilde{G}$. That commutator therefore lands in $\underline{A}$. The map $[k, -] : \tilde{G}_\omega \to \underline{A}$ is a homomorphism:

$$(kg_1k^{-1}g_1^{-1})(kg_2k^{-1}g_2^{-1}) = kg_1k^{-1}(kg_2k^{-1}g_2^{-1})g_1^{-1} = k(g_1g_2)k^{-1}(g_1g_2)^{-1} \quad \checkmark$$

and descends to a homomorphism $\tilde{G} \to \underline{A}$. But $\tilde{G}$ is connected and $\mathfrak{g}_{ab} = 0$, so there are no non-trivial homomorphisms from $\tilde{G}$ to an abelian group. Therefore K is central in G and $K \to \tilde{G}_\omega \to G$ is a central extension.

It remains to show that $K \cong \underline{A} \times \pi_1(G)$, i.e., that the sequence $\underline{A} \to K \to \pi_1(G)$ splits. This follows from the general structure theory of abelian groups, using that $\underline{A}$ is divisible and hence an injective abelian group.

4. If $A \to \tilde{\mathfrak{g}}_\omega \to \mathfrak{g}$ is universal, then for any central extension $B \to \tilde{G} \to G$ with associated Lie algebra central extension $\mathfrak{b} \to \tilde{\mathfrak{g}} \to \mathfrak{g}$, there is a unique map

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \tilde{\mathfrak{g}}_\omega & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ & & {\scriptstyle f}\big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & \mathfrak{b} & \longrightarrow & \tilde{\mathfrak{g}} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0. \end{array}$$

Since $\tilde{G}_\omega$ is connected, there is at most one homomorphism $\tilde{G}_\omega \to \tilde{G}$ that integrates the map $\tilde{\mathfrak{g}}_\omega \to \tilde{\mathfrak{g}}$. So all we need to do is construct such a homomorphism.

The canonical splitting $\mathfrak{g} \to \tilde{\mathfrak{g}}_\omega$ induces a splitting $s : \mathfrak{g} \to \tilde{\mathfrak{g}}$. Letting $F : A \to B$ be the homomorphism which integrates f, the map $\tilde{G}_\omega \to \tilde{G}$ is given by

$$[(\gamma, a)] \mapsto \delta(1) \cdot F(a)$$

where $\delta : [0,1] \to \tilde{G}$ is the unique solution of $\delta(t)^{-1}\frac{d}{dt}\delta(t) = s\big(\gamma(t)^{-1}\frac{d}{dt}\gamma(t)\big)$. *[Here, $\gamma(t)^{-1}\frac{d}{dt}\gamma(t)$ denotes the left-translate of $\frac{d}{dt}\gamma(t) \in T_{\gamma(t)}G$ back to $T_eG = \mathfrak{g}$.]* $\square$

Remark. *The splitting of $0 \to \underline{A} \to K \to \pi_1(G) \to 0$ is not canonical, so the center of $\tilde{G}_\omega$ is only non-canonically isomorphic to $\underline{A} \times \pi_1(G)$.*

The above proposition takes care of the central extension of $\mathrm{Diff}(S^1)$ (the top row in (7)). But the case of $\mathrm{Ann}(S^1)$ is more tricky because the various tangent spaces of $\mathrm{Ann}(S^1)$ are no longer all isomorphic (or rather, left translation is not an isomorphism). So we can't talk about the left-invariant 2-form associated to a Lie algebra 2-cocycle. To go around this difficulty, we use a little trick. Given an annlus $A \in \mathrm{Ann}(S^1)$, let

$$\mathrm{Ann}^{\leq A} := \big\{A_1 \in \mathrm{Ann}(S^1) \,\big|\, \exists A_2 : A_1A_2 = A\big\} \cong \big\{\gamma : S^1 \hookrightarrow A \,\big|\, \gamma \text{ “wraps around } A\text{”}\big\}.$$

When thinking in terms of maps $\gamma : S^1 \to A$, the tangent space of $\mathrm{Ann}^{\leq A}$ is easy to compute, and we see that the map $\mathrm{Ann}^{\leq A_2} \to \mathrm{Ann}^{\leq A}$ given by $B \mapsto A_1B$ induces an isomorphism of tangent spaces

$$T_1(\mathrm{Ann}^{\leq A_2}) = \mathfrak{X}_{\mathbb{C}}(S^1) \xrightarrow{\cong} T_{A_1}(\mathrm{Ann}^{\leq A}).$$

So $\underline{\omega_{Vir}}$ makes sense as a 2-form on $\mathrm{Ann}^{\leq A}$, and we can define

$${}^{\mathbb{C}\times\mathbb{Z}}\mathrm{Ann}(S^1) := \left\{ (A,\gamma,a) \,\middle|\, \begin{array}{l} A \in \mathrm{Ann}(S^1),\ a \in \mathbb{C} \\ \gamma : [0,1] \to \mathrm{Ann}^{\leq A}, \\ \gamma(0) = 1,\ \gamma(1) = A \end{array} \right\} \Big/ \begin{array}{l} \big(\gamma, a\big) \sim \big(\gamma', a + \int_h \underline{\omega_{Vir}}\big), \\ h \text{ a homotopy from } \gamma \text{ to } \gamma' \end{array}$$

very much like what we did in Proposition 8.

Here's a big chart with all the groups and semigroups related to the Virasoro algebra:

$$
\begin{tikzcd}
 & & {}^{i\mathbb{R}\oplus\mathbb{Z}}\mathrm{Diff}(S^1) \arrow[rr] \arrow[dd] \arrow[dddr] & & {}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S^1) \arrow[rr] \arrow[dd] \arrow[dddr] & & {}^{\mathbb{Z}}\mathrm{Diff}(S^1) \arrow[dd] \arrow[dddr] & \\
{}^{\mathbb{Z}}PSU(1,1) \arrow[urr, dashed] \arrow[urrrrrr] \arrow[dd] \arrow[dddr] & & & & & & & \\
 & & {}^{i\mathbb{R}}\mathrm{Diff}(S^1) \arrow[rr] \arrow[dddr] & & {}^{U(1)}\mathrm{Diff}_c(S^1) \arrow[rr] \arrow[dddr] & & \mathrm{Diff}(S^1) \arrow[dddr] & \\
PSU(1,1) \arrow[urr] \arrow[urrrrrr] \arrow[dddr] & & & {}^{\mathbb{C}\oplus\mathbb{Z}}\mathrm{Ann}(S^1) \arrow[rr] \arrow[dd] & & {}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S^1) \arrow[rr] \arrow[dd] & & {}^{\mathbb{Z}}\mathrm{Ann}(S^1) \arrow[dd] \\
 & {}^{\mathbb{Z}}\mathrm{Univ}(\mathbb{D}) \arrow[urr] \arrow[urrrrrr] \arrow[dd] & & & & & & \\
 & & & {}^{\mathbb{C}}\mathrm{Ann}(S^1) \arrow[rr] & & {}^{\mathbb{C}^\times}\mathrm{Ann}_c(S^1) \arrow[rr] & & \mathrm{Ann}(S^1) \\
 & \mathrm{Univ}(\mathbb{D}) \arrow[urr] \arrow[urrrrrr] & & & & & &
\end{tikzcd}
$$

The dotted map exists because the universal cover of $PSU(1,1)$ is also its universal central extension. This allows us to identify a canonical copy of $\mathbb{Z}$ inside the center of ${}^{i\mathbb{R}\oplus\mathbb{Z}}\mathrm{Diff}(S^1)$, and to define ${}^{i\mathbb{R}}\mathrm{Diff}(S^1)$ as the quotient by that $\mathbb{Z}$. Similarly, ${}^{\mathbb{C}}\mathrm{Ann}(S^1)$ is the quotient of ${}^{\mathbb{C}\oplus\mathbb{Z}}\mathrm{Ann}(S^1)$ by that same copy of $\mathbb{Z}$.

Loop groups

There is another class of infinite dimensional Lie groups which are very important in conformal field theory, and to which Proposition 8 readily applies: loop groups.
Fix:

- A finite dimensional, compact, simple, simply connected Lie group G, called the *gauge group*.
- A positive integer $k \in \mathbb{N}$, called the *level*.

The *loop group* of G is the group of smooth maps from S^1 to G:

$$LG := \mathrm{Map}_{C^\infty}(S^1, G),$$

and its Lie algebra $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ is called the loop algebra. Let $\omega : L\mathfrak{g} \times L\mathfrak{g} \to i\mathbb{R}$ be the cocycle given by

$$\omega(f, g) := \tfrac{1}{2\pi i} \int_{S^1} \langle f, dg \rangle, \tag{10}$$

where $\langle\ ,\ \rangle$ is the *basic inner product* on $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{su}(2)$ (also for $\mathfrak{g} = \mathfrak{su}(n)$), the basic inner product is given by $\langle X, Y \rangle := -\mathrm{tr}(XY)$. For other simple Lie algebras, the basic inner product is the smallest G-invariant inner product whose restriction to any $\mathfrak{su}(2) \subset \mathfrak{g}$ is a positive integer multiple of the basic inner product of $\mathfrak{su}(2)$.

Here, $\langle f, dg \rangle \in \Omega^1(S^1)$ is a somewhat funny notation. It denotes the image of $f dg \in \Omega^1(S^1; \mathfrak{g} \otimes \mathfrak{g})$ under the map $\Omega^1(S^1; \mathfrak{g} \otimes \mathfrak{g}) \to \Omega^1(S^1; \mathbb{R})$ induced by $\langle\ ,\ \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$.

Let's quickly check that ω is a 2-cocycle:

$$\oint \langle [f,g], dh\rangle \overset{\text{integration by parts}}{=} -\oint \langle [df,g],h\rangle - \oint \langle [f,dg],h\rangle \underset{\langle\ ,\ \rangle \text{ is } G\text{-invariant} \Leftrightarrow \langle [X,Y],Z\rangle = \langle X,[Y,Z]\rangle\ \forall X,Y,Z\in\mathfrak{g}}{=} -\oint \langle df,[g,h]\rangle + \oint \langle dg,[f,h]\rangle \quad \checkmark$$

The central extension of $L\mathfrak{g}$ induced by the above cocycle is called the (real form of the) *affine Lie algebra*. That same algebra goes by various names: it also called the *current algebra*, and also the *affine Kac-Moody algebra*[5]. I will denote it here by $\widetilde{L\mathfrak{g}}$.

Theorem 9 *The second continuous cohomology $H^2_{cts}(L\mathfrak{g},\mathbb{R})$ is one dimensional, and the cocycle $(f,g)\mapsto \int_{S^1}\langle f,dg\rangle$ represents a generator.*

As a corollary, we learn that $\widetilde{L\mathfrak{g}}$ is a universal central extension of $L\mathfrak{g}$.

Proof. By the same argument as in the proof of Theorem 6,

$$H^2_{cts}(L\mathfrak{g};\mathbb{R}) = \frac{\{\,G\text{-invariant 2-cocycles}\,\}}{\{d\mu \,|\, \mu : L\mathfrak{g}\to\mathbb{R},\ \ \mu \text{ is } G\text{-invariant}\}}$$

where G acts of $L\mathfrak{g}$ by its adjoint action on $\mathfrak{g}$. The space of G-invariant linear functionals $\mu : L\mathfrak{g}\to\mathbb{R}$ is trivial, so all we need to show is that the space of G-invariant 2-cocycles is one dimensional. We already know that it's at least one dimensional. So we need to show that it's at most one dimensional. At this point, it becomes convenient to complexify. Given $X\in\mathfrak{g}_\mathbb{C}$, let us introduce the notation X_n for $Xz^n\in L\mathfrak{g}_\mathbb{C}$. The Lie bracket of such elements is given by $[X_m,Y_n]=[X,Y]_{m+n}$.

Let ω be a G-invariant 2-cocycle. By continuity, ω is entirely determined by its restriction to the X_n's. Write $c_{m,n}$ for the map $X,Y\mapsto\omega(X_m,Y_n)$. Since ω is G-invariant, so is $c_{m,n}$. The space of G-invariant bilinear forms $\mathfrak{g}_\mathbb{C}\times\mathfrak{g}_\mathbb{C}\to\mathbb{C}$ is one dimensional, spanned by the basic inner product. So the $c_{m,n}$ are multiples of the basic inner product. In particular, they satisfy $c_{m,n}(X,Y)=c_{m,n}(Y,X)$. By the antisymmetry of ω, we then have

$$c_{m,n}(X,Y)=\omega(X_m,Y_n)=-\omega(Y_n,X_m)=-c_{n,m}(Y,X)=-c_{n,m}(X,Y).$$

So $c_{m,n}=-c_{n,m}$.

Since ω is a 2-cocycle, the $c_{m,n}$ satisfy

$$\begin{aligned} 0 &= c_{m+n,p}([X,Y],Z)+c_{p+m,n}([Z,X],Y)+c_{n+p,m}([Y,Z],X)\\ \text{because the } c_{m,n} \text{ are } G\text{-invariant} \quad &= c_{m+n,p}([X,Y],Z)+c_{p+m,n}([X,Y],Z)+c_{n+p,m}([X,Y],Z)\\ \text{commutators span } \mathfrak{g}_\mathbb{C} \quad &\Rightarrow c_{m+n,p}+c_{p+m,n}+c_{n+p,m}=0. \end{aligned}$$

Setting $m=n=0$, we get

$$c_{0,p}+2c_{p,0}=0 \quad\Rightarrow\quad c_{0,p}=0,\ \ \forall p.$$

[5]To be precise, the term 'affine Kac-Moody algebra' usually refers to the semidirect product $\mathbb{C}\ltimes\widetilde{L\mathfrak{g}_\mathbb{C}}$.

Setting $n = 1$ and $p = r - (m + 1)$, we get

$$c_{m+1,r-(m+1)} = c_{1,r-1} + c_{m,r-m} \quad \Rightarrow \quad c_{m,r-m} = m \cdot c_{1,r-1}, \;\; \forall m.$$

Setting $m = r$ in the last equation, we get

$$0 = c_{r,r-r} = r \cdot c_{1,r-1} \quad \Rightarrow \quad c_{1,r-1} = 0, \;\; \forall r \neq 0.$$

So $c_{m,n} = 0$ when $n \neq -m$, and $c_{m,-m} = m \cdot c_{1,-1}$. The cocycle ω is therefore entirely determined by the value of $c_{1,-1}$, and the space of G-invariant 2-cocycles is at most one dimensional. □

We now wish to apply Proposition 8 to the cocycle (10).

