

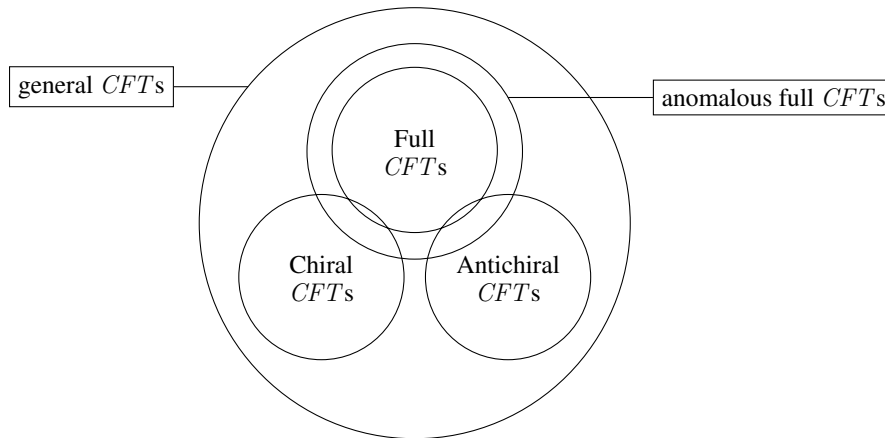
Chiral conformal field theory

Course notes

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Introduction

These notes are concerned with two-dimensional conformal field theory.

In 1987, Graeme Segal circulated a preprint that contained a transformative idea: a quantum field theory is a functor from a category whose objects are manifolds and whose morphisms are cobordisms to the category of vector spaces. Specifically, these should be symmetric monoidal functors, meaning they send disjoint unions of manifolds to tensor products of vector spaces. This formalism is now known as the *functorial approach* to quantum field theory. The type of quantum field theory Segal had in mind when developing his formalism were two-dimensional conformal field theories, in which case the relevant cobordism category is the two-dimensional complex cobordism category, Cob^{conf} , whose objects are (compact, smooth, oriented) 1-manifolds, and whose morphisms are Riemann surfaces with boundary. One arrives at the following

Definition sketch: *A conformal field theory is a symmetric monoidal functor*

$$Z : Cob^{\text{conf}} \rightarrow \text{Vec}.$$

One reason this is still preliminary is that conformal field theories are in fact projective functors, meaning that if Σ is a cobordism between two 1-manifolds S_1 and S_2 , then the linear map $Z(\Sigma) : Z(S_1) \rightarrow Z(S_2)$ is only well defined up to a positive scalar. This can be equivalently formulated by introducing a certain central extension $\widetilde{Cob}^{\text{conf}}$ of Cob^{conf} , and declaring a conformal field theory to be a symmetric monoidal functor $\widetilde{Cob}^{\text{conf}} \rightarrow \text{Vec}$.

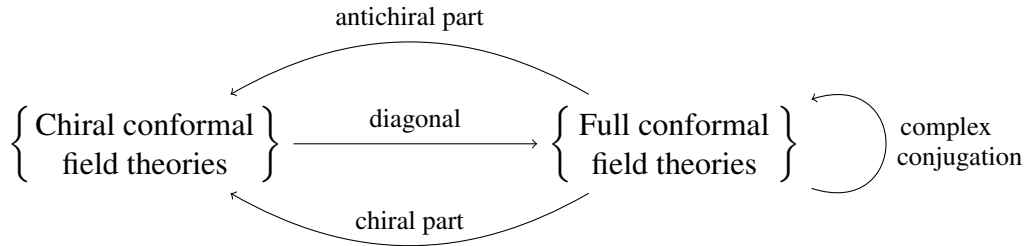
Confusingly, there are two things which go by the name “conformal field theory” (CFT), and which are quite distinct: chiral conformal field theories and full conformal field theories. The above definition-sketch is that of a full conformal field theory. Unlike full CFTs, which are projective functors $Cob^{\text{conf}} \rightarrow \text{Vec}$, chiral CFTs are (non-projective) functors from the same cobordism category to the 2-category of *concrete linear categories*:

Definition sketch: *A chiral conformal field theory is a symmetric monoidal functor*

$$Cob^{\text{conf}} \rightarrow \text{LinCat}^{\text{concrete}}.$$

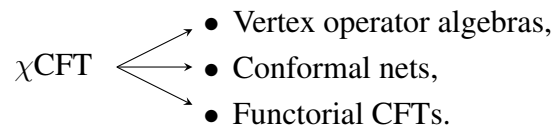
Here, a concrete linear category is a pair (\mathcal{C}, U) consisting of a linear category \mathcal{C} together with a faithful functor U from \mathcal{C} to the category of vector spaces [Think: \mathcal{C} is the category of representations of some group or some algebra, and U the functor which sends a representation to its underlying vector space], and a *concrete functor* $(\mathcal{C}_1, U_1) \rightarrow (\mathcal{C}_2, U_2)$ is a pair consisting of a linear functor $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ and a linear natural transformation $Z : U_1 \rightarrow U_2 \circ F$.

Chiral and full CFTs are furthermore related by constructions¹



Note: Graeme Segal uses the term *weak CFT* for what we here call a ‘functorial chiral CFT’. There is in a general notion, which we call *general CFT* for lack of better name, which contains both chiral and full CFTs as special cases. The Venn diagram on page 2 indicates how all these notions fit together. (The classes of chiral, and antichiral CFTs are disjoint, with the exception of the trivial CFT which is omitted from the diagram.) Very loosely speaking, a general CFT is *chiral* if ‘stuff depends holomorphically on the Riemann surfaces’, *antichiral* if ‘stuff depends anti-holomorphically’, *full* if it’s single-valued on all Riemann surfaces up to a real-valued Weyl anomaly, and *anomalous full* when the Weyl anomaly is complex, meaning the left and right central charges are distinct.