Unlike $\mathrm{Diff}(S^1)$, whose fundamental group was non-trivial but whose higher homotopy groups were all trivial, the loop group LG is simply connected but has lots of non-trivial higher homotopy groups. We care about $\pi_2(LG)$. As a manifold, LG is diffeomorphic to $G \times \Omega G$, where ΩG denotes the *based* loop group of G. So

$$\pi_2(LG) = \pi_2(G \times \Omega G) = \underbrace{\pi_2(G)}_{=\pi_1(\Omega G)=0} \times \underbrace{\pi_3(G)}_{=\pi_2(\Omega G)=\mathbb{Z}} = \mathbb{Z}.$$

[The computations $\pi_1(\Omega G) = 0$ and $\pi_2(\Omega G) = \mathbb{Z}$ are rather non-trivial. They can be done by applying Morse theory to ΩG, with respect to a suitable Morse function. (A suitable Morse function can be obtained by taking the energy functional $\gamma \mapsto \int_{S^1} \|\gamma^{-1}\gamma'\|^2$, which is not Morse, and deforming it a bit.) One the other hand, they're fairly easy to do for $G = SU(n)$ by using the fiber sequences $SU(n-1) \to SU(n) \to S^{2n-1}$ and the fact that $SU(2) = S^3$.]

To go further, we need to compute the group of periods of $\underline{\omega}$. It's a pretty annoying computation which I'll do in a moment. The answer turns out to be that the periods of $\underline{\omega}$ are equal to $2\pi i\mathbb{Z} \subset i\mathbb{R}$.

So, by Proposition 8, we get a central extension of LG by $i\mathbb{R}/2\pi i\mathbb{Z} = U(1)$, which is also its universal central extension. We call it the *level 1 central extension* of the loop group, and denote it $\widetilde{LG}$. The level k central extension is then obtained as a pushout:

$$\begin{tikzcd} 0 \arrow[r] & U(1) \arrow[r] \arrow[d, "z \mapsto z^n"'] \arrow[dr, phantom, "\ulcorner"] & \widetilde{LG} \arrow[r] \arrow[d] & LG \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & U(1) \arrow[r] & \widetilde{LG}_k \arrow[r] & LG \arrow[r] & 0 \end{tikzcd}$$

The cocycle which is adapted to the canonical basis element of the Lie algebra of $U(1) \subset \widetilde{LG}_k$ is given by

$$\omega_k(f, g) := \tfrac{k}{2\pi i} \int_{S^1} \langle f, dg \rangle.$$

We write $\widetilde{L\mathfrak{g}}_k$ for the corresponding central extension of $L\mathfrak{g}$. (It is isomorphic to $L\mathfrak{g}$ as a mere Lie algebra, but not as a central extension of $L\mathfrak{g}$ by $i\mathbb{R}$.)

Let's now compute the periods of $\underline{\omega}$. For any simple group G, there is a homomorphism $SU(2) \to G$ that represents a generator of $\pi_3(G)$ (recall that $SU(2) \cong S^3$). Moreover, the restriction of the basic inner product of G is the basic inner product of $SU(2)$. So it's enough to deal with the case $G = SU(2)$. Let's identify $G = SU(2)$ with the group $\{q \in \mathbb{H} : |q| = 1\}$ of unit quaternions via (the $\mathbb{R}$-linear extension of)

$$1 \leftrightarrow \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad i \leftrightarrow \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right), \quad j \leftrightarrow \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \quad k \leftrightarrow \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right).$$

The Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ then corresponds to the space of imaginary quaternions, and the basic inner product is given by $\langle i, i\rangle = \langle j, j\rangle = \langle k, k\rangle = 2$.

Trying to integrate $\underline{\omega}$ over a generator of $\pi_2(LG)$ is a genuine nightmare (I tried, and I failed). But integrating it over *twice* a generator turns out to be feasible. Let us model S^2 as the space $\{q = ai + bj + ck \,:\, |q|^2 = 1\}$ of unit imaginary quaternions, and let us take S^1 to be $[0, 2\pi]/\sim$. The following map represents twice a generator of $\pi_2(LG)$:

$$h : S^2 \to LG$$
$$h(q) := \big(\theta \mapsto \cos(\theta) + q\sin(\theta)\big).$$

By symmetry considerations, the 2-form $h^*\underline{\omega}$ is a constant multiple of the standard volume form of S^2. So it's enough to consider what happens at the point $i \in S^2$. The tangent space at i is spanned by j and k, and one easily computes $\frac{\partial h}{\partial j}(\theta) = j\sin(\theta)$ and $\frac{\partial h}{\partial k}(\theta) = k\sin(\theta)$. Translating back to the origin, we get

$$h^{-1}\tfrac{\partial h}{\partial j}(\theta) = \big(\cos\theta - i\sin\theta\big) j \sin\theta = \tfrac{1}{2}\big(j\sin 2\theta + k(\cos 2\theta - 1)\big)$$
$$h^{-1}\tfrac{\partial h}{\partial k}(\theta) = \big(\cos\theta - i\sin\theta\big) k \sin\theta = \tfrac{1}{2}\big(k\sin 2\theta + j(1 - \cos 2\theta)\big)$$

So *twice* the smallest period of $2\pi i\underline{\omega}$ is given by

$$\begin{aligned}
&\mathrm{vol}(S^2)\cdot \int_0^{2\pi} \Big\langle h^{-1}\tfrac{\partial h}{\partial j}, \tfrac{d}{d\theta}\big(h^{-1}\tfrac{\partial h}{\partial k}\big)\Big\rangle\, d\theta \\
&= 4\pi \int_0^{2\pi} \Big\langle \tfrac{1}{2}\big(j\sin 2\theta + k(\cos 2\theta - 1)\big), \tfrac{d}{d\theta}\tfrac{1}{2}\big(k\sin 2\theta + j(1-\cos 2\theta)\big)\Big\rangle \\
&= 2\pi \int_0^{2\pi} \Big\langle j\sin 2\theta + k(\cos 2\theta - 1), k\cos 2\theta + j\sin 2\theta\Big\rangle \\
&= 2\pi \int_0^{2\pi} \sin^2(2\theta)\langle j, j\rangle + \cos^2(2\theta)\langle k, k\rangle \;= 2\pi\int_0^{2\pi} 2 = 8\pi^2.
\end{aligned}$$

So the periods of $2\pi i\underline{\omega}$ are $4\pi^2\mathbb{Z}$, and the periods of $\underline{\omega}$ are $2\pi i\mathbb{Z} \subset i\mathbb{R}$.

Examples of chiral CFTs

In this section, we introduce two classes of chiral CFTs:

- the unitary *chiral minimal models*
  (there's one such model for every $c = 1 - \frac{6}{m(m+1)}$, for $m = 2, 3, 4, \ldots$)
- the *chiral WZW models*
  (there's one such model for every choice of gauge group G and level k), and

At first, we will describe the linear categories that those models assign to 1-manifolds. In the case of a chiral minimal model, the categories $\mathcal{C}(S)$ are given by

$$\mathcal{C}(S) = \mathrm{Rep}\big(\mathit{Vir}_c(S)\big)$$

where $\mathit{Vir}_c(S)$ denotes the appropriate (universal) central extension of $\mathfrak{X}_{\mathbb{C}}(S)$ by $\mathbb{C}$. (And we insist that the central $\mathbb{C}$ acts in the standard way.)

And in the chiral WZW model, these categories are (roughly) given by

$$\mathcal{C}(S) = \mathrm{Rep}\big(\widetilde{L_S G_k}\big)$$

where $\widetilde{L_S G_k}$ denotes the appropriate central extension of $L_S G := \mathrm{Map}_{C^\infty}(S, G)$ by $U(1)$. (And we insist that the central $U(1)$ acts in the standard way.)

The above descriptions are not very precise, because we haven't said anything about the class of representations that we're allowing. And without any specifications, those categories are huge. So we *do* need to say a bit more. Let us introduce a couple of technical conditions:

Definition: A representation of Vir_c has *positive energy* if the associated operator L_0 has discrete spectrum, the spectrum is bounded from below, and all the (generalized) eigenspaces are finite dimensional.

In our case of interest, we always have $e^{2\pi i L_0} = \theta_\lambda$ (where θ_λ is the conformal spin). So L_0 is in fact diagonalizable, and there is no need to talk about generalized eigenspaces.

Remark. The operator L_0 is obviously coordinate dependent. However, assuming the action of Vir_c integrates to an action of ${}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S^1)$, the property of being positive energy is independent of the choice of coordinate, because one can conjugate any coordinate into any other coordinate by an element of $\mathrm{Diff}(S^1)$.

[I believe that every positive energy representation of Vir_c integrates to a representation of ${}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S^1)$, and even to a representation of ${}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S^1)$. But this result probably depends on the type of topological vector spaces that one allows.]

Definition: An irreducible representation of $\widetilde{L\mathfrak{g}_k}$ has *positive energy* if it extends to a representation of $\mathit{Vir}_c \ltimes \widetilde{L\mathfrak{g}_k}$ for some c, and the Virasoro action has positive energy. A positive energy representation of $\widetilde{L\mathfrak{g}_k}$ is a finite direct sum of irreducible positive energy representations of $\widetilde{L\mathfrak{g}_k}$. A representation of $\widetilde{L\mathfrak{g}_k}$ is called *integrable* if it integrates to a representation of $\widetilde{LG_k}$.

Factoid: *A positive energy representation of $\widetilde{L\mathfrak{g}_k}$ is integrable iff it is unitary. [Once again, this probably depends on the type of topological vector spaces that one allows]*

Given the above definitions, we can go back and re-define the categories $\mathrm{Rep}\big(\mathit{Vir}_c(S)\big)$ and $\mathrm{Rep}\big(\widetilde{L_S G_k}\big)$, with more attention to detail. In place of $\mathrm{Rep}\big(\mathit{Vir}_c(S)\big)$, we should have written

$$\begin{aligned}
\mathrm{Rep}^{\text{unitary}}_{\substack{pos\\en.}}\big(\mathit{Vir}_c(S)\big) &:= \big\{\text{positive energy unitary reps of } \mathit{Vir}_c(S)\big\}\\
&= \big\{\text{positive energy unitary reps of } {}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S)\big\}\\
&= \big\{\text{positive energy unitary reps of } {}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S)\big\},
\end{aligned}$$

Similarly, in place of $\mathrm{Rep}(\widetilde{L_SG_k})$, we should have written

$$\begin{aligned}\mathrm{Rep}_{\substack{pos\\en.}}(\widetilde{L_SG_k}) &:= \{\text{positive energy representations of } \widetilde{L_SG_k}\}\\ &= \{\text{positive energy integrable reps of } \widetilde{L_S\mathfrak{g}_k}\}\\ &= \{\text{positive energy unitary reps of } \widetilde{L_S\mathfrak{g}_k}\},\end{aligned}$$

where $\widetilde{L_S\mathfrak{g}_k}$ is the central extension of $L_S\mathfrak{g} := C^\infty(S, \mathfrak{g})$ by $i\mathbb{R}$ defined by the cocycle $\omega_k(f, g) := \frac{k}{2\pi i}\int_S \langle f, dg\rangle$.

Remark 10 When working with nuclear Fréchet vector spaces, one should add the condition that the representations are *smooth*, meaning that the map $\mathcal{G} \times V \to V$ is smooth (where $\mathcal{G}$ is either $\widetilde{L_SG_k}$ or ${}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S)$), and that the (semi-)norms $\|\ \|_n : V \to \mathbb{R}_+$, $n \in \mathbb{N}$, which define the Fréchet topology may be chosen in such a way that the maps $\mathcal{G} \times (V, \|\ \|_n) \to (V, \|\ \|_n)$ are continuous. *[I'm not really sure that these conditions are strong enough to exclude all unwanted examples. I'm just guessing that they are, by analogy with the Casselman-Wallach theorem in the representation theory of finite dimensional Lie groups.]*

When working with Hilbert spaces, one should be aware that the actions of $\widetilde{L_S\mathfrak{g}_k}$ and of $Vir_c(S)$ are by *unbounded* operators, and one should be very careful with the domains of definition of these operators.

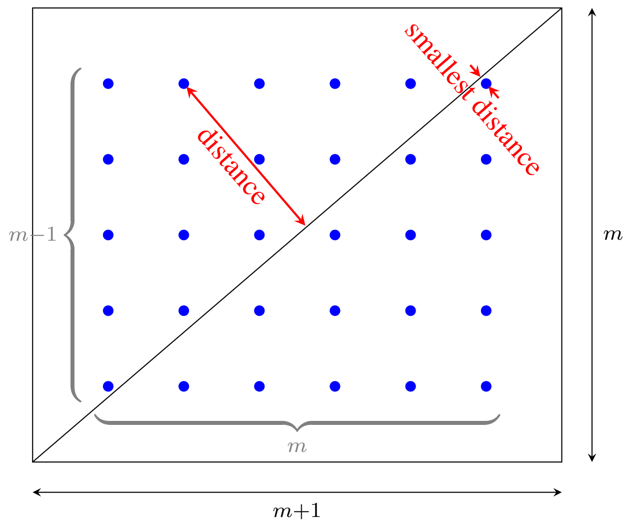
Unitary representations of the Virasoro and affine Lie algebras

Let us explain a bit what the representation theory of the Virasoro algebra looks like. For $c \geq 1$, the irreducible unitary representations of Vir_c are classified by their minimal energy, which can take any value in $\mathbb{R}_{\geq 0}$. The corresponding chiral CFT is not rational, and is called chiral Liouville theory (at least for $c > 1$).

In the range $c < 1$, there exists a discrete set of values of c for which the Virasoro algebra admits unitary representations (outside of that set, Vir_c has no unitary representations at all). These are the numbers of the form $c = 1 - \frac{6}{m(m+1)}$, for $m = 2, 3, 4, \ldots$

$$c \;=\; 0,\ \tfrac{1}{2},\ \tfrac{7}{10},\ \tfrac{4}{5},\ \tfrac{6}{7},\ \tfrac{25}{28},\ \tfrac{11}{12},\ \tfrac{14}{15},\ \tfrac{52}{55},\ \tfrac{21}{22},\ \tfrac{25}{26},\ \tfrac{88}{91},\ \ldots$$

For such a value of the central charge, Vir_c has exactly $m(m-1)/2$ irreducible unitary representations. They are classified by their minimal energy, which can take any value of the form $\frac{[(m+1)p-mq]^2-1}{4m(m+1)}$, for $1 \leq p \leq m-1$ and $1 \leq q \leq m$. A good mnemonic for remembering these numbers is to note that they're equal to the square-distance to the diagonal minus the smallest square-distance to the diagonal in the following rectangular array of dots:



(normalized so that the smallest square-distance is $\frac{1}{4m(m+1)}$).

We now spend a couple of words describing the unitary representations of $\widetilde{L\mathfrak{g}}_k$. Given a finite dimensional simple Lie algebra $\mathfrak{g}$ of rank r, with Cartan subalgebra $\mathfrak{h}$, the *Weyl alcove* is the r-dimensional simplex $\mathsf{A} \subset \mathfrak{h}^*$ bound by the r walls of the Weyl chamber, and by the hyperplane that bisects the segment $[0, \alpha_0]$, where $\alpha_0 \in \mathfrak{h}^*$ is the highest root.

For any level k, the set of simple objects of $\mathcal{C}_{\mathfrak{g},k} := \mathrm{Rep}_{\substack{pos\\en.}}(\widetilde{LG}_k) = \mathrm{Rep}^{\text{unitary}}_{\substack{pos\\en.}}(\widetilde{L\mathfrak{g}}_k)$ is in canonical bijection with the set of (finite dimensional) irreducible representations of $\mathfrak{g}$ whose highest weight is in $k\mathsf{A}$. The bijection sends a representation of $\widetilde{L\mathfrak{g}}_k$ to its lowest energy subspace.

In Table 2, on the next page, we illustrate the set of simple objects of $\mathcal{C}_{\mathfrak{g},k}$ with some examples.

By definition, an irreducible positive energy representation of $\widetilde{L\mathfrak{g}}_k$ extends to $Vir_c \ltimes \widetilde{L\mathfrak{g}}_k$ for some c. The commutation relations in the latter are given by:

$$\begin{aligned} [X_m, Y_n] &= [X,Y]_{m+n} + kn\langle X, Y\rangle \delta_{m+n,0} \\ [L_m, X_n] &= -nX_{m+n} \qquad \longleftarrow \boxed{\text{since } -z^{m+1}\tfrac{\partial}{\partial z} z^n = -nz^{m+n}} \\ [L_m, L_n] &= (m-n)L_{m+n} + \tfrac{c}{12}(m^3-m)\delta_{m+n,0} \end{aligned} \tag{11}$$

where X_n stands for $Xz^n \in L\mathfrak{g} \subset \widetilde{L\mathfrak{g}}_k$. It turns out that, when (V, ρ) is an irreducible representation of $\widetilde{L\mathfrak{g}}_k$, the action of the L_m and the value of the central charge c are uniquely determined. We prove this in the following two lemmas, and below:

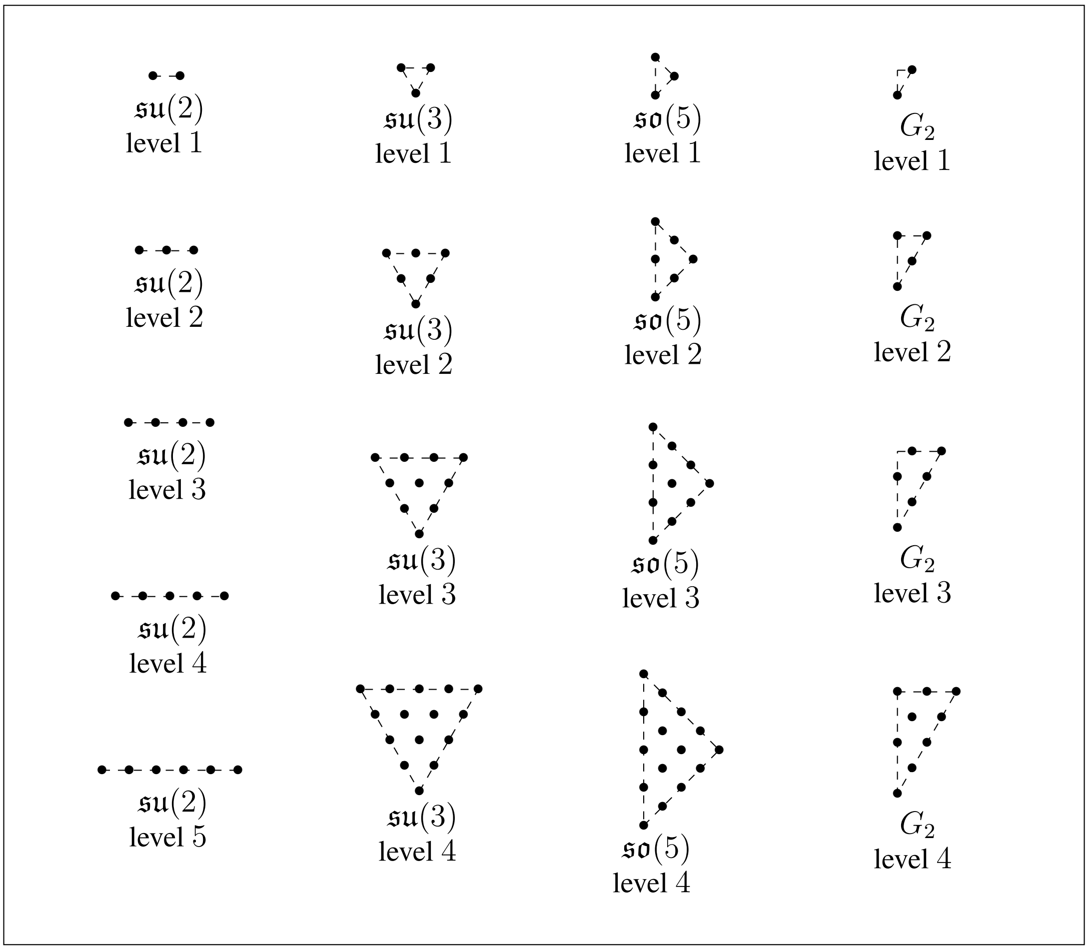


Table 2.

Lemma. *The equation* $[\rho(L_m), \rho(X_n)] = -n\rho(X_{m+n})$ *uniquely determines* $\rho(L_m)$ *up to the addition of a scalar.*

Proof. Suppose $\rho(L_m)$ and $\rho'(L_m)$ are two solutions. Then $[\rho'(L_m) - \rho(L_m), \rho(X_n)] = 0$. So $\rho'(L_m) - \rho(L_m) : V \to V$ is a morphism of $\widetilde{L\mathfrak{g}}_k$-representations. By Schur's lemma, $\rho'(L_m) = \rho(L_m) + \text{cst}$. $\square$

Lemma. *If the operators* $\rho(L_m)$ *satisfy* *then,* $[\rho(L_m), \rho(L_n)] = \rho([L_m, L_n]) + \text{cst}$.

Proof.

$$[[\rho(L_m), \rho(L_n)], \rho(X_r)] = \underbrace{[\rho(L_m), \underbrace{[\rho(L_n), \rho(X_r)]}_{-r\rho(X_{n+r})}]}_{-(n+r)\cdot(-r)\rho(X_{m+n+r})} - \underbrace{[\rho(L_n), \underbrace{[\rho(L_m), \rho(X_r)]}_{-r\rho(X_{m+r})}]}_{-(m+r)\cdot(-r)\rho(X_{m+n+r})} =$$

$= (m-n) \cdot (-r\rho(X_{m+n+r}))$.

So $[\rho(L_m), \rho(L_n)]$ satisfies the same commutation relations as $\rho([L_m, L_n])$, and we're done by the previous lemma. $\square$

If we make a random pick for the $\rho(L_m)$'s, subject to the relations $[\rho(L_m), \rho(X_n)] = -n\rho(X_{m+n})$, then we won't quite get a representation of the Witt algebra. (The constants which appear in the last lemma are the 2-cocycle that measures the failure of ρ being a representation.) What we get instead is a representation of a central extension $\widetilde{\mathbb{W}} \to \mathbb{W}$. Now, since Vir is the universal central extension of $\mathbb{W}$, we get a unique Lie algebra homomorphism $Vir \to \widetilde{\mathbb{W}}$ that commutes with the projection to $\mathbb{W}$. We check where the central element of Vir goes, and that gives us the central charge c.

This now raises two questions: what is c, and how does one compute it? The only way I know to compute it is to write down the L_m's explicitly, in terms of the X_n's, and to check that they satisfy the desired commutation relations (the ones in the first of the above two lemmas). The construction is called the *Segal-Sugawara construction*, and looks as follows:

$$L_m := \tfrac{1}{2(k+h^\vee)} \sum_{X \in \mathcal{B}} \left(\sum_{n<0} X_n X_{m-n} + \sum_{n \geq 0} X_{m-n} X_n \right).$$

Here, $\mathcal{B}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the basic inner product, and $h^\vee$ is the dual Coxeter number. *[The dual Coxeter number can be defined in multiple ways. One way is to say that $2h^\vee$ is the ratio between the Killing form and the basic inner product on $\mathfrak{g}$. Another way is to say that $h^\vee$ is the smallest level k such that $k\mathsf{A}$ contains an element of the weight lattice in its interior. By the way, that element in the interior is usually called ρ.]* The central charge is given by the formula $c = \frac{k \cdot \dim(\mathfrak{g})}{k+h^\vee}$, and the minimal energy h_λ of the $\widetilde{L\mathfrak{g}}_k$-rep with highest weight λ is given by the formula $h_\lambda = \frac{\|\lambda+\rho\|^2 - \|\rho\|^2}{2(k+h^\vee)}$.

The positive energy condition

In the previous sections, we motivated the introduction of the positive energy condition by saying "otherwise, there's too many representations". But there's a much better reason to include that condition. That's because, in a Segal CFT, that condition is *forced* on you:

Theorem 11 *In a chiral Segal CFT, the action of the Virasoro algebra on any sector always has positive energy.*

The proof will be based on the following result:

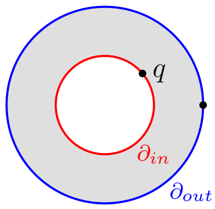
Proposition 12 *If Σ is a thick complex cobordism, then $Z_\Sigma : U(\lambda) \to U(F_\Sigma(\lambda))$ is a trace-class operator.*

We'll give the definition of trace-class a bit later. For the moment, we just need to know that, for a diagonal operator on a topological vector space, we have

$$\mathrm{Diag}(\alpha_1, \alpha_2, \ldots) \text{ is trace-class} \qquad \Longrightarrow \qquad \lim_{n\to\infty} \alpha_n = 0.$$

When our vector space is a Hilbert space, we have the much stronger result that an operator of the form $\mathrm{Diag}(\alpha_1, \ldots)$ is trace-class iff $\sum |\alpha_n| < \infty$. But things don't work quite as nicely when dealing with more general types of vector spaces. *[The implication $\sum |\alpha_n| < \infty \Rightarrow \mathrm{Diag}(\alpha_1, \ldots)$ is trace-class only holds when the diagonal operator is defined w.r.t. an unconditional basis. The implication $\mathrm{Diag}(\alpha_1, \ldots)$ trace-class $\Rightarrow \sum |\alpha_n| < \infty$ almost never holds.]*

Proof of Theorem 11. For q a complex number, $|q| < 1$, let $A_q := \{z \in \mathbb{C} : |q| \le |z| \le 1\}$, with boundary parametrizations $\varphi_{in} : z \mapsto qz : S^1 \to \partial_{in} A$ and $\varphi_{out} = \mathrm{id}_{S^1}$:

$A_q =$ 

Pick a lift of $\tilde{A}_q \in \mathrm{Univ}(\mathbb{D})^{\mathbb{Z}}$ of A_q to the universal cover of $\mathrm{Univ}(\mathbb{D})$. Equivalently, pick a logarithm of q. The corresponding operator on $U(\lambda)$ (defined in Equation (3)) is then given by

$$U(T_{\tilde{A}_q}) \circ Z_{A_q} = q^{L_0} := e^{\log(q) L_0}.$$

The operator $U(T_{\tilde{A}_q}) \circ Z_{A_q}$ is trace-class by Proposition 12. In particular, its sequence of eigenvalues (counted with multiplicity) tends to zero. This is equivalent to the spectrum of L_0 being discrete, bounded from below, and all its eigenspaces being finite dimensional. $\square$

Before discussing the proof of Proposition 12, we recall some definitions from functional analysis. From now on, we assume that all our vector spaces are complete locally convex topological vector spaces.[6]

Definition: An operator $f : V \to W$ is *trace-class*[7] if it is in the image of the map

$$\mathrm{E} : W \otimes_\pi V' \to \mathcal{L}(V, W).$$

Here, V' is the continuous dual of V (the set of continuous linear maps $V \to \mathbb{C}$), and $\otimes_\pi$ is the projective tensor product of topological vector spaces (defined by the universal property that continuous bi-linear maps out of the product are the same thing as continuous linear maps out of the tensor product). If $f : V \to V$ is trace-class, then its trace $\mathrm{tr}(f) \in \mathbb{C}$ is the image of $\mathrm{E}^{-1}(f)$ under the evaluation map $V \otimes V' \cong V' \otimes V \xrightarrow{ev} \mathbb{C}$.

Warning: *When working with general topological vector spaces, the map* E *can fail to be injective; this can already happen with Banach spaces. When this happens,* $\mathrm{tr}(f)$ *typically fails to be well defined. But everything is ok (i.e., the map* E *is injective) when the spaces have bases. (A subset* $(b_n)_{n \in \mathbb{N}}$ *of a topological vector space* V *is a* basis *if for every* $v \in V$ *there is a unique sequence of numbers* (a_n) *such that* $v = \sum a_n b_n$*.)*

[6] A topology is called ‘locally convex’ if it is generated by a set of (semi-)norms.

[7] When working with Hilbert spaces, one typically uses the term ‘trace class’. When working with more general topological vector spaces, one typically uses the word ‘nuclear’ for that same notion.

Lemma 13 *A linear map $f : V \to W$ is trace-class if and only if there exists a space X, and linear maps $a : \mathbb{C} \to W \otimes_\pi X$, and $b : X \otimes_\pi V \to \mathbb{C}$, such that*

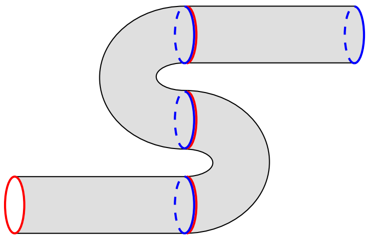
$$f = \Big(V \xrightarrow{a \otimes \mathrm{id}} W \otimes X \otimes V \xrightarrow{\mathrm{id} \otimes b} W\Big).$$

Proof. $\Rightarrow$: Take $X = V'$, $a = \mathrm{E}^{-1}(f)$, and $b = ev : V' \otimes V \to \mathbb{C}$.
$\Leftarrow$: b induces a map $\tilde{b} : X \to V'$. Then f is the image of $(\mathrm{id}_W \otimes \tilde{b})a \in W \otimes_\pi V'$. $\square$

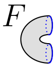
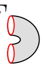
As a corollary, trace-class maps form an ideal: if a map f is trace-class, then so is $f \circ g$, and so is $h \circ f$.

Remark. The statement of Lemma 13 also holds true when V, W, and X are Hilbert spaces, and the projective tensor product is replaced by the Hilbert space tensor product. But the proof is rather different (it relies on the fact that the composition of two Hilbert-Schmidt operators is always trace-class).

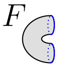
The proof of Proposition 12 will be based on the fact that every thick annulus can be decomposed as follows:



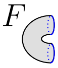
(12)

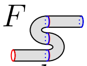
We first compute F and F. Let $1_\emptyset := \mathbb{C}$ be the canonical simple object of $\mathcal{C}(\emptyset) = \mathsf{Vec}_{\mathrm{f.d.}}$.