The definition-sketches presented above are those of *functorial* full/chiral CFT. These notes are mainly concerned with chiral CFT. There exist three main mathematical formalizations of chiral CFT:



Terminology warning: Whereas the term ‘vertex operator algebra’ (VOA) unambiguously refers to chiral CFTs, there exist variants of the notions of conformal net and of functorial CFT which model the notion of full CFT. In order to avoid any ambiguity, it is therefore preferable to use the terminology ‘chiral conformal net’ and ‘functorial chiral CFT’. (There also exists a variant of the notion of vertex operator algebra which formalizes full CFTs, and which goes by the name ‘full field algebra’.)

The notions of vertex operator algebra, of chiral conformal net, and of functorial chiral CFT are expected/conjectured to be equivalent, provided appropriate qualifiers are

¹From a mathematical point of view, the existence of these constructions should be treated as conjectures.

added. Note that these notions cannot be completely equivalent because:

- *Unitarity* is built into conformal nets, but not into VOAs, nor functorial CFTs.
- *Rationality* is built into functorial CFTs, but not into VOAs, nor conformal nets (rationality is a certain finiteness condition that a chiral CFT might or might not satisfy).
- There exists a certain equivalence between functorial CFTs called *infinitesimal equivalence*. Infinitesimally equivalent functorial CFTs model the same physics and should therefore be treated as ‘the same’.

We propose:

Conjecture 1 (i) *There is a bijection*

$$\text{unitary VOAs} \Leftrightarrow \text{chiral conformal nets.}$$

(ii) *There is a bijection*

$$\text{rational VOAs} \Leftrightarrow \text{functorial chiral CFTs up to infinitesimal equivalence.}$$

(iii) *There is a bijection*

$$\text{rational conformal nets} \Leftrightarrow \text{unitary functorial chiral CFTs.}$$

There exist a couple of constructions in the literature which connect VOAs, chiral conformal nets, and functorial chiral CFTs. But these constructions only work in special cases, and it is fair to say that the above conjecture is wide open.

The formalism of functorial chiral CFTs, is the *least developed* of the above three mathematical formalizations of chiral CFT (for example, the only chiral CFTs which have been constructed so far, or proven to exist as functorial CFTs are free field CFTs). But this is also, in some sense, the most powerful one of the above three formalisms, and we expect that it should be easy (in comparison to other constructions) to construct a VOA or a conformal net from a functorial chiral CFT.

Complex cobordisms

The definition of functorial CFT (for both chiral and full) is based on the notion of *complex cobordisms*, which are Riemann surfaces with boundary equipped with a decomposition of their boundary into a part labelled ‘in’ and a part labelled ‘out’.

Let $\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im}(z) \geq 0\}$ be the complex upper half plane, and let $\overset{\circ}{\mathbb{H}} := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ be its interior.

Definition: *A Riemann surface with boundary is a ringed space² $(\Sigma, \mathcal{O}_\Sigma)$ which is locally isomorphic to $(\mathbb{H}, \mathcal{O}_\mathbb{H})$, where $\mathcal{O}_\mathbb{H}$ is the sheaf on \mathbb{H} given by*

$$\mathcal{O}_\mathbb{H}(U) := \left\{ f : U \rightarrow \mathbb{C} \mid \begin{array}{l} f|_{U \cap \overset{\circ}{\mathbb{H}}} \text{ is holomorphic,} \\ \exists V \subset \mathbb{C} \text{ open and } g \in C^\infty(V) \text{ s.t. } f = g|_U \end{array} \right\}$$

for $U \subset \mathbb{H}$ an open subset.

²All our ringed spaces will in fact be locally ringed spaces.

By a classical result known as Borel’s lemma, for an open subset $U \subset \mathbb{H}$, the condition that a function $f : U \rightarrow \mathbb{C}$ be the restriction a C^∞ function defined on some open $V \subset \mathbb{C}$ is equivalent to f being smooth all the way to the boundary. Here, ‘smooth all the way to the boundary’ is just the usual notion of smoothness, adapted to the case of manifolds with boundary (when writing down the limits which are used to define the derivative of a function, restrict the domain of the limit to just one side if necessary, so as to not fall outside of the manifold).

An equivalent definition of the sheaf $\mathcal{O}_{\mathbb{H}}$ is to declare $\mathcal{O}_{\mathbb{H}}(U)$ to be the set of continuous functions on U which are holomorphic when restricted to $U \cap \mathring{\mathbb{H}}$, and smooth when restricted to $U \cap \partial\mathbb{H}$:

$$\mathcal{O}_{\mathbb{H}}(U) = \left\{ f \in C^0(U, \mathbb{C}) \mid \begin{array}{l} f|_{U \cap \mathring{\mathbb{H}}} \text{ is holomorphic,} \\ f|_{U \cap \partial\mathbb{H}} \text{ is smooth} \end{array} \right\}.$$

The equivalence between the above two definitions of $\mathcal{O}_{\mathbb{H}}$ will be proven below, in Lemma 2. We first state an important theorem:

Theorem.³ (*Riemann mapping theorem for simply connected domains with smooth boundary*) Let $D \subset \mathbb{C}$ be a compact simply connected domain with smooth boundary. Then there exists an isomorphism

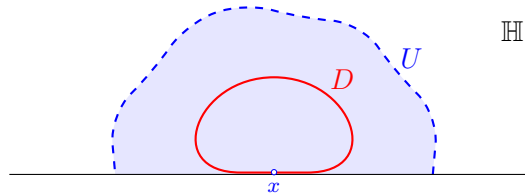
$$D \cong \mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$$

which is holomorphic in the interior, and smooth all the way to the boundary. Moreover, that isomorphism is unique up to an element of

$$\text{Aut}(\mathbb{D}) = PSU(1, 1) = \left\{ z \mapsto \frac{az+b}{bz+\bar{a}} : |a|^2 - |b|^2 = 1 \right\}.$$

Lemma 2 Let $U \subset \mathbb{H}$ be an open subset, and let $f : U \rightarrow \mathbb{C}$ be a continuous function such that $f|_{U \cap \mathring{\mathbb{H}}}$ is holomorphic and $f|_{U \cap \partial\mathbb{H}}$ is smooth. Then f is smooth all the way to the boundary.