Proposition. *There exists a canonical involution $\lambda \mapsto \bar{\lambda}$, called charge conjugation, on the set of isomorphism classes of simple objects of $\mathcal{C}(S^1)$, such that*

F$(1_\emptyset) = \bigoplus_\lambda \lambda \otimes \bar{\lambda}$ *and* F$(\mu \otimes \nu) = \delta_{\mu,\bar{\nu}} 1_\emptyset$.

Proof. Write

F$(1_\emptyset) = \bigoplus_{\lambda,\mu} a_{\lambda,\mu} \lambda \otimes \mu$ and F$(\mu \otimes \nu) = b_{\mu,\nu} 1_\emptyset$.

The triviality of F means that the matrices $a = (a_{\lambda,\mu})$ and $b = (b_{\mu,\nu})$ satisfy $ab = 1$. Similarly, we have $ba = 1$. Since the entires of a and of b all lie in $\mathbb{N}$, they are therefore permutation matrices.

Finally, the axiom according to which for every complex diffeomorphism $\phi : \Sigma' \to \Sigma$, we have an invertible natural transformation $F_{\Sigma'} \circ (\partial_{in}\phi)^* \cong (\partial_{out}\phi)^* \circ F_\Sigma$ tells us that a and b are involutions (and hence that $a = b$). $\square$

Proof of Proposition 12 . Since trace-class maps form an ideal, and since the tensor product of two trace-class maps is again trace-class, it's enough to show that Z_Σ is trace-class when $\Sigma = A$ is an annulus. Cutting A as in (12), we can decompose Z as:

$$\begin{tikzcd}
 & H_\lambda \otimes H_{\bar\lambda} \otimes H_\lambda \arrow[dr, dashed] & \\
H_\lambda \arrow[r, "Z"'] & \bigoplus_\mu H_\mu \otimes \underbrace{H_{\bar\mu} \otimes H_\lambda}_{\in \ker(F) \text{ unless } \mu=\lambda} \arrow[u, two heads] \arrow[r, "Z"'] & H_\lambda
\end{tikzcd}$$

We are then done by Lemma 13, and the fact that trace-class maps form an ideal. $\square$

Primary fields

Let $\Delta \in \mathbb{Z}$ be an integer.

Definition: *A primary field of conformal dimension Δ is a gadget φ that assigns to every complex cobordism Σ equipped with:*

- *points $z_1, \ldots, z_n \in \Sigma$, and*
- *tangent vectors $v_i \in T_{z_i}\Sigma$,*

and to every object $\lambda \in \mathcal{C}(\partial_{in}\Sigma)$, a linear map

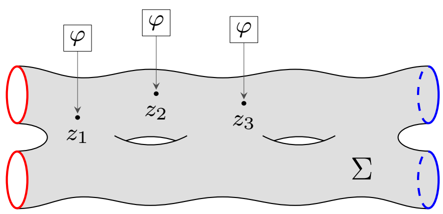
$$Z_{\Sigma,\varphi(z_1;v_1),\ldots,\varphi(z_n;v_n)} : U(\lambda) \to U(F_\Sigma(\lambda)).$$

These maps are homogeneous of degree Δ in the v_i's:

$$Z_{\Sigma,\varphi(z_1;v_1),\ldots,\varphi(z_i;av_i),\ldots,\varphi(z_n;v_n)} = a^\Delta Z_{\Sigma,\varphi(z_1;v_1),\ldots,\varphi(z_n;v_n)} \qquad \forall a \in \mathbb{C}, \tag{13}$$

and agree with Z_Σ when $n = 0$. Moreover, they satisfy the same axioms that the Z_Σ satisfy (naturality in λ and in Σ, compatibility with disjoint union, and with composition of cobordisms).

The map $Z_{\Sigma,\varphi(z_1;v_1),\ldots,\varphi(z_n;v_n)}$ is called a *propagator with field insertions*:



$$Z_{\Sigma,\varphi(z_1;v_1),\ldots,\varphi(z_n;v_n)} : U(\lambda) \longrightarrow U(F_\Sigma(\lambda))$$

Example: The *vacuum field* Ω

$$Z_{\Sigma,\Omega(z_1;v_1),\dots,\Omega(z_n;v_n)} := Z_\Sigma$$

is a primary field of conformal dimension zero.

We will often suppress the vectors v_i from the notation, and write $Z_{\Sigma,\varphi(z_1),\dots,\varphi(z_n)}$ instead of $Z_{\Sigma,\varphi(z_1;v_1),\dots,\varphi(z_n;v_n)}$.

$\boxed{1 \in \mathbb{C} = U(1_\emptyset)}$

When Σ is a closed surface and $\lambda = 1_\emptyset$, then $Z_{\Sigma,\varphi(z_1),\dots,\varphi(z_n)}(1)$ is called a *correlator*, and denoted

$$\big\langle\varphi(z_1),\dots,\varphi(z_n)\big\rangle_\Sigma \in H_\Sigma := U(F_\Sigma(1_\emptyset)).$$

A linear functional $\mathcal{B} : H_\Sigma \to \mathbb{C}$ is called a *conformal block*[8], and we write

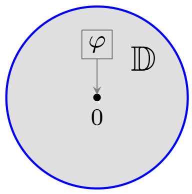
$$\big\langle\varphi(z_1),\dots,\varphi(z_n)\big\rangle_{\Sigma,\mathcal{B}} \in \mathbb{C} \tag{14}$$

for the image of $\langle\varphi(z_1),\dots,\varphi(z_n)\rangle_\Sigma$ under $\mathcal{B}$. When thought of as a function of the z_i's, the expression (14) is also called a *correlation function*.

Theorem. (State-field correspondence) *There is a natural bijection*

$$\left\{\begin{array}{c}\text{Primary fields of}\\ \text{conformal dimension } \Delta\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{States } \xi \in H_0 \text{ such that}\\ L_0(\xi) = \Delta\xi \text{ and } L_n(\xi) = 0\,\forall n > 0\end{array}\right\}.$$

Proof. Given a field φ, the corresponding state $\xi \in H_0$ is given by

$\xi :=$  $= Z_{\mathbb{D},\varphi(0;1)}(1) \in H_0.$

$\boxed{1 \in \mathbb{C} = U(1_\emptyset)}$ $\boxed{1 \in T_0\mathbb{D} = \mathbb{C}}$

We need to show that

$$L_0(\xi) = \Delta\xi \qquad \text{and} \qquad L_n(\xi) = 0\,\forall n > 0. \tag{15}$$

Let $\mathrm{Univ}_0(\mathbb{D}) := \{f \in \mathrm{Univ}(\mathbb{D})\,|\,f(0) = 0\}$ be the semigroup associated to the Lie algebra $Vir_{\geq 0} := \mathrm{Span}\{L_n\}_{n\geq 0}$. The conditions (15) are equivalent to

$$Z_A\xi = f'(0)^\Delta\xi \qquad \forall f \in \mathrm{Univ}_0(\mathbb{D}),\ A = \mathbb{D} \setminus f(\mathring{\mathbb{D}}).$$

We can then compute:

$$Z_A\xi = Z_A Z_{\mathbb{D},\varphi(0;1)}(1) = Z_{A\cup\mathbb{D},\varphi(0;1)}(1) = Z_{\mathbb{D},\varphi(0;f'(0))}(1) = f'(0)^\Delta\xi.$$

[8] Some people would call the expression (14) (viewed as a function of the z_i) the 'conformal block'.

Conversely, starting from a vector $\xi \in H_0$ that satisfies the equations (15), we proceed as follows. Given a complex cobordism Σ together with points $z_1, \ldots, z_n$ and tangent vectors $v_i \in T_{z_i}\Sigma$, choose disjoint embeddings $f_i : \mathbb{D} \to \Sigma$, $f_i(0) = z_i$, and let

$$\Sigma^0 := \Sigma \setminus (f_1(\mathring{\mathbb{D}}) \sqcup \ldots \sqcup f_n(\mathring{\mathbb{D}})).$$

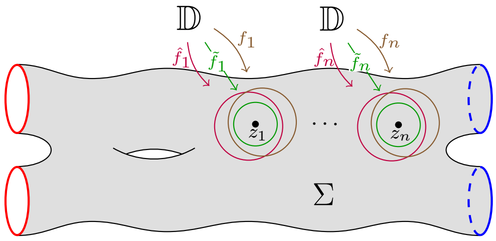
We then define

$$Z_{\Sigma,\varphi(z_1),\ldots,\varphi(z_n)} := \prod \Big(\frac{v_i}{f_i'(0)}\Big)^{\Delta} Z_{\Sigma^0}\big(\xi \otimes \ldots \otimes \xi \otimes -\big). \tag{16}$$

Let's check that this map lands in the right place:

$$\begin{aligned} Z_{\Sigma^0} : \underset{\xi}{\underset{\cup\!\!\!\shortmid}{H_0}} \otimes \underset{\cdots}{\ldots} \otimes \underset{\xi}{\underset{\cup\!\!\!\shortmid}{H_0}} \otimes U(\lambda) &= U\big(F_{\mathbb{D}}(1) \otimes \ldots \otimes F_{\mathbb{D}}(1) \otimes \lambda\big) \\ &\longrightarrow U\big(F_{\Sigma^0}(F_{\mathbb{D}}(1) \otimes \ldots \otimes F_{\mathbb{D}}(1) \otimes \lambda)\big) \\ &= U\big(F_{\Sigma^0 \cup (\mathbb{D} \sqcup \ldots \sqcup \mathbb{D})}(1 \otimes \ldots \otimes 1 \otimes \lambda)\big) = U(F_\Sigma(\lambda)). \quad \checkmark \end{aligned}$$

We need to show that the map (16) is independent of the choice of f_i. Let $\hat{f} : \mathbb{D} \to \Sigma$, $\hat{f}_i(0) = z_i$, be another set of embeddings, and let $\hat{Z}_{\Sigma,\varphi(z_1),\ldots,\varphi(z_n)} : U(\lambda) \to U(F_\Sigma(\lambda))$ be the corresponding map, defined as in (16). We wish to show that $\hat{Z}_{\Sigma,\ldots} = Z_{\Sigma,\ldots}$. In order to do so, we introduce yet another set of embeddings $\tilde{f} : \mathbb{D} \to \Sigma$, $\tilde{f}_i(0) = z_i$, that satisfy $\tilde{f}_i(\mathbb{D}) \subset f_i(\mathbb{D}) \cap \hat{f}_i(\mathbb{D})$. Let $\tilde{Z}_{\Sigma,\ldots}$ be the corresponding map. We will show that $\hat{Z}_{\Sigma,\ldots} = \tilde{Z}_{\Sigma,\ldots} = Z_{\Sigma,\ldots}$.



(17)

Clearly, it's enough to show that $\tilde{Z}_{\Sigma,\ldots} = Z_{\Sigma,\ldots}$ (the other equality follows by symmetry). Let

$$\psi_i := f_i^{-1} \circ \tilde{f}_i \qquad \text{and let} \qquad A_i := \mathbb{D} \setminus \psi_i(\mathring{\mathbb{D}}) = f_i(\mathbb{D}) \setminus \tilde{f}_i(\mathring{\mathbb{D}})$$

be the corresponding annuli. We then have $Z_{A_i}(\xi) = \psi_i'(0)^\Delta \xi = \big(\frac{\tilde{f}_i'(0)}{f_i'(0)}\big)^\Delta \xi$, from which we get:

$$\begin{aligned} \tilde{Z}_{\Sigma,\varphi(z_1),\ldots,\varphi(z_n)} &= \prod \Big(\frac{v_i}{\tilde{f}_i'(0)}\Big)^{\Delta} Z_{\tilde{\Sigma}^0}\big(\xi \otimes \ldots \otimes \xi \otimes -\big) \\ &= \prod \Big(\frac{v_i}{\tilde{f}_i'(0)}\Big)^{\Delta} Z_{\Sigma^0}\big(Z_{A_1}(\xi) \otimes \ldots \otimes Z_{A_n}(\xi) \otimes -\big) \\ &= \prod \Big(\frac{v_i}{\tilde{f}_i'(0)}\Big)^{\Delta} \prod \Big(\frac{\tilde{f}_i'(0)}{f_i'(0)}\Big)^{\Delta} Z_{\Sigma^0}\big(\xi \otimes \ldots \otimes \xi \otimes -\big) = Z_{\Sigma,\varphi(z_1),\ldots,\varphi(z_n)}. \end{aligned} \tag{18}$$

$\square$

Given a bunch of primary fields $\varphi_1, \ldots, \varphi_n$, with corresponding vectors $\xi_1, \ldots, \xi_n \in H_0$, it's now easy to adapt the definition (16):

$$Z_{\Sigma,\varphi_1(z_1;v_i),\ldots,\varphi_n(z_n;v_n)} := \prod \Big(\frac{v_i}{f_i'(0)}\Big)^{\Delta_i} Z_{\Sigma^0}\big(\xi_1 \otimes \ldots \otimes \xi_n \otimes -\big). \tag{19}$$

Here, as before, $\Sigma^0 = \Sigma \setminus (f_1(\mathring{\mathbb{D}}) \sqcup \ldots \sqcup f_n(\mathbb{D}))$ for some $f_i : \mathbb{D} \to \Sigma$ satisfying $f_i(0) = z_i$.

It is also fruitful to allow the ξ_i to take their values in other sectors than the vacuum sector. The corresponding fields are called *charged fields*. Given irreducible objects $\mu_i \in \mathcal{C}(S^1)$, and vectors $\xi_i \in U(\mu_i)$ satisfying the same conditions (15) as before, the definition (19) still makes sense, even thought it's no longer a map $U(\lambda) \to U(F_\Sigma(\lambda)$. Instead, it's a map:

$$Z_{\Sigma,\varphi_1(z_1),\ldots,\varphi_n(z_n)} \,:\, U\big(\lambda\big) \;\to\; U\big(F_{\Sigma,\mu_1(z_1),\ldots,\mu_n(z_n)}(\lambda)\big),$$

where $F_{\Sigma,\mu_1(z_1),\ldots,\mu_n(z_n)}(\lambda) := F_{\Sigma^0}(\mu_1 \otimes \ldots \otimes \mu_n \otimes \lambda)$.

As before, $Z_{\Sigma,\varphi_1(z_1),\ldots,\varphi_n(z_n)}$ depends on the choice of tangent vectors $v_i \in T_{z_i}\Sigma$. Similarly, the functor

$$F_{\Sigma,\mu_1(z_1),\ldots,\mu_n(z_n)} : \mathcal{C}(\partial_{in}\Sigma) \to \mathcal{C}(\partial_{out}\Sigma)$$

depends on some extra choices at the points z_i. But what it depends on is somewhat weaker than tangent vectors: the functor $F_{\Sigma,\ldots}$ only depends on rays $\rho_i \subset T_{z_i\Sigma}$ [a *ray* in a vector space V is an element of the quotient $(V \setminus \{0\})/\mathbb{R}_+$]. Also, when defining Σ^0, it was important to have used embeddings $f_i : \mathbb{D} \to \Sigma$ which satisfied $f_i'(0) \in \rho_i$.