Proof. Let $x \in U \cap \partial\mathbb{H}$ be a point, and let $D \subset U$ be a neighbourhood of x which is compact, simply connected, and with smooth boundary:



We also assume that $[a, b] := \partial D \cap \partial\mathbb{H}$ is a connected interval. Let $\psi : \mathbb{D} \rightarrow D$ be a uniformizing map, which is smooth by the previous theorem. The function $g := \psi^* f$ is

³S. Bell. *Mapping problems in complex analysis and the ∂ -problem*. Bull. Amer. Math. Soc., 22(2):233–259, 1990.

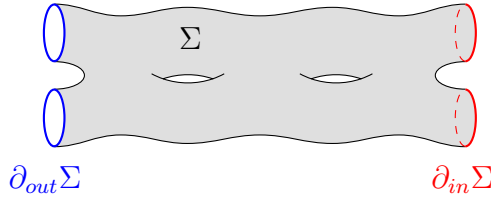
continuous, holomorphic in the interior of \mathbb{D} , and smooth on the boundary \triangleleft . The Taylor coefficients a_n of $g(z) = \sum a_n z^n$ satisfy

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_{|z|=r} g(z) z^{-(n+1)} dz && \text{for any } r < 1 \\ &= \frac{1}{2\pi i} \oint_{|z|=1} g(z) z^{-(n+1)} dz && \text{since } g \text{ is continuous} \\ &= \frac{\pm 1}{n(n-1)\dots(n-k+1)} \cdot \frac{1}{2\pi i} \oint_{|z|=1} (g|_{\partial\mathbb{D}})^{(k)}(z) \cdot z^{-(n-k+1)} dz && \forall k \leq n. \end{aligned}$$

It follows that $|a_n| \leq \frac{1}{n(n-1)\dots(n-k+1)} \cdot \|(g|_{\partial\mathbb{D}})^{(k)}\|_\infty$. The coefficients a_n decay faster than any power of n , so $g(z)$ is smooth all the way to the boundary. The same therefore holds for f around x .

\triangleleft There is a gap in the above proof, because we don't know that $f|_{\partial D}$ is smooth at the two boundary points of the interval $[a, b] = \partial D \cap \partial\mathbb{H}$. But we can fix that gap. Let $h \in \mathcal{O}_{\mathbb{H}}(\mathbb{H})$ be an auxiliary function with zeros of infinite order at a and b (for example, $h(z) = e^{-\frac{1+i}{\sqrt{z-a}} - \frac{1+i}{\sqrt{z-b}}}$). We run the same argument with $\tilde{f} := hf$ (the function $\tilde{f}|_{\partial D}$ is now smooth at a and b because it vanishes to infinite order), deduce that \tilde{f} is smooth all the way to the boundary, and divide by h to get the result. \square

Definition: A *complex cobordism* is a Riemann surface with boundary equipped with a decomposition of its boundary into a disjoint union $\partial\Sigma = \partial_{in}\Sigma \sqcup \partial_{out}\Sigma$:



We equip $\partial_{out}\Sigma$ with the orientation induced by that of Σ , and we equip $\partial_{in}\Sigma$ with the opposite of that orientation.

Given oriented 1-manifolds S_1 and S_2 , a *complex cobordism from S_1 to S_2* is a triple $(\Sigma, \varphi_{in}, \varphi_{out})$ where Σ is a complex cobordism as defined above, and $\varphi_{in} : S_1 \rightarrow \partial_{in}\Sigma$ and $\varphi_{out} : S_2 \rightarrow \partial_{out}\Sigma$ are diffeomorphisms.

One shortcoming with the above notion of complex cobordism is that it does not allow for identity cobordisms. This means that Cob^{conf} is a *non-unital* category.

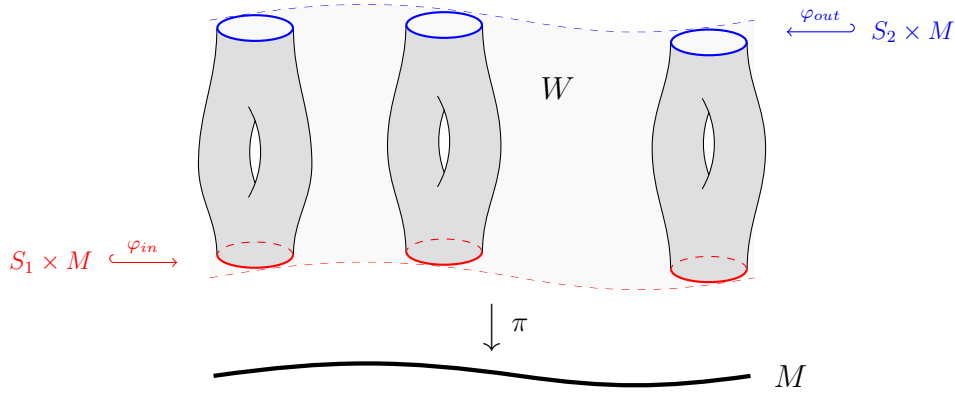
For the purposes of chiral CFT, it is important to understand not just individual complex cobordisms but also the *moduli spaces* thereof. Let S_1 and S_2 be closed 1-manifolds. The space⁴ $Cob^{\text{conf}}(S_1, S_2)$ that parametrizes complex cobordisms from S_1 to S_2 is an infinite dimensional smooth manifold, which is furthermore equipped with a complex structure. Indeed, the moduli space $Cob^{\text{conf}}(S_1, S_2)$ can be written as an open subset

⁴The part of the moduli space that contains closed components is in fact not a space but a stack. If one insists that all components have non-empty boundary, then the stackiness goes away and the moduli space is just a space.

of some infinite dimensional complex manifold⁵ (see p. 15 for a discussion in the special case of annuli). We will take the point of view that, in order to describe the complex structure (/ the smooth structure / the topology) on $Cob^{conf}(S_1, S_2)$, it is enough to understand what it means for a map

$$M \rightarrow Cob^{conf}(S_1, S_2) \quad (1)$$

from a finite dimensional complex manifold (/ smooth manifold / topological space) to be holomorphic (/ smooth / continuous). By definition, a holomorphic map from an n -dimensional complex manifold M into the moduli space is the same thing as a diagram



where W is an $(n + 1)$ -dimensional complex manifold with boundary (a space locally isomorphic to $f^{-1}(\mathbb{R}_{\geq 0})$, for $f : \mathbb{C}^{n+1} \rightarrow \mathbb{R}$, $f^{-1}(0)$ smooth), π is a proper holomorphic submersion, and $\varphi = \varphi_{in} \sqcup \varphi_{out} : (S_1 \sqcup S_2) \times M \rightarrow W$ is an isomorphism onto ∂W satisfying $\pi \circ \varphi = pr_2$. Moreover, and very importantly: *for every point x of S_1 or S_2 , the section $\varphi|_{\{x\} \times M} : M \rightarrow W$ of π should be holomorphic.*

If we only require M to be a smooth manifold and only require W to be equipped with a complex structure along the fibers of the projection π , then we obtain the notion of a smooth map (1). And if M is a topological space, and the fibers of π are equipped with continuously varying complex structures, then that's what it means for a map (1) to be continuous.

Conformal welding

Defining complex cobordisms via ringed spaces allows for a particularly elegant description of the operation of composition of cobordisms, as a pushout in the category of ringed spaces.

Theorem. (*Conformal welding*) Let Σ_1 and Σ_2 be complex cobordisms, and let $\phi : \partial_{in}\Sigma_1 \rightarrow \partial_{out}\Sigma_2$ be an orientation preserving diffeomorphism. Then $(\Sigma_1 \cup_{\phi} \Sigma_2, \mathcal{O}_{\Sigma_1 \cup_{\phi} \Sigma_2})$

⁵Again, this is only true in the absence of closed components. In the presence of closed components, the statement remains true provided one replaces the word “manifold” by “orbifold”.

with

$$\begin{aligned} \partial_{in}(\Sigma_1 \cup_\phi \Sigma_2) &= \partial_{in}\Sigma_2, & \partial_{out}(\Sigma_1 \cup_\phi \Sigma_2) &= \partial_{out}\Sigma_1, \\ \mathcal{O}_{\Sigma_1 \cup_\phi \Sigma_2}(U) &:= \{f : U \rightarrow \mathbb{C} \mid f|_{U \cap \Sigma_i} \in \mathcal{O}_{\Sigma_i}(U \cap \Sigma_i), \text{ for } i = 1, 2\} \end{aligned} \quad (2)$$

is a complex cobordism. (I.e., the ringed space $(\Sigma_1 \cup_\phi \Sigma_2, \mathcal{O}_{\Sigma_1 \cup_\phi \Sigma_2})$ is isomorphic to an open subset of $(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$ in a neighbourhood of the image of $\partial_{in}\Sigma_1$.)

Moreover, the image of $\partial_{in}\Sigma_1$ inside $\Sigma_1 \cup_\phi \Sigma_2$ (equivalently, the image of $\partial_{out}\Sigma_2$) is a smooth curve.

Proof. The problem being local, it is enough to treat the case

$$\begin{aligned} \Sigma_1 = \mathbb{D}_- &:= \{z \in \mathbb{C} \cup \{\infty\} : |z| \geq 1\}, & \Sigma_2 = \mathbb{D}_+ &:= \{z \in \mathbb{C} : |z| \leq 1\}, \\ \phi : S^1 = \partial\mathbb{D}_- &\xrightarrow{\cong} \partial\mathbb{D}_+ = S^1. \end{aligned}$$

We will construct a homeomorphism $f : \mathbb{D}_- \cup_\phi \mathbb{D}_+ \rightarrow \mathbb{C}\mathbb{P}^1$ that is holomorphic on the interiors $\mathring{\mathbb{D}}_-$ and $\mathring{\mathbb{D}}_+$, and smooth on \mathbb{D}_- and \mathbb{D}_+ all the way to the boundary.

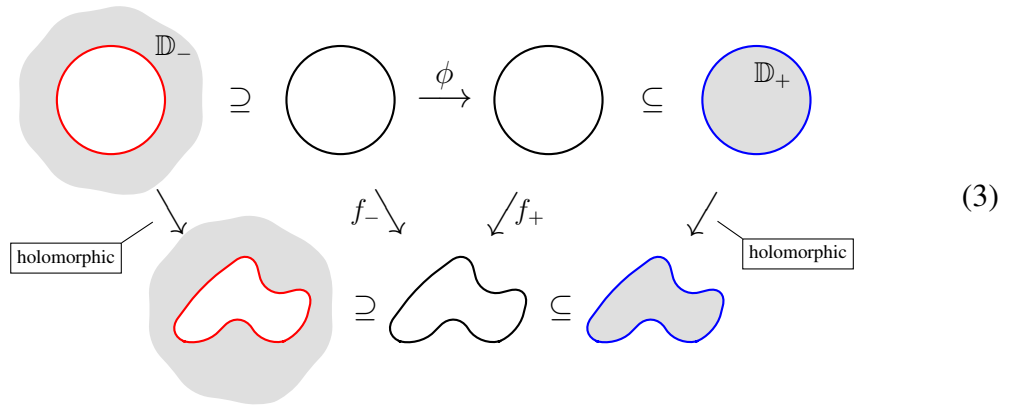
The space of such isomorphisms is three dimensional (corresponding to the fact that $\text{Aut}(\mathbb{C}\mathbb{P}^1) = PSL(2, \mathbb{C})$). In order to force the solution to be unique, we will also insist that $f : \infty \in \mathbb{D}_- \mapsto \infty \in \mathbb{C}\mathbb{P}^1$, with first derivative $f'(\infty) = 1$, and second derivative $f''(\infty) = 0$.