Let us show that $F_{\Sigma,\mu_1(z_1),\ldots,\mu_n(z_n)}$ doesn't depend of the choice of f_i (up to canonical iso). Let $\hat{f}$ be another choice, and let $\hat{F}_{\Sigma,\ldots}$ be the corresponding functor. To compare $\hat{F}_{\Sigma,\ldots}$ and $F_{\Sigma,\ldots}$, we pick a third set of maps $\tilde{f}$ as in (17), and let $\tilde{F}_{\Sigma,\ldots}$ be the corresponding functor. It's enough to identify $\tilde{F}_{\Sigma,\ldots}$ with $F_{\Sigma,\ldots}$. Let $\psi_i = f_i^{-1} \circ \tilde{f}_i$ and $A_i = \mathbb{D} \setminus \psi_i(\mathring{\mathbb{D}})$. Since $\psi_i' \in \mathbb{R}_+$, the annulus A_i comes with a canonical lift $\tilde{A}_i$ to the universal cover of $\mathrm{Univ}_0(\mathbb{D})$. The desired identification is then given by:

$$\begin{aligned}
\tilde{F}_{\Sigma,\mu_1(z_1),\ldots,\mu_n(z_n)} &= F_{\tilde{\Sigma}^0}\big(\mu_1 \otimes \ldots \otimes \mu_n \otimes -\big) \\
&= F_{\Sigma^0}\big(F_{A_1}(\mu_1) \otimes \ldots \otimes F_{A_n}(\mu_n) \otimes -\big) \\
&\qquad\qquad \downarrow T_{\tilde{A}_1} \qquad\qquad\quad \downarrow T_{\tilde{A}_n} \\
&\quad\; F_{\Sigma^0}\big(\;\; \mu_1 \quad \otimes \quad \ldots \quad \otimes \quad \mu_n \quad \otimes \; - \;\big) = F_{\Sigma,\mu_1(z_1),\ldots,\mu_n(z_n)}.
\end{aligned}$$

Descendant fields

Our next goal is to generalize (19) to the case when the condition $Vir_{>0}\,\xi = 0$ is no longer satisfied. These are called *descendant fields*. More precisely, I'd call (the field associated to) $\xi \in H_0$ a *descendant* (of ξ_0) if $\xi = L_{m_1} \ldots L_{m_k}\xi_0$ for some $m_1, \ldots, m_k \leq 0$, and some primary ξ_0. When dealing with descendant fields, we need to replace the vectors v_i by elements of the *jet space*:

Definition: Let Σ be a Riemann surface. The *jet space* of order d of Σ is given by:

$$J_z^d\Sigma := \left\{ j : \underset{0\ \mapsto\ z}{\mathbb{C} \dashrightarrow \Sigma} \;\middle|\; \begin{matrix} j \text{ is holomorphic,} \\ j'(0) \neq 0 \end{matrix} \right\} \Big/ j_1 \sim j_2 \text{ if } j_1(z) = j_2(z) + o(z^d),$$
$$J^d\Sigma := \bigcup\nolimits_{z\in\Sigma} J_z^d\Sigma\,.$$

Here, the notation $j : \mathbb{C} \dashrightarrow \Sigma$ means that j is only defined in a neighbourhood of 0.

Consider the tower of Lie algebras

$$Vir_{[0]} \twoheadleftarrow Vir_{[0,1]} \twoheadleftarrow Vir_{[0,2]} \twoheadleftarrow Vir_{[0,3]} \twoheadleftarrow \ldots\ldots \qquad \leftarrow Vir_{\geq 0},$$

where $Vir_{[0,d-1]} := Vir_{\geq 0}/Vir_{\geq d} = \mathrm{Span}\{L_0, \ldots, L_{d-1}\}$. They integrate to a tower of Lie groups:

$$\mathbb{C}^\times = G_1 \twoheadleftarrow G_2 \twoheadleftarrow G_3 \twoheadleftarrow G_4 \twoheadleftarrow \ldots\ldots \qquad \leftarrow \mathrm{Univ}_0(\mathbb{D}),$$

where $G_d = J_0^d\mathbb{C}$, with group operation given by composition of functions. In other words:

$$G_d = \left\{ \begin{matrix} \text{changes of coordinate} \\ \text{defined up to degree } d \end{matrix} \right\}.$$

One can also describe that group more algebraically, as $\mathrm{Aut}(\mathbb{C}[z]/z^{d+1})$.

Definition: A vector $\xi \in H_0$ is called a *finite energy vector* if it is a finite linear combination of eigenvectors of L_0.

We now generalize (19) to the case when ξ_i are arbitrary finite energy vectors. By the positive energy condition, since the L_n for $n > 0$ are lowering operators, the action of $Vir_{\geq 0}$ on ξ_i generates a finite dimensional subspace. Call it $V_i \subset H_0$. The action of $Vir_{\geq 0}$ on V_i factors through a finite quotient $Vir_{[0,d_i-1]}$, and integrates to an action of G_{d_i}. [A priori, one might expect the action to only integrate to an action of the universal cover $\widetilde{G}_{d_i}$ of G_{d_i}. But the subalgebra $Vir_{[0]} \subset Vir_{[0,d_i-1]}$ integrates to a $\mathbb{C}^\times$. So the action of $\widetilde{G}_{d_i}$ on V_i descends to G_{d_i}.]

Instead of (19), we can then write:

$$Z_{\Sigma,\varphi_1(z_1;j_i),\ldots,\varphi_n(z_n;j_n)} := Z_{\Sigma^0}\big(g_1\xi_1 \otimes \ldots \otimes g_n\xi_n \otimes -\big), \tag{20}$$

where $g_i := f_i^{-1} \circ j_i \in G_{d_i}$ and, as before, $\Sigma^0 = \Sigma \setminus (f_1(\mathring{\mathbb{D}}) \sqcup \ldots \sqcup f_n(\mathbb{D}))$. Once again, we abbreviate things by writing $Z_{\Sigma,\varphi_1(z_1),\ldots,\varphi_n(z_n)}$ instead of $Z_{\Sigma,\varphi_1(z_1;j_i),\ldots,\varphi_n(z_n;j_n)}$.

In order to check that the map (20) is well defined (independent of the f_i), we proceed along the same lines as the previous proof. The analog of (18) (the most relevant part of the computation) is given by:

$$\begin{aligned}\tilde{Z}_{\Sigma,\varphi_1(z_1),\ldots,\varphi_n(z_n)} &= Z_{\tilde{\Sigma}^0}\big(\tilde{g}_1\xi_1 \otimes \ldots \otimes \tilde{g}_n\xi_n \otimes -\big)\\ &= Z_{\Sigma^0}\big(Z_{A_1}(\tilde{g}_1\xi_1) \otimes \ldots \otimes Z_{A_n}(\tilde{g}_n\xi_n) \otimes -\big)\\ &= Z_{\Sigma^0}\big(\psi_1\tilde{g}_1(\xi_1) \otimes \ldots \otimes \psi_n\tilde{g}_n(\xi_n) \otimes -\big)\\ &= Z_{\Sigma^0}\big(g_1\xi_1 \otimes \ldots \otimes g_n\xi_n \otimes -\big)\\ &= Z_{\Sigma,\varphi_1(z_1),\ldots,\varphi_n(z_n)},\end{aligned}$$

where $\psi_i = f_i^{-1} \circ \tilde{f}_i$ and $A_i = f_i(\mathbb{D}) \setminus \tilde{f}_i(\mathring{\mathbb{D}})$.

Lemma 14 *Let $\xi \in H_0$ be a finite energy vector, with corresponding field φ. Let $g \in G_d$ be a group element, and let $g\varphi$ be the field that corresponds to $g\xi$. Then we have:*

$$\varphi(z; j \circ g) = (g\varphi)(z; j).$$

Proof. $Z_{\Sigma,\varphi(z;j\circ g),\ldots} = Z_{\Sigma^0}((f^{-1}jg)\xi \otimes \ldots) = Z_{\Sigma^0}((f^{-1}j)(g\xi) \otimes \ldots) = Z_{\Sigma,(g\varphi)(z;j),\ldots}$ where, as before, $\Sigma^0 = \Sigma \setminus (f(\mathring{\mathbb{D}}) \sqcup \ldots)$ for some embeddings $f : \underset{0\,\mapsto\, z}{\mathbb{D} \to \Sigma}, \ldots$ ☐

When ξ is an eigenvector of L_0, one can also describe these more general types of fields axiomatically. In the definition of primary field, just replace the tangent vector v_i by a local coordinate $j_i : \underset{0\,\mapsto\, z_i}{\mathbb{C} \dashrightarrow \Sigma}$, and require the equation $\varphi(z; j \circ (z \mapsto az)) = a^{\Delta}\varphi(z; j)$ to hold:

$$Z_{\Sigma,\varphi(z_1;j_1),\ldots,\varphi(z_i;j_i\circ(z\mapsto az)),\ldots,\varphi(z_n;j_n)} = a^{\Delta} Z_{\Sigma,\varphi(z_1;j_1),\ldots,\varphi(z_n;j_n)} \qquad \forall a \in \mathbb{C}^{\times}.$$

We call such a thing a *field of conformal dimension* Δ. Similarly to the case of primary fields, we then have:

Theorem. (State-field correspondence) *There is a natural bijection:*

$$\Big\{ \text{Fields of conformal dimension } \Delta \Big\} \longleftrightarrow \Big\{ \xi \in H_0 \,\Big|\, L_0(\xi) = \Delta\xi \Big\}.$$

The proof goes along the same lines as the one in the previous section.

One of the defining properties of chiral CFT is that the fields $\varphi(z)$ 'depend holomorphically on z'. We formalize this in the following proposition:

Proposition. *Let φ be a field. Then the map*

$$\begin{array}{ccc} J^d(\Sigma \setminus \{\ldots\}) & \longrightarrow & \mathrm{Hom}\big(U(\lambda), U(F_\Sigma(\lambda))\big) \\ j \in J^d_z\Sigma & \mapsto & Z_{\Sigma,\varphi(z;j),\ldots} \end{array} \tag{21}$$

is holomorphic. (Here, the "$\setminus\{\ldots\}$" refers to the finitely many points where the other field insertions take place.)

Proof. Holomorphicity is a local condition. For every disc $D \subset \Sigma \setminus \{\ldots\}$, we have $Z_{\Sigma,\varphi(z;j),\ldots} = Z_{\Sigma \setminus D,\ldots} \circ Z_{D,\varphi(z;j)}$. So it's enough to prove the statement when Σ is a disc.

When $\Sigma = \mathbb{D}$, we can identify a jet $j \in J^d_z(\mathbb{D})$ with an element $g \in G_d$. By Lemma 14, for every $r < 1$, and every z of norm at most $1 - r$, we have

$$Z_{\mathbb{D},\varphi(z;j)} = Z_{\mathbb{D},(g\varphi)(z;w \mapsto w+z)} = Z_{\mathbb{D}\setminus(r\mathbb{D}+z)} \circ Z_{r\mathbb{D},(g\varphi)(0;\mathrm{id})}.$$

$Z_{\mathbb{D}\setminus(r\mathbb{D}+z)}$ depends holomorphically on z because $\mathbb{D} \setminus (r\mathbb{D} + z)$ does, and $Z_{r\mathbb{D},(g\varphi)(0;\mathrm{id})}$ depends holomorphically on j because $g\varphi$ does. So $Z_{\mathbb{D},\varphi(z;j)}$ depends holomorphically on z and on j. $\square$

Examples of fields

- If our conformal field theory is a chiral WZW model, or anything that admits affine Lie algebra symmetries, then, for every $X \in \mathfrak{g}$, we can consider the element $X_{-1}\Omega \in H_0$. The associated field is called a current and is denoted $J^X(z)$. This is a primary field of conformal dimension one:

$$L_1 X_{-1}\Omega = X_{-1}\underbrace{L_1\Omega}_{=0 \text{ because that's in degree } -1} + \underbrace{X_0\Omega}_{=0 \text{ because } \Omega \text{ is } \mathfrak{g}\text{-invariant}} = 0.$$

- If our conformal field theory is a chiral minimal model, or anything that contains Virasoro algebra symmetries (which is to say... any chiral CFT), then we can consider the element $\omega := L_{-2}\Omega \in H_0$, the so-called "conformal vector". The associated field is called the stress-energy tensor and is denoted $T(z)$. This field is also called the 'energy-momentum tensor' or 'Virasoro field', and it is *not* primary, unless $c = 0$:

$$\begin{aligned} L_1\omega &= L_1L_{-2}\Omega = 3\underbrace{L_{-1}\Omega} = 0 \\ L_2\omega &= L_2L_{-2}\Omega = 4\underbrace{L_0\Omega}_{=0 \text{ because } \Omega \text{ is } PSU(1,1)\text{-invariant}} + \tfrac{c}{12}\cdot 6 \cdot \Omega = \tfrac{c}{2}\cdot\Omega \end{aligned}$$

The action of $Vir_{\geq 0}$ on ω generates a two dimensional subspace $\mathrm{Span}\{\Omega, \omega\} \subset H_0$, on which the Lie algebra $Vir_{[0,2]}$ acts by

n	0	1	2
$L_n\Omega$	0	0	0
$L_n\omega$	2ω	0	$\frac{c}{2}\Omega$

(22)

Claim: At the Lie group level, this integrates to the action of $G_3 = \mathrm{Aut}(\mathbb{C}[z]/z^4)$ given by:

$$\begin{cases} g \cdot \Omega = \Omega \\ g \cdot \omega = g'(0)^2\omega + \frac{c}{12}\left(\frac{g'''(0)}{g'(0)} - \frac{3}{2}\left(\frac{g''(0)}{g'(0)}\right)^2\right)\Omega. \end{cases} \tag{23}$$

In terms of the basis $\{\Omega, \omega\}$, that representation can be equivalently described as:

$$g \;\mapsto \begin{pmatrix} 1 & \frac{c}{12}\left(\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2\right) \\ 0 & g'^2 \end{pmatrix}_{z=0}$$

Here, the expression is called the *Schwarzian derivative*, and is often denoted by the symbol $\{g, z\}$.