Remark. When $\phi : S^1 \rightarrow S^1$ is real analytic, the problem is easy. Extend ϕ to a biholomorphic map $\tilde{\phi} : U \rightarrow V$ from an open neighbourhood U of $S^1 = \partial\mathbb{D}_-$ to an open neighbourhood V of $S^1 = \partial\mathbb{D}_+$. We may then rewrite $\mathbb{D}_- \cup_\phi \mathbb{D}_+$ as $(\mathbb{D}_- \cup U) \cup_{\tilde{\phi}} (\mathbb{D}_+ \cup V)$, which is now obviously a complex manifold. The latter is then isomorphic to $\mathbb{C}\mathbb{P}^1$ by the Riemann uniformization theorem. \diamond

Letting $f_\pm := f|_{\mathbb{D}_\pm}$, we may rephrase the problem as that of finding a pair of functions $f_+, f_- \in \mathcal{C}^\infty(S^1)$ of the form

$$f_+(z) = a_0 + a_1z + a_2z^2 + \dots \quad f_-(z) = z + b_1z^{-1} + b_2z^{-2} + \dots$$

that make the following diagram commute:



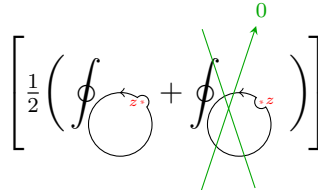
Let $\mathcal{H} = \{ \sum_{n \geq 0} a_n z^n \} \subset \mathcal{C}^\infty(S^1)$ be the Hardy space, and $\mathcal{H}^\perp = \{ \sum_{n < 0} b_n z^n \}$ its orthogonal complement.

Claim. For $f \in \mathcal{C}^\infty(S^1)$

$$Tf := \frac{1}{2\pi i} \text{P.V.} \int_{w \in S^1} \frac{f(w)}{w-z} dw = \begin{cases} \frac{1}{2}f & \text{if } f \in \mathcal{H} \\ -\frac{1}{2}f & \text{if } f \in \mathcal{H}^\perp \end{cases}$$


where $\text{P.V.} \int_{w \in S^1} := \lim_{\varepsilon \rightarrow 0} \int_{w \in S^1, |w-z| \geq \varepsilon}$, and ‘P.V.’ stand for ‘principal value’.

Proof of claim: We prove the claim for f real analytic; the general case follows by continuity. If $f \in \mathcal{H}$, then f extends to a holomorphic function on a neighborhood of \mathbb{D}_+ . We then have

$$\begin{aligned} \frac{1}{2\pi i} \text{P.V.} \int_{S^1} \frac{f(w)dw}{w-z} &= \frac{1}{2\pi i} \left[\frac{1}{2} \left(\oint_{\text{clockwise}} + \oint_{\text{counterclockwise}} \right) \right] \frac{f(w)dw}{w-z} \\ &= \frac{1}{4\pi i} \cdot 2\pi i \text{Res}_z \frac{f(w)}{w-z} = \frac{1}{2}f(z). \end{aligned}$$


If $f \in \mathcal{H}^\perp$, it extends to a holomorphic function on a neighborhood of \mathbb{D}_- and vanishes at infinity. The 1-form $\frac{f(w)dw}{w-z}$ is regular at infinity (dw has a double pole at infinity while $\frac{f(w)}{w-z}$ has at least a double zero) so, by the same argument as above with \mathbb{D}_- instead of \mathbb{D}_+ , we get

$$\begin{aligned} \frac{1}{2\pi i} \text{P.V.} \int_{S^1} \frac{f(w)dw}{w-z} &= \frac{1}{2\pi i} \left[\frac{1}{2} \left(\oint_{\text{counterclockwise}} + \oint_{\text{clockwise}} \right) \right] \frac{f(w)dw}{w-z} \\ &= \frac{1}{4\pi i} \cdot (-2\pi i \text{Res}_z \frac{f(w)}{w-z}) = -\frac{1}{2}f(z). \end{aligned}$$

The contour  runs clockwise around \mathbb{D}_-

◇

Going back to conformal welding, let $V_\phi : \mathcal{C}^\infty(S^1) \rightarrow \mathcal{C}^\infty(S^1)$ be the operator of precomposition by ϕ . We may rewrite the conditions in (3) as

$$V_\phi f_+ = f_- \quad T f_+ = \frac{1}{2}f_+ \quad T f_- = z - \frac{1}{2}f_- . \quad (4)$$

Any solution of these equations must satisfy $V_\phi T V_\phi^{-1} f_- = V_\phi T f_+ = \frac{1}{2}V_\phi f_+ = \frac{1}{2}f_-$. In particular, $(T - V_\phi T V_\phi^{-1})f_- = z - f_-$. Hence

$$\left[1 - (T - V_\phi T V_\phi^{-1}) \right] (f_-) = z. \quad (5)$$

We may perform a variable substitution in the singular integral to get the following general formula:

$$\begin{aligned} V_\phi T V_\phi^{-1} f(z) &= T V_\phi^{-1} f(\phi(z)) \\ &= \frac{1}{2\pi i} \text{P.V.} \int \frac{V_\phi^{-1} f(w)}{w - \phi(z)} dw \\ &= \frac{1}{2\pi i} \text{P.V.} \int \frac{f(\phi^{-1}(w))}{w - \phi(z)} dw = \frac{1}{2\pi i} \text{P.V.} \int \frac{f(u)}{\phi(u) - \phi(z)} \phi'(u) du. \end{aligned}$$

The integral kernel of $T - V_\phi T V_\phi^{-1}$ is therefore given by

$$K(z, w) = \frac{1}{w - z} - \frac{\phi'(w)}{\phi(w) - \phi(z)}.$$

The singularities of $\frac{1}{w-z}$ and $\frac{\phi'(w)}{\phi(w)-\phi(z)}$ exactly cancel out. So, despite its appearance, $K(z, w)$ is in fact a smooth function, which makes $T - V_\phi T V_\phi^{-1}$ a *smoothing operator*⁶! For ϕ close enough to the identity, the operator $1 - (T - V_\phi T V_\phi^{-1})$ is therefore invertible, and we may solve Equation (5) by simply writing

$$f_- := \left[1 - (T - V_\phi T V_\phi^{-1}) \right]^{-1} (z).$$

It remains to check that f_- as above together with $f_+ := V_\phi^{-1} f_-$ form a solution of the conformal welding problem (4). This is obvious when ϕ is analytic (because then (3) admits a unique solution, by the Riemann uniformization theorem), and follows by continuity for arbitrary smooth ϕ . This finishes the proof of the theorem for ϕ close to the identity.

When ϕ is not necessarily close to the identity, write it as a composite $\phi = \phi'' \circ \phi'$ of an analytic diffeomorphism ϕ' followed by a diffeomorphism ϕ'' that is close to the identity. The surface $\mathbb{D}_- \cup_\phi \mathbb{D}_+$ can be obtained from $\mathbb{D}_- \cup_{\phi'} \mathbb{D}_+ \cong \mathbb{C}\mathbb{P}^1$ by cutting it along an analytically embedded circle $S \subset \mathbb{C}\mathbb{P}^1$ (the image of $\partial\mathbb{D}_-$) and regluing the two halves using ϕ'' . Locally around S , this is equivalent to the problem of constructing $\mathbb{D}_- \cup_{\phi''} \mathbb{D}_+$, which we have just solved above.

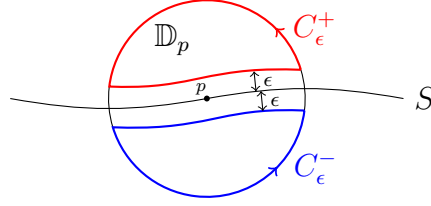
It remains to identify the sheaf (2) with the sheaf of holomorphic functions on the welded surface. This is the content of the next lemma. \square

Lemma 3 *Let $S \subset \mathbb{C}$ be a smooth curve, and $U \subset \mathbb{C}$ an open. If $f : U \rightarrow \mathbb{C}$ is a continuous function whose restriction to $U \setminus S$ is analytic, then f is analytic.*

Proof: We work in the neighbourhood of a point $p \in S$. Let $\mathbb{D}_p \subset U$ be a disc centred

⁶Such operators act on L^2 functions as compact operators. Indeed, they act as compact operators on any Sobolev space of functions.

at p . Consider the following contours:



By Cauchy's residue formula, we have

$$f(z) = \frac{1}{2\pi i} \oint_{C_\epsilon^+ \cup C_\epsilon^-} \frac{f(w)}{w-z} dw$$

for every $z \in \mathring{\mathbb{D}}_p$ whose distance from S is at least ϵ . Taking the limit as $\epsilon \rightarrow 0$ and noting that the two contributions along S cancel each other, we get

$$f(z) = \frac{1}{2\pi i} \oint_{\partial \mathring{\mathbb{D}}_p} \frac{f(w)}{w-z} dw$$

for every $z \in \mathring{\mathbb{D}}_p \setminus S$. By continuity, the last formula also holds for $z \in S$. To finish the argument, we note that the right hand side is an analytic function of z , because each of the functions $z \mapsto \frac{f(w)}{w-z}$ is analytic. \square

The conformal welding map

$$-\cup_{S_2} - : Cob^{\text{conf}}(S_1, S_2) \times Cob^{\text{conf}}(S_2, S_3) \rightarrow Cob^{\text{conf}}(S_1, S_3)$$

is **smooth**, meaning that a pair of smooth maps

$$M \rightarrow Cob^{\text{conf}}(S_1, S_2) \quad \text{and} \quad M \rightarrow Cob^{\text{conf}}(S_2, S_3)$$

composes to a smooth map

$$M \rightarrow Cob^{\text{conf}}(S_1, S_3).$$

The simple reason is that all the ingredients used in this section depend smoothly on parameters. (Specifically, the ingredients are: the Riemann mapping theorem for domains with smooth boundary⁷, and taking the inverse of an operator of the form identity plus smoothing operator.)

The conformal welding map is furthermore **holomorphic**, meaning that it maps holomorphic families to holomorphic families. This will be proven later, in Proposition 5 (on p. 16) in the special case of annuli. [Note that one may not argue as above, as the Riemann mapping theorem does *not* depend holomorphically on parameters!]

Now observe that the question of whether holomorphic families of complex cobordisms can be welded into a holomorphic family is a local one. Any complex cobordism is locally isomorphic to a disc, and ditto for families thereof. The special case of discs is therefore sufficient to prove the general result.

⁷S. Bell, *The Cauchy Transform, Potential Theory and Conformal Mapping*, Theorem 28.1.

The semigroup of annuli

The genus zero part of $Cob^{\text{conf}}(S^1, S^1)$ is called the *semigroup of annuli*, and denoted $\text{Ann}(S^1)$. It was introduced by G. Segal and Y. Neretin, and plays a special role in the definition of functorial chiral CFT. One of their great insights is that this semigroup, even though it is not a group, is a kind of complexification of the group $\text{Diff}(S^1)$ of orientation preserving diffeomorphisms of S^1 .