Now, if someone hands you a representation of a Lie group sand says *"that's the representation which integrates the following Lie algebra rep"*, how do you check it? One way to proceed is as follows:

(1) You compute the formula to first order for an element close to the identity, and check that it agrees with the given Lie algebra representation. Computing modulo ε^2, we get:

$$\text{If } g(z) = z + \varepsilon z, \text{ then } \{g, z\} = 0 \quad \rightsquigarrow \quad g \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1+2\varepsilon \end{smallmatrix}\right). \qquad \checkmark$$

$$\text{If } g(z) = z + \varepsilon z^2, \text{ then } \{g, z\} = 0 \quad \rightsquigarrow \quad g \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right). \qquad \checkmark$$

$$^{9}\text{If } g(z) = z + \varepsilon z^3, \text{ then } \{g, z\} = 6\varepsilon \quad \rightsquigarrow \quad g \mapsto \left(\begin{smallmatrix} 1 & \frac{c}{2}\varepsilon \\ 0 & 1 \end{smallmatrix}\right). \qquad \checkmark$$

(2) You check that it's indeed a representation:

$$\begin{pmatrix} 1 & \frac{c}{12}\left(\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2\right) \\ 0 & g'^2 \end{pmatrix}_{z=0} \begin{pmatrix} 1 & \frac{c}{12}\left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right) \\ 0 & f'^2 \end{pmatrix}_{z=0} \overset{?}{=} \begin{pmatrix} 1 & \frac{c}{12}\left(\frac{(g\circ f)'''}{(g\circ f)'} - \frac{3}{2}\left(\frac{(g\circ f)''}{(g\circ f)'}\right)^2\right) \\ 0 & (g\circ f)'^2 \end{pmatrix}_{z=0}$$

This looks like an annoying computation, but it's actually not too bad. To begin with, we compute the first, second, and third derivatives of $g \circ f$ at zero:

$$\begin{aligned} (g\circ f)' &=_{z=0} g'f' \\ (g\circ f)'' &=_{z=0} g''f'^2 + g'f'' \\ (g\circ f)''' &=_{z=0} g'''f'^3 + 3g''f'f'' + g'f'''. \end{aligned}$$

Ignoring the $\frac{c}{12}$, the upper right corners of the two sides of the above equation are given by:

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 + \frac{g'''f'^2}{g'} - \frac{3}{2}\left(\frac{g''f'}{g'}\right)^2$$

and

$$\frac{g'''f'^3 + 3g''f'f'' + g'f'''}{g'f'} - \frac{3}{2}\left(\frac{g''f'^2 + g'f''}{g'f'}\right)^2$$

$$= \frac{g'''f'^2}{g'} + \frac{3g''f''}{g'} + \frac{f'''}{f'} - \frac{3}{2}\left(\frac{g''f'}{g'}\right)^2 - 3\frac{g''f''}{g'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \qquad \| \;\checkmark$$

[9] We write $z+\varepsilon z^3$ as opposed to $z-\varepsilon z^3$ for the infinitesimal transformation corresponding to L_2 because of the issue pointed out in the remark on page 16.

Those two expressions are indeed equal.

This finishes the proof that the action (22) of $Vir_{[0,2]}$ integrates to the action (23) of the group G_3.

As an upshot of the above computation, we get the following special case of Lemma 14, known as the *'anomalous transformation of the stress-energy tensor'*: For $f : \mathbb{C} \dashrightarrow \mathbb{C}$, $f(0) = 0$, we have

$$T\big(w; j \circ f\big) \;=\; f'(0)^2 T\big(w; j\big) + \tfrac{c}{12}\{f, z\}_{z=0}\,\Omega.$$

For $w \in \mathbb{C}$, this becomes:

$$T\big(w; z \mapsto f(z)\big) \;=\; f'(0)^2 T\big(w; z \mapsto z + w\big) + \tfrac{c}{12}\{f, z\}_{z=0}\,\Omega.$$

In physics lingo, this is usually expressed in the following terms: "Under the map $z \to f(z)$, the stress-energy tensor transforms as $T(z) \to (\partial f)^2 T(f(z)) + \frac{c}{12}\{f, z\}$."

Remark 15 *The Schwarzian derivative also appears in the formulas which describe the action of $\varphi \in \mathrm{Diff}(S^1)$ on the universal central extensions of $\mathfrak{X}_{\mathbb{C}}(S^1)$. Namely, for $(f\frac{\partial}{\partial z}, a) \in {}^{\mathbb{C}}\mathfrak{X}_{\mathbb{C}}(S^1)$, we have $\varphi^*(f\frac{\partial}{\partial z}, a) = (\frac{f \circ \varphi}{\varphi'}\frac{\partial}{\partial z}, a + \frac{1}{12}\int_{S^1} \frac{f \circ \varphi(z)}{\varphi'(z)}\{\varphi, z\}\frac{dz}{2\pi i})$.*

Because the Schwarzian derivative vanishes on all infinitesimal coordinate transformations of the form $z \to z + \varepsilon$, $z \to z + \varepsilon z$ and $z \to z + \varepsilon z^2$, it vanishes on the group generated by them. Namely, on the group $PSL(2, \mathbb{C})$ of fractional linear transformations.

***Slogan*:** The Schwarzian derivative $\{f, z\}$ is a version of the third derivative which measures the failure of f being a fractional linear transformation (just like f''' measures the failure of f being a quadratic polynomial).

Let $\varphi(z)$ be a field of conformal dimension Δ. One way to say that $\varphi(z)$ is primary is to say that it satisfies

$$\varphi(z; j \circ f) = f'(0)^{\Delta}\varphi(z; j) \qquad \forall f : \mathbb{C} \underset{0 \mapsto 0}{\dashrightarrow} \mathbb{C}.$$

That's just a complicated way of saying that $\varphi(z)$ only depends on the vector $j'(0) \in T_z\Sigma$. An arbitrary field $\varphi(z)$ of conformal dimension Δ only satisfies

$$\varphi(z; j \circ (z \mapsto az)) = a^{\Delta}\varphi(z; j) \qquad \forall a \in \mathbb{C}^{\times}.$$

There's also an intermediate condition which is useful:

Definition: A field $\varphi(z)$ of conformal dimension Δ is called quasi-primary if

$$\varphi(z; j \circ f) = f'(0)^{\Delta}\varphi(z; j) \qquad \forall f \in PSL(2, \mathbb{C}),\; f(0) = 0.$$

The stress-energy tensor $T(z)$ is a quasi-primary field of conformal dimension 2.

The state-field correspondence for quasi-primary fields reads as follows:

Theorem. (State-field correspondence) *There is a natural bijection*

$$\left\{ \begin{array}{c} \text{Quasi-primary fields} \\ \text{of conformal dimension } \Delta \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{States } \xi \in H_0 \text{ s.t.} \\ L_0(\xi) = \Delta\xi \text{ and } L_1(\xi) = 0 \end{array} \right\}.$$

The Segal commutation relations

Let $\xi \in H_0$ be a finite energy vector, let φ be the associated field, let $V \subset H_0$ be the $Vir_{\geq 0}$-module generated by ξ, and let $d \in \mathbb{N}$ be such that the action of $Vir_{\geq 0}$ on V descends to an action of $Vir_{[0,d-1]}$ and hence to an action of the group G_d. Let Σ be a complex cobordism and let $\lambda \in \mathcal{C}(\partial_{in}\Sigma)$. By Lemma 14, we have a commutative diagram:

$$\begin{tikzcd}[column sep=large] J^d\mathring{\Sigma} \arrow[r, "{j \,\mapsto\, Z_{\Sigma,\varphi(j(0);j)}}"'] \arrow[d, "{\substack{\downarrow \\ [(j,\xi)]}}"] & \mathrm{Hom}\Big(U(\lambda), U(F_\Sigma(\lambda))\Big) \\ V_{\mathring{\Sigma}} := J^d\mathring{\Sigma} \underset{G_d}{\times} V \arrow[ur, dashed, "\Phi_{\mathring{\Sigma}}"'] & \end{tikzcd} \tag{24}$$

The dotted map is given by $\Phi_{\mathring{\Sigma}} : [(j,\eta)] \mapsto Z_{\Sigma,\varphi_\eta(j(0);j)}$, where φ_η is the field associated to η. The map $\Phi_{\mathring{\Sigma}}$ is holomorphic, and linear on the fibers of the vector bundle $V_{\mathring{\Sigma}} \to \mathring{\Sigma}$.

Our next goal is to generalize the above picture to allow the insertion point $z = j(0)$ to be on the boundary $\partial\Sigma$. Assuming Σ is equipped with *collars* (see the picture (27) below), we'll upgrade (24) to a map

$$\Phi_\Sigma : V_\Sigma := J^d\Sigma \underset{G_d}{\times} V \to \mathrm{Hom}\Big(\check{U(\lambda)}, U\widehat{(F_\Sigma(\lambda))}\Big) \tag{25}$$

Here, the cech on $U(\lambda)$ means that we make the space a little bit 'thinner', in a way that we'll explain below, and the hat on $U(F_\Sigma(\lambda))$ means that we make the space a bit 'fatter'.

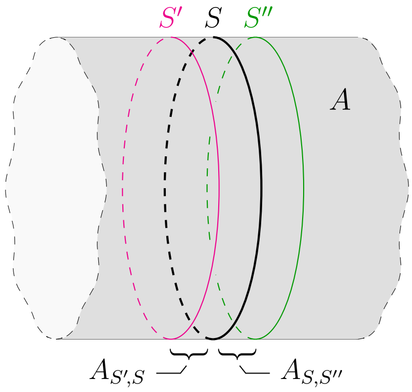
Remark: If φ is primary of conformal dimension Δ, then the vector space V is one dimensional, and $V_\Sigma = T^{\otimes\Delta}\Sigma$.

Let S be a circle. A collar is a piece of Riemann surface A in which S is analytically embedded. Two collars $S \hookrightarrow A$ and $S \hookrightarrow B$ are equivalent if there exist open subsets $U \subset A$ and $V \subset B$ containing S and an isomorphism $U \cong V$ that restricts to the identity on S (by analytic continuation, such an isomorphism is unique provided it exists.) An equivalence class of collars is the same thing as an *analytic structure* on S, i.e., a subsheaf $\mathcal{O}_S^{an} \subset \mathcal{O}_S$ of the sheaf of smooth functions on S which is locally isomorphic to the sheaf of analytic functions on $\mathbb{R}$.

Let $S \hookrightarrow A$ be a collar. Given two circles $S_1, S_2 \subset A$ that are isotopic to S and such that S_2 is 'in the future' of S_1, we write A_{S_1,S_2} for the part of A which lies between S_1 and S_2 (A_{S_1,S_2} is a thick annulus). For convenience, we abbreviate $F_{A_{S_1,S_2}}$ by F_{S_1,S_2} and $Z_{A_{S_1,S_2}}$ by Z_{S_1,S_2}. For any object $\lambda \in \mathcal{C}(S)$, we can then define

$$\check{H}_\lambda := \varinjlim_{\substack{S' \text{ in the} \\ \text{past of } S}} U\big(F^{-1}_{S',S}(\lambda)\big) \qquad \text{and} \qquad \widehat{H}_\lambda := \varprojlim_{\substack{S'' \text{ in the} \\ \text{future of } S}} U\big(F_{S,S''}(\lambda)\big)$$

where the maps used to define the limits are given by $Z_{S_1',S_2'} : U(F^{-1}_{S_1',S}(\lambda)) \to U(F^{-1}_{S_2',S}(\lambda))$ and $Z_{S_1'',S_2''} : U(F_{S,S_1''}(\lambda)) \to U(F_{S,S_2''}(\lambda))$, respectively.

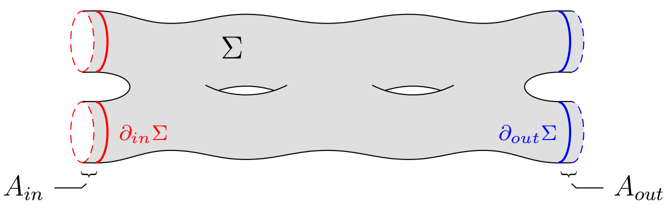


We then have dense inclusions $\check{H}_\lambda \subset H_\lambda \subset \widehat{H}_\lambda$. [10]

The advantage of working with the $\check{}$ and $\hat{}$ versions is that the map

$$Z_{\Sigma,\varphi_1(z_1),\ldots,\varphi_n(z_n)} : \widecheck{U(\lambda)} \longrightarrow \widehat{U(F_\Sigma(\lambda))} \tag{26}$$

is now well defined for all $z_1, \ldots, z_n$, including on the boundary of Σ. It is induced by the maps $Z_{\Sigma^+,\varphi_1(z_1),\ldots,\varphi_n(z_n)} : U(F_{A_{in}}^{-1}(\lambda)) \to U(F_{A_{out}}(F_\Sigma(\lambda)))$, where $\Sigma^+ = A_{in} \cup \Sigma \cup A_{out}$, and A_{in} and A_{out} are thin collars on the outside of Σ:



(27)

We also have maps

$$\check{Z}_{\Sigma,\ldots} : \widecheck{U(\lambda)} \to \widecheck{U(F_\Sigma(\lambda))} \qquad \text{and} \qquad \hat{Z}_{\Sigma,\ldots} : \widehat{U(\lambda)} \to \widehat{U(F_\Sigma(\lambda))},$$

defined in the obvious way.

As a particular case of (26), for every circle with collar S, and every point with local coordinate $z \in S$, we have a map

$$\boxed{\varphi(z) : \check{H}_\lambda \to \widehat{H}_\lambda}$$

given by $Z_{\mathrm{id}_S,\varphi(z)}$.

Let $V_S := J^d_{\mathbb{C}} S \times_{G_d} V$ be as in (24), where $J^d_{\mathbb{C}}$ denotes the complexified jet space of S, and let $\Phi_S : V_S \to \mathrm{Hom}(\check{H}_\lambda, \widehat{H}_\lambda)$ be as in (25). Given a smooth section $f \in \Gamma(\Omega^1_S \otimes V_S)$, we define the *smeared field* $\varphi[f] : \check{H}_\lambda \to \widehat{H}_\lambda$ to be the image of f under the map

[10] G. Segal suggests to add the axiom $H_\lambda = \widehat{H}_\lambda$ to the definition of a chiral CFT. Unfortunately, I think that this is incompatible with the requirement that the representations be smooth in the sense of Remark 10.

$$\Gamma(\Omega^1_S \otimes V_S) \xrightarrow{\Phi_S} \Gamma\big(\Omega^1_S \otimes \mathrm{Hom}(\check{H}_\lambda, \widehat{H}_\lambda)\big) \xrightarrow{\int_S} \mathrm{Hom}(\check{H}_\lambda, \widehat{H}_\lambda).$$

Remark: When φ is primary of conformal dimension Δ, then f is just a section of $T^{\otimes(\Delta-1)}\Sigma$ (in particular, when φ is a current f is just a function). In that case, we can rewrite $\varphi[f]$ in the following more intuitive form:

$$\varphi[f] = \int_S f(z)\varphi(z)dz.$$

A priori, $\varphi[f]$ is only a map from $\check{H}_\lambda$ to $\widehat{H}_\lambda$. However, as far as I understand, when working with nuclear Fréchet spaces, this always extends by continuity to a map

$$\boxed{\varphi[f] : H_\lambda \to H_\lambda}$$

In that sense, quantum fields are *operator valued distributions*. They are things which take a test function f as input and produce an operator $H_\lambda \to H_\lambda$ as output.