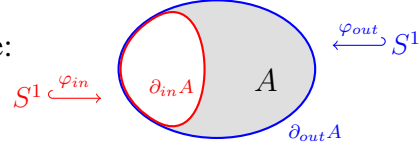
Of course, for $\text{Ann}(S^1)$ to qualify as a complexification of sorts of $\text{Diff}(S^1)$, there needs to at least be a map $\text{Diff}(S^1) \hookrightarrow \text{Ann}(S^1)$, which currently there isn't since $\text{Ann}(S^1)$ doesn't even have an identity element! To fix that problem, we construct a version of the semigroup of annuli which contains annuli that are allowed to be 'thin', meaning that the incoming and outgoing boundaries are allowed to touch each other.

Given an embedding $\gamma : S^1 \hookrightarrow \mathbb{C}$, let $\text{Int}(\gamma) \subset \mathbb{C}$ denote the open disc bounded by the image of γ . An *embedded partially thin annulus* is a subset of \mathbb{C} of the form

$$A = \overline{\text{Int}(\gamma_{\text{out}})} \setminus \text{Int}(\gamma_{\text{in}}) \subset \mathbb{C},$$

where $\gamma_{\text{in}}, \gamma_{\text{out}} : S^1 \hookrightarrow \mathbb{C}$ are embeddings with winding number 1. Let us write $\varphi_{\text{in/out}} : S^1 \rightarrow A$ for the maps $\gamma_{\text{in/out}}$ when thought of as maps $S^1 \rightarrow A$.

Here's what a partially thin annulus can look like:



We equip A with the sheaf \mathcal{O}_A of functions that are continuous on A , holomorphic on \mathring{A} , and smooth on $\partial_{\text{in}}A$ and on $\partial_{\text{out}}A$. Namely, for an open $U \subset A$, we set

$$\mathcal{O}_A(U) := \left\{ \begin{array}{l} \text{continuous functions } U \rightarrow \mathbb{C} \text{ that are holomorphic on } \\ \mathring{A} \cap U, \text{ smooth on } \partial_{\text{in}}A \cap U, \text{ and smooth on } \partial_{\text{out}}A \cap U \end{array} \right\} \quad (6)$$

An *abstract partially thin annulus* is a triple $(A, \varphi_{\text{in}}, \varphi_{\text{out}})$ where A is a ringed space isomorphic to an embedded partially thin annulus, and $\varphi_{\text{in/out}} : S^1 \hookrightarrow A$ are the two boundary inclusions. A partially thin annulus A is called *thick* if $\partial_{\text{in}}A \cap \partial_{\text{out}}A = \emptyset$ (in which case it is diffeomorphic to $S^1 \times [0, 1]$).

Definition: The **completed semigroup of annuli** $\overline{\text{Ann}}(S^1)$ is the set of isomorphism classes of abstract partially thin annuli. It contains the subset $\text{Ann}(S^1) \subset \overline{\text{Ann}}(S^1)$ of thick annuli as a subsemigroup. The composition operation

$$\cup : \overline{\text{Ann}}(S^1) \times \overline{\text{Ann}}(S^1) \rightarrow \overline{\text{Ann}}(S^1)$$

is defined as in (2) as a pushout in the category of ringed spaces. The fact that the result of this pushout is again an element of $\overline{\text{Ann}}(S^1)$ will be proven in Proposition 4.

More generally, if S is a *circle* (by which we mean an oriented manifold diffeomorphic to S^1), we can define $\text{Ann}(S)$ and $\overline{\text{Ann}}(S)$ in terms of annuli with boundary parametrised by S .

The group $\text{Diff}(S^1)$ admits an obvious embedding $\text{Diff}(S^1) \hookrightarrow \overline{\text{Ann}}(S^1)$ which sends a diffeomorphism ψ to the completely thin annulus $(A=S^1, \varphi_{in}=\psi, \varphi_{out}=\text{id})$. So we get two inclusions:

$$\text{Diff}(S^1) \subset \overline{\text{Ann}}(S^1) \supset \text{Ann}(S^1)$$

We will later check that the induced map at the level of tangent spaces $T_1\text{Diff}(S^1) \hookrightarrow T_1\overline{\text{Ann}}(S^1)$ is the inclusion of a vector space into its complexification. This justifies the claim that:

$$\overline{\text{Ann}}(S^1) \text{ is a complexification of } \text{Diff}(S^1)$$

There is of course a problem, because $\overline{\text{Ann}}(S^1)$ is not a manifold. It is some kind of infinite dimensional manifold with boundary (or rather with corners), and its identity element sits right on the boundary (or rather at some corner of infinite codimension). So it is not clear a priori that the tangent space $T_1\overline{\text{Ann}}(S^1)$ is a well defined concept. But things are not too bad. Letting P be the set of smooth paths $\gamma : [0, \infty) \rightarrow \overline{\text{Ann}}(S^1)$ with $\gamma(0) = \mathbf{1}$, the tangent space $T_1\overline{\text{Ann}}(S^1)$ can be defined as the quotient of the free vector space $\mathbb{R}[P]$ by the relation which declares $a\gamma_1 + b\gamma_2 = \gamma_3$ if there exists a smooth map $\Gamma : [0, \infty)^2 \rightarrow \overline{\text{Ann}}(S^1)$ satisfying $\gamma_1(t) = \Gamma(t, 0)$, $\gamma_2(t) = \Gamma(0, t)$, $\gamma_3(t) = \Gamma(at, bt)$. For manifolds, this procedure recovers the usual notion of tangent space.