Remark. When working with Hilbert spaces, a smeared field $\varphi[f]$ is typically not an operator $H_\lambda \to H_\lambda$. It is only a map $\check{H}_\lambda \to H_\lambda$, as well as a map $H_\lambda \to \widehat{H}_\lambda$. In other words, it is an *unbounded* operator on H_λ.

Proposition. (Segal commutation relations) *Let Σ be a complex cobordism, and let φ be a field. Then, for every holomorphic section $f \in \Gamma_{hol}(\Omega^1_\Sigma \otimes V_\Sigma)$, letting $f_{in} := f|_{\partial_{in}\Sigma}$ and $f_{out} := f|_{\partial_{out}\Sigma}$, we have:*

$$\varphi[f_{out}] \circ \check{Z}_\Sigma \;=\; \widehat{Z}_\Sigma \circ \varphi[f_{in}]. \tag{28}$$

Proof. Consider the image of f under the map

$$\Gamma_{hol}\big(\Omega^1_\Sigma \otimes V_\Sigma\big) \xrightarrow{\Phi_\Sigma} \Gamma_{hol}\Big(\Omega^1 \otimes \mathrm{Hom}\big(\widecheck{U(\lambda)}, U\big(\widehat{F_\Sigma(\lambda)}\big)\big)\Big).$$

Then $\Phi_\Sigma(f)$ is a $\mathrm{Hom}\big(\widecheck{U(\lambda)}, U\big(\widehat{F_\Sigma(\lambda)}\big)\big)$-valued 1-form which is holomorphic on all of Σ. The integrals $\widehat{Z}_\Sigma \circ \varphi[f_{in}] = \int_{\partial_{in}\Sigma} \Phi_\Sigma(f)$ and $\varphi[f_{out}] \circ \check{Z}_\Sigma = \int_{\partial_{out}\Sigma} \Phi_\Sigma(f)$ are therefore equal by Cauchy's theorem. $\square$

Assuming that all smeared fields extend to maps $H_\lambda \to H_\lambda$, the commutation relations (28) simplify to:

$$\boxed{\varphi[f_{out}] \circ Z_\Sigma \;=\; Z_\Sigma \circ \varphi[f_{in}]}$$

It is expected that, when $\partial\Sigma \neq \emptyset$, the functor $F_\Sigma : \mathcal{C}(\partial_{in}\Sigma) \to \mathcal{C}(\partial_{out}\Sigma)$ is universal with respect to the Segal commutation relations:

Conjecture 16 Fix a chiral Segal CFT. Let Σ be a complex cobordism with non-empty bondary, and let $\lambda \in \mathcal{C}(\partial_{in}\Sigma)$. Then, for every object $\mu \in \mathcal{C}(\partial_{out}\Sigma)$ and every linear map $\zeta : U(\lambda) \to U(\mu)$,

<u>**if**</u> for every field φ of the CFT and every holomorphic section $f \in \Gamma_{hol}(\Omega^1_\Sigma \otimes V_\Sigma)$, the equation

$$\varphi[f_{out}] \circ \zeta \;=\; \zeta \circ \varphi[f_{in}]$$

holds,

<u>**then**</u> there exists a unique morphism $\kappa : F_\Sigma(\lambda) \to \mu$ such that

$$\zeta = U(\kappa) \circ F_\Sigma.$$

The definition of the WZW model

Fix a gauge group G, with Lie algebra $\mathfrak{g}$, and a level $k \in \mathbb{N}$. Recall that for every $X \in \mathfrak{g}$, the field associated to $X_{-1}\Omega \in H_0$ is denoted $J^X(z)$. Before going on, we'll need the following useful fact:

Proposition. *For every smooth function $f(z) = \sum_{n\in\mathbb{Z}} f_n z^n$ on S^1, we have*

$$\tfrac{1}{2\pi i} J^X[f] \;=\; \sum_{n\in\mathbb{Z}} f_n X_n. \tag{29}$$

Recall that, by definition, $\frac{1}{2\pi i} J^X[f] = \frac{1}{2\pi i} \int_{S^1} f(z) J^X(z) dz$, and that

$$\tfrac{1}{2\pi i} \int_{S^1} f(z)\Big(\sum_{n\in\mathbb{Z}} X_n z^{-n-1}\Big) dz = \sum_{m,n\in\mathbb{Z}} f_m \underbrace{\Big(\tfrac{1}{2\pi i}\int_{S^1} z^m z^{-n-1} dz\Big)}_{=\delta_{m,n}} X_n = \sum_{n\in\mathbb{Z}} f_n X_n.$$

So (29) is equivalent to the statement

$$J^X(z) \;=\; \sum_{n\in\mathbb{Z}} X_n z^{-n-1}. \tag{30}$$

Unfortunately, I can't really prove this... mostly because we haven't defined the WZW model yet (but also because I just don't know how to prove it). I can only offer the following partial proof:

Partial proof. We prove (29) and (30) in the case of the vacuum sector, and after applying these operators to the vacuum vector. Namely, we prove that the equation $J^X(z)\Omega = \sum X_n z^{-n-1}\Omega$ holds. Equivalently,

$$\widehat{Z}_{\mathbb{D}, J^X(z)} \;=\; \sum_{n\geq 0} z^n X_{-n-1}\Omega. \tag{31}$$

Both sides of (31) make sense for $z \in \mathbb{D}$. So, by continuity, it's enough to show that the equation holds for $z \in \mathring{\mathbb{D}}$. And by analyticity, it's enough to show that the equation holds for $z \in (-1, 1)$. First of all, the two sides of (31) agree when z is zero: $\widehat{Z}_{\mathbb{D},J^X(0)} = X_{-1}\Omega$. Letting $z_t = \tanh(t)$, we show that both

$$(1 - z_t^2)\widehat{Z}_{\mathbb{D},J^X(z_t)} \qquad \text{and} \qquad (1 - z_t^2)\sum_{n\geq 0} z_t^n X_{-n-1}\Omega \tag{32}$$

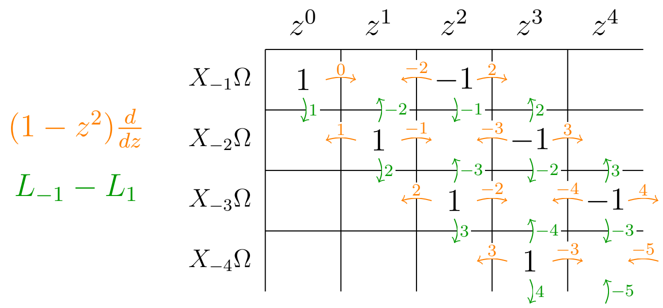
satisfy the ODE $\frac{d}{dt}F(t) = (L_{-1} - L_1)F(t)$, and therefore agree for $t \in \mathbb{R}$, i.e., for $z_t \in (-1, 1)$. The vector field $L_{-1} - L_1 = (z^2 - 1)\frac{\partial}{\partial z}$ integrates to the homomorphism

$$t \in \mathbb{R} \mapsto U_t := \tfrac{\cosh(t)z + \sinh(t)}{\sinh(t)z + \cosh(t)} \in PSU(1, 1)$$

(up to a minus sign; see the remark on page 16), so the LHS of (32) satisfies the ODE because

$$(1 - z_t^2)\widehat{Z}_{\mathbb{D},J^X(z_t)} \overset{J^X \text{ is primary of dimension one}}{=} U_t\big(\widehat{Z}_{\mathbb{D},J^X(0)}\big).$$

The RHS satisfies the equivalent ODE $(1 - z^2)\frac{d}{dz}F(z) = (L_{-1} - L_1)F(z)$ by direct computation, using the commutation relations (11). Here's a picture of the computation:



So the two sides of (31) agree for $z \in (-1, 1)$. By analytic continuation, they therefore agree for all $z \in \mathring{\mathbb{D}}$. $\square$

What we can now do with the above proposition is use it to turn Conjecture 16 into a definition of the WZW models (at least a definition of the functors F_Σ and of the linear maps Z_Σ). Recall that the category associated to a 1-manifold S is given by $\mathrm{Rep}^{\text{unit.}}_{\substack{pos\\en.}}(\widetilde{L_S\mathfrak{g}_k})$, where $\widetilde{L_S\mathfrak{g}_k}$ is the central extension of $L_S\mathfrak{g} = C^\infty(S, \mathfrak{g})$ associated to the cocycle $(f, g) \mapsto \frac{k}{2\pi i}\int_S \langle f, dg\rangle$.

Definition: (definition of the WZW model) For Σ a complex cobordism, let us abbreviate $\mathrm{Rep}^{\text{unit.}}_{\substack{pos\\en.}}(\widetilde{L_{\partial_{in}\Sigma}\mathfrak{g}_k})$ by $\mathrm{Rep}(\widetilde{L\mathfrak{g}}_{k,in})$ and $\mathrm{Rep}^{\text{unit.}}_{\substack{pos\\en.}}(\widetilde{L_{\partial_{out}\Sigma}\mathfrak{g}_k})$ by $\mathrm{Rep}(\widetilde{L\mathfrak{g}}_{k,out})$. Given a positive energy representation

$$(V, \rho) = \lambda \in \mathrm{Rep}(\widetilde{L\mathfrak{g}}_{k,in}),$$

its image

$$\Big((W, \pi) = F_\Sigma(\lambda) \in \mathrm{Rep}(\widetilde{L\mathfrak{g}}_{k,out}), \quad Z_\Sigma : V \to W\Big)$$

under (F_Σ, Z_Σ) satisfies the Segal commutation relations

$$\pi(f_{out}) \circ Z_\Sigma = Z_\Sigma \circ \rho(f_{in}) \qquad \forall f \in \mathcal{O}(\Sigma, \mathfrak{g}_\mathbb{C}),$$

and it is universal in the sense that for every object $(W', \pi') \in \mathrm{Rep}\big(\widetilde{L\mathfrak{g}}_{k,out}\big)$ and every linear map $Z' : V \to W'$ that satisfies

$$\pi'(f_{out}) \circ Z' = Z' \circ \rho(f_{in}) \qquad \forall f \in \mathcal{O}(\Sigma, \mathfrak{g}_{\mathbb{C}}),$$

there exists a unique $\widetilde{L\mathfrak{g}}_{k,out}$-linear map $\kappa : W \to W'$ such that $Z' = \kappa \circ Z_\Sigma$.

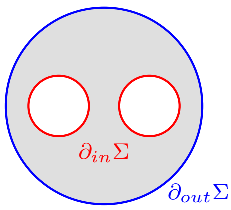
Open problem: Given two composable cobordisms Σ_1 and Σ_2, prove that the natural map $F_{\Sigma_1 \cup \Sigma_2}(\lambda) \to F_{\Sigma_1} \circ F_{\Sigma_2}(\lambda)$ is an isomorphism.

Now, why is this difficult?...
Well... for a universal construction to be well behaved, one needs the category in which it takes place to be "big enough". And, from that point of view, the positive energy condition is very awkward. So what we'd like is to be able to perform the universal construction in a bigger category (one which doesn't include the the positive energy condition), and have the result naturally satisfy the positive energy condition. But it's not clear that that's the case...

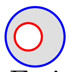
The fusion product

When Σ is a pair of pants, F_Σ is called the *fusion product*. This immediately raises the question of... *which* pair of pants?

Let us use pairs of pants Σ embedded in $\mathbb{C}$, where the boundary circles are round and parametrized by $z \mapsto az + b$ with $a, b \in \mathbb{R}$:

$\Sigma \;=$ 

(33)

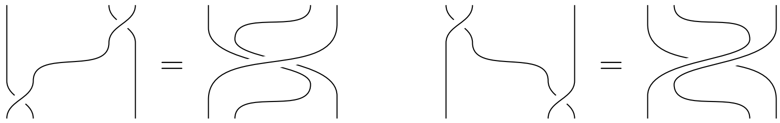
Note that if $A =$  is an annulus with boundary parametrized by $z \mapsto az + b$ with a and b real, then F_A is canonically trivialized. The trivialization is given by $T_{\tilde{A}}$, where $\tilde{A} \in \mathrm{Univ}(\mathbb{D})^{\mathbb{Z}}$ is the canonical lift of $A \in \mathrm{Univ}(\mathbb{D})$ to an element of the universal cover (using that a and b are real). By composing and un-composing a pair of pants (33) with such annuli, one can reach any other pair of pants of the form (33) in a way which is unique 'up to homotopy'. So the fusion product

$$F_\Sigma : \mathcal{C}(S^1) \times \mathcal{C}(S^1) \to \mathcal{C}(S^1)$$

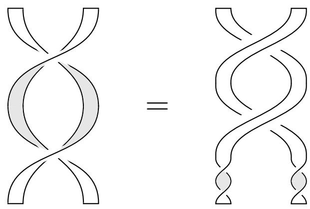
is well-defined, canonically up to canonical isomorphism.

The fusion product is visibly associative and unital, and it endows $\mathcal{C}(S^1)$ with the structure of a monoidal category. But it's more. It's also braided and balanced.

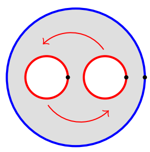
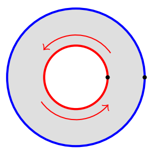
Definition: A monoidal category $(\mathcal{C}, \otimes)$ is *braided* if it's equipped with a family of natural isomorphisms $\beta_{\lambda,\mu} : \lambda \otimes \mu \to \mu \otimes \lambda$ that satisfy the two hexagon axioms:



Definition: A braided monoidal category $(\mathcal{C}, \otimes, \beta)$ is *balanced* if it's equipped with a family of natural isomorphisms $\theta_\lambda : \lambda \to \lambda$ that satisfy $\theta_{\lambda\otimes\mu} = \beta_{\mu,\lambda} \circ \beta_{\lambda,\mu} \circ (\theta_\lambda \otimes \theta_\mu)$. The isomorphism θ_λ is called the *twist*, are is denoted graphically by the full twist of a ribbon. With that graphical notation in mind, the above axiom becomes:



In terms of circles with holes, the braiding β and the twist θ correspond to the motions

 and 

In order to deal with such motions, it's important to relax the condition that the boundary parametrizations be of the form $z \mapsto az + b$ with $a, b \in \mathbb{R}$, and also allow $a, b \in \mathbb{C}$. (The little black dots in the above picture are indicators of where $1 \in S^1$ goes under the boundary parametrizations.)

Let $\mathcal{C} := \mathcal{C}(S^1)$, and let's introduce the following moduli space:

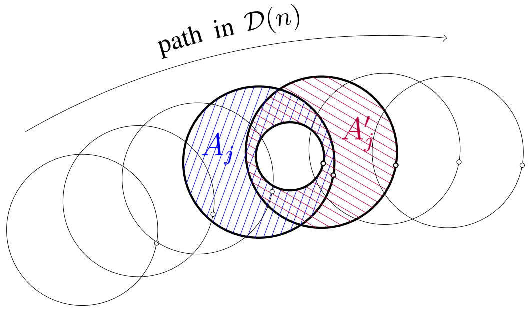
$$\mathcal{D}(n) := \left\{ \begin{matrix} n \text{ non-overlapping round circles in } \mathbb{D} \text{ with} \\ \partial \text{ parametrized by } z \mapsto az + b \text{ with } a, b \in \mathbb{C}. \end{matrix} \right\} \tag{34}$$

For each disc configuration $P \in \mathcal{D}(n)$, we get a functor $F_P : \mathcal{C}^n \to \mathcal{C}$, compatibly with composition. Let us write $P \prec P'$ if every circle of P is contained in the corresponding circle of P'.

Claim: for each homotopy class of path $\gamma : [0, 1] \to \mathcal{D}(n)$ from P_1 to P_2, there is an associated invertible natural transformation $T_\gamma : F_{P_1} \to F_{P_2}$.

The construction of T_γ goes as follows. Subdivide $[0, 1]$ into small intervals $[t_i, t_{i+1}]$ and let $P_i := \gamma(t_i)$. If the subdivision is fine enough, the circles of P_i and of P_{i+1} will have large overlaps. Pick $P'_i \in \mathcal{D}(n)$ such that $P_i \succ P'_i \prec P_{i+1}$, and write $P'_i = P_i \cup (A_1 \sqcup \ldots \sqcup A_n)$ and $P'_i = P_{i+1} \cup (A'_1 \sqcup \ldots \sqcup A'_n)$ for suitable annuli A_j and A'_j. Provided we pick P'_i close enough to P_i and to P_{i+1}, the annuli $A_j, A'_j \in \mathrm{Univ}(\mathbb{D})$ come with preferred lifts $\widetilde{A}_j, \widetilde{A}'_j$ to the universal cover of $\mathrm{Univ}(\mathbb{D})$. We go from F_{P_i} to

$F_{P_i'} = F_{P_i} \circ F_{A_1 \sqcup \ldots \sqcup A_n}$ by composing with the trivializations $T_{\widetilde{A}_j}$, and we then go back to $F_{P_{i+1}}$ by composing with the trivializations $T_{\widetilde{A}'_j}$.



(35)

By finely triangulating the domain a homotopy $h : [0,1]^2 \to \mathcal{D}(n)$ between two paths γ_0 and γ_1 from P_1 to P_2 and playing a similar game as above, we can see that $T_\gamma : F_{P_1} \to F_{P_2}$ only depends on the homotopy class of γ.

The above arguments show that $\mathcal{C}$ is not only monoidal, but also braided, and balanced. But it's even more:

Definition: A monoidal category $(\mathcal{C}, \otimes)$ is called *rigid* if every object $\lambda \in \mathcal{C}$ has a left dual and a right dual. Here, a left dual is an object $\lambda^\vee \in \mathcal{C}$ together with maps $\mathrm{ev} : \lambda^\vee \otimes \lambda \to 1$ and $\mathrm{coev} : 1 \to \lambda \otimes \lambda^\vee$ satisfying $(1_\lambda \otimes \mathrm{ev}) \circ (\mathrm{coev} \otimes 1_\lambda) = 1_\lambda$ and $(\mathrm{ev} \otimes 1_{\lambda^\vee}) \circ (1_{\lambda^\vee} \otimes \mathrm{coev}) = 1_{\lambda^\vee}$. Right duals are defined similarly. Even though this is not obvious from the definition, being rigid is just a property (it's not extra structure). In other words, if an object has a dual (say a left dual), then any two duals are canonically isomorphic.

Definition: A braided tensor category is called *ribbon* if it is balanced, rigid, and for every object $\lambda \in \mathcal{C}$, we have $\mathrm{ev} \circ (\theta_{\lambda^\vee} \otimes 1_\lambda) = \mathrm{ev} \circ (1_{\lambda^\vee} \otimes \theta_\lambda)$.

Definition: A braided tensor category is called *modular* if it is ribbon and the S-matrix $\big[$  $\big]_{\lambda\mu}$ is invertible.

The category $\mathcal{C} = \mathcal{C}(S^1)$ that a chiral Segal CFT assigns to a circle is always modular, but this is hard to prove. Showing that $\mathcal{C}$ is rigid is already very non-trivial. The only proof that I know of genuinely uses the CFT (i.e., it uses the right column of the table on page 9).[11] It has been conjectured that the mere fact that $\mathcal{C}$ is part of a modular functor (the left column of the table on page 9) should already imply that $\mathcal{C}$ is modular. But this is an open question. It's actually one of the big open questions in the field.

Remark. The fact that the circles were round in the definition of $\mathcal{D}(n)$ is not so important. What is important is that the parametrizations of the incoming circles extend to holomorphic maps on $\mathbb{D}$, so as to have A_j, A'_j in $\mathrm{Univ}(\mathbb{D})$ in (35).

[11]The proof is performed in the language of VOAs.

2d chiral CFT as a boundary of 3d TQFT

We can generalize the moduli space (34) by replacing the disc $\mathbb{D}$ by an arbitrary complex cobordism. Given a complex cobordism Σ, let:

$$\mathcal{D}_\Sigma(n) := \left\{ \begin{matrix} n \text{ holomorphic embeddings } \mathbb{D} \to \Sigma \\ \text{with non-overlapping images.} \end{matrix} \right\}$$

Then, for every $P \in \mathcal{D}_\Sigma(n)$, we get a functor

$$F_{\Sigma,P} : \mathcal{C}(\partial_{in}\Sigma) \otimes \mathcal{C}^n \longrightarrow \mathcal{C}(\partial_{out}\Sigma).$$

And for every path $\gamma : [0,1] \to \mathcal{D}_\Sigma(n)$ from P_1 to P_2 we get an isomorphism $T_\gamma : F_{\Sigma,P_1} \to F_{\Sigma,P_2}$. Moreover, the isomorphism T_γ only depends on the path γ up to homotopy.

If we fix objects $\mu_1, \dots, \mu_n \in \mathcal{C}$ and interior points $z_1, \dots, z_n \in \mathring{\Sigma}$ together with rays $\rho_i \subset T_{z_i}\Sigma$, we can always find embeddings $f_i : \mathbb{D} \to \Sigma$ satisfying $f_i(0) = z_i$ and $f_i'(0) \in \rho_i$. This yields a point $P \in \mathcal{D}_\Sigma(n)$, well defined up to contractible choice. The functor $F_{\Sigma,P}(- \otimes \mu_1 \otimes \dots \otimes \mu_n)$ then agrees with what we had denoted

$$F_{\Sigma,\mu_1(z_1;\rho_1),\dots,\mu_n(z_n;\rho_n)} : \mathcal{C}(\partial_{in}\Sigma) \to \mathcal{C}(\partial_{out}\Sigma).$$

Moreover, given finite energy vectors $\xi_i \in H_{\mu_i}$, the corresponding charged fields φ_i are maps

$$Z_{\Sigma,\varphi_1(z_1;j_1),\dots,\varphi_n(z_n;j_n)} \;:\; U(\lambda) \;\to\; U\big(F_{\Sigma,\mu_1(z_1;\rho_1),\dots,\mu_n(z_n;\rho_n)}(\lambda)\big), \tag{36}$$

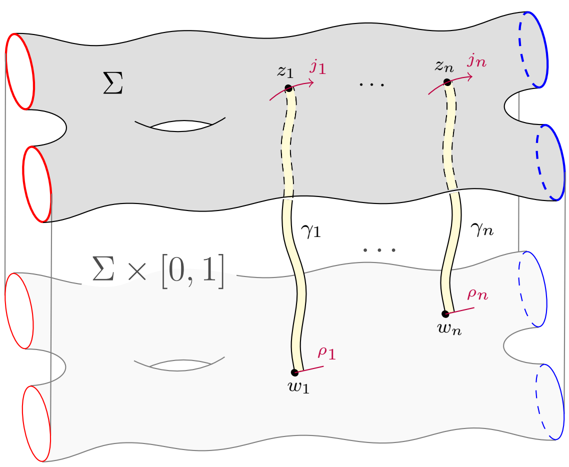
provided $j_i'(0) \in \rho_i$.

We'd like to say that $\varphi_i(z_i)$ depends holomorphically on z_i (and on j_i). This is a little bit tricky to formulate given that the place in which (36) takes its values depends on z_i (and on j_i). But it depends in a *flat* way, i.e., it's a vector bundle with flat connection over the moduli space of z_i's and ρ_i's. So we can locally trivialize the right hand side of (36) and pretend that all the $\varphi_i(z_i)$ take their values in the same space, at least locally in z_i and ρ_i.

Thinking more globally, we can make sense of the statement

$$Z_{\Sigma,\varphi_1(z_1;j_1),\dots,\varphi_n(z_n;j_n)}(\eta) \;\in\; U\big(F_{\Sigma,\mu_1(w_1;\rho_1),\dots,\mu_n(w_n;\rho_n)}(\lambda)\big) \quad \text{for} \quad \eta \in U(\lambda) \tag{37}$$

whenever we are provided with a (homotopy class of) path from $(z_i; j_1'(0)), \dots, (z_n; j_n'(0))$ to $(w_1; \rho_1), \dots, \mu_n(w_n; \rho_n)$. The picture which I wish to associate to (37) is the following:



Here, the path $\gamma : [0,1] \to \mathcal{D}(n)$ is interpreted as a ribbon braid $\gamma = (\gamma_1, \ldots, \gamma_n)$ inside $\Sigma \times [0,1]$, connecting the points $(w_1, \ldots, w_n) \subset \Sigma \times \{0\}$ to the points $(z_1, \ldots, z_n) \subset \Sigma \times \{1\}$. The fact that we could draw things as we did is a manifestation of the fact that *a chiral conformal field theory sits at the boundary of a 3d topological field theory*. The surface on which we drew the z_i's carries the chiral CFT, whereas the bulk, namely $\Sigma \times [0,1)$, is where the 3d TQFT lives.

In the special case $\partial\Sigma = \emptyset$, $\lambda = 1_\emptyset$, $\eta = 1$, one can rewrite (37) as follows:

$$\big\langle \varphi_1(z_1), \ldots, \varphi_n(z_n) \big\rangle_{\Sigma, \gamma_1, \ldots, \gamma_n} \in H_{\Sigma, \mu_1(w_1), \ldots, \mu_n(w_n)}. \tag{38}$$

Here, $H_{\Sigma, \mu_1(w_1), \ldots, \mu_n(w_n)}$ is just an equivalent name for the finite dimensional vector space $U(F_{\Sigma, \mu_1(w_1), \ldots, \mu_n(w_n)}(1_\emptyset))$. For charged fields, the correlator $\langle \varphi_1(z_1), \ldots, \varphi_n(z_n) \rangle$ doesn't just depend on the points z_i (and the local coordinates j_i), but also on the ribbon braid γ. (And if you were to try to think of it as a function of just the z_i and the j_i, then it would become a multivalued function.)

As before, a *conformal block* is a linear map $\mathcal{B} : H_{\Sigma, \mu_1(w_1), \ldots, \mu_n(w_n)} \to \mathbb{C}$, and the correlation function

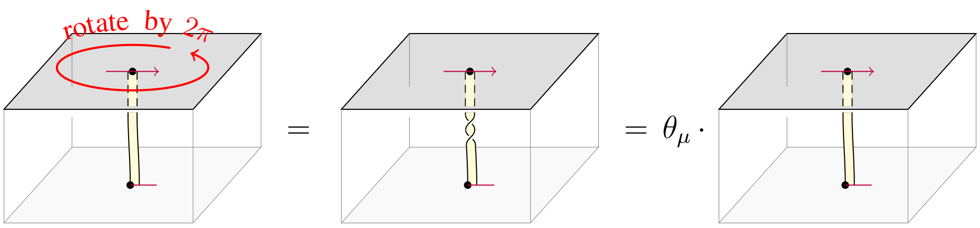
$$\big\langle \varphi_1(z_1), \ldots, \varphi_n(z_n) \big\rangle_{\Sigma, \mathcal{B}, \gamma_1, \ldots, \gamma_n} \in \mathbb{C}.$$

is the image of (38) under that map.

Let $\varphi(z)$ be a field of charge μ which is primary. Then we have

$$\varphi(z; av) = a^\Delta \varphi(z; v) \qquad \text{for } v \in T_z\Sigma \tag{39}$$

as in (13). But the point z now has a ribbon attached to it, so a lives no longer in $\mathbb{C}^\times$ but instead in the universal cover of $\mathbb{C}^\times$. Correspondingly, the conformal dimension Δ is no longer an integer. It is an eigenvalue of the action of L_0 on H_μ. Namely, it is a number of the form $h_\mu + n$, where h_μ is the minimal energy of H_μ and $n \in \mathbb{N}$ is a natural number. For example, if $a = e^{2\pi i}$ then $a^\Delta = (e^{2\pi i})^\Delta = \theta_\mu$ is the conformal spin of μ (which is equal to the twist of μ, coming from the fact that $\mathcal{C}(S^1)$ is balanced). Equation (39) becomes:



Let $\mathcal{M}_{g,n}$ be the moduli space of closed Riemann surfaces Σ of genus g with n marked points $w_1, \ldots, w_n$ together with rays $\rho_i \subset T_{z_i}\Sigma$. The fiber of the map $\mathcal{M}_{g,n} \to \mathcal{M}_g := \mathcal{M}_{g,0}$ *[Note that $\mathcal{M}_g$ is not a space but something slightly more general, called a stack]* over a Riemann surface $\Sigma \in \mathcal{M}_g$ is the configuration space $\mathrm{Conf}_\Sigma(n)$. The latter is a space which is homotopy equivalent to the space $\mathcal{D}_\Sigma(n)$ defined above. So we have a fiber bundle

$$\begin{array}{ccc} \mathrm{Conf}_\Sigma(n) & \to & \mathcal{M}_{g,n} \\ & & \downarrow \\ & & \mathcal{M}_g \end{array} \tag{40}$$

Fix objects $\mu_1, \ldots, \mu_n \in \mathcal{C}$. Then

$$\big(\Sigma, (w_1, \ldots, w_n)\big) \;\mapsto\; H_{\Sigma, \mu_1(w_1), \ldots, \mu_n(w_n)}$$

is a vector bundle of finite rank over $\mathcal{M}_{g,n}$ called (the dual of) the *bundle of conformal blocks*. Its fibers are called the (dual) *spaces of conformal blocks*.

The trivializations $T_{\tilde{A}}$ (see Table 1 on page 9) equip the bundle of conformal blocks with a flat projective connection. What this means is that the mapping class group $\pi_1(\mathcal{M}_{g,n})$ acts projectively on spaces of conformal blocks. Moreover, by the construction described in (35), that flat projective connection admits a lift to an honest (i.e. non-projective) flat connection in the direction of the fibers of (40). What this means is the above action of the mapping class group restricts to an honest action of the surface ribbon braid group $\pi_1(\mathrm{Conf}_\Sigma(n))$.