The Lie algebra of $\text{Diff}(S^1)$ is the Lie algebra $\mathfrak{X}(S^1) := \{f(z)\frac{\partial}{\partial z} \mid \frac{f(z)}{z} \in i\mathbb{R}\}$ of vector fields on S^1 . We recall the well-known formula for the Lie bracket of vector fields:

$$\left[f(z)\frac{\partial}{\partial z}, g(z)\frac{\partial}{\partial z} \right]_{Lie} = (fg' - gf')\frac{\partial}{\partial z} \quad (7)$$

Remark. *It is a great annoyance in differential geometry that, for a manifold M , the Lie algebra of $\text{Diff}(M)$ is not $\mathfrak{X}(M)$ but instead the Lie algebra of vector fields equipped with the opposite of the usual Lie bracket of vector fields.*

In order to avoid this annoyance and the various minus signs that it creates, we will always be endowing $\mathfrak{X}(S^1)$ with the following Lie bracket:

$$\left[f(z)\frac{\partial}{\partial z}, g(z)\frac{\partial}{\partial z} \right] := \left[f(z)\frac{\partial}{\partial z}, g(z)\frac{\partial}{\partial z} \right]_{Lie}^{op} = (gf' - fg')\frac{\partial}{\partial z}. \quad (8)$$

The Lie algebra $\mathfrak{X}_{\mathbb{C}}(S^1)$ of complexified vector fields on S^1 admits a topological basis given by the vector fields⁸

$$\ell_n := z^{n+1}\frac{\partial}{\partial z}.$$

These generate an algebra known as the *Witt algebra*:

$$[\ell_m, \ell_n] = (m - n)\ell_{m+n}.$$

Remark. *When acting on the sectors of a chiral CFT, the ℓ_n with $n < 0$ will be acting as creation operators, whereas ℓ_n with $n > 0$ will be acting as annihilation operators. The operator associated to ℓ_0 will have an interpretation as “the energy”, and will always have positive spectrum.*

⁸Had we been working with the Lie bracket (7), we would have written $\ell_n = -z^{n+1}\frac{\partial}{\partial z}$.

In view of the claim that $\overline{\text{Ann}}(S^1)$ is a complexification of $\text{Diff}(S^1)$, it is perhaps reasonable to ask whether there exists a complex Lie **group** which is a complexification of $\text{Diff}(S^1)$? Equivalently: is there a Lie group that integrates the Lie algebra of $\mathfrak{X}_{\mathbb{C}}(S^1)$ of complexified vector fields on S^1 ? We will see that the answer is no. Indeed, the subalgebra $\text{Span}\{\ell_{-n}, \ell_0, \ell_n\}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ via the map

$$\frac{1}{n}\ell_{-n} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \frac{1}{n}\ell_0 \mapsto \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \frac{1}{n}\ell_n \mapsto \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \quad (9)$$

If G was a Lie group that integrates $\mathfrak{X}_{\mathbb{C}}(S^1)$, then the above map (9) would integrate to a homomorphism $SL(2, \mathbb{C}) \rightarrow G$ (since $SL(2, \mathbb{C})$ is simply connected). But the relation $\exp\left(4\pi i \cdot \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = 1$ holds in $SL(2, \mathbb{C})$. Therefore, for every n , the relation $\exp\left(4\pi i \cdot \frac{1}{n} \ell_0\right) = 1$ would have to hold in G . Clearly impossible. ζ

We now address the question of the well-definedness of composition in $\overline{\text{Ann}}(S^1)$:

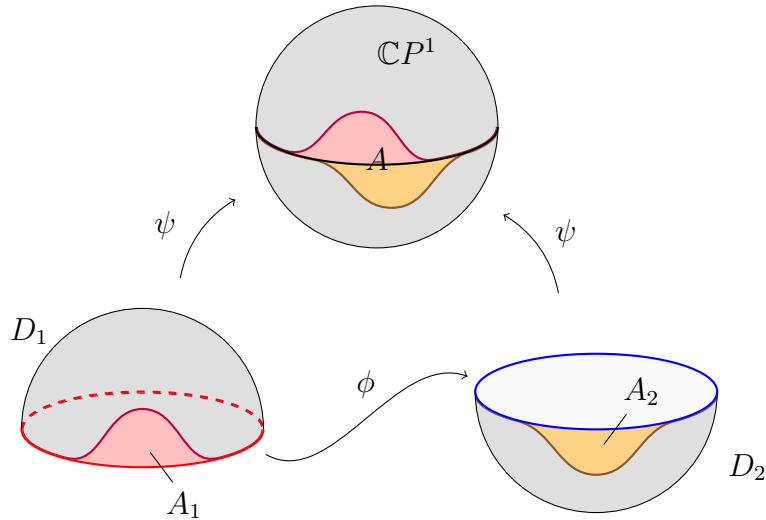
Proposition 4 (*Conformal welding of partially thin annuli*) Let $A_1, A_2 \in \overline{\text{Ann}}(S^1)$, and let $\phi : \partial_{in}A_1 \xrightarrow{\cong} \partial_{out}A_2$ be the diffeomorphism induced by the boundary parametrisations. Then the topological space $A_1 \cup_{\phi} A_2$, along with its boundary parametrisations $\varphi_{in/out} : S^1 \rightarrow A_1 \cup_{\phi} A_2$ inherited from those of A_1 and A_2 , and its structure sheaf $\mathcal{O}_{A_1 \cup_{\phi} A_2}$

$$\mathcal{O}_{A_1 \cup_{\phi} A_2}(U) := \left\{ f : U \rightarrow \mathbb{C} \mid f|_{U \cap A_i} \in \mathcal{O}_{A_i}(U \cap A_i), \text{ for } i = 1, 2 \right\} \quad (10)$$

is again an element of $\overline{\text{Ann}}(S^1)$.

Proof. Write A_1 as $D_1 \setminus \dot{D}'_1$ for some discs $D'_1 \subset D_1 \subset \mathbb{C}$. Similarly, write $A_2 = D_2 \setminus \dot{D}'_2$ for $D'_2 \subset D_2 \subset \mathbb{C}P^1$, where we now insist that $\infty \in D'_2$. By conformal welding, we may pick an isomorphism $\psi : D_1 \cup_{\phi} D_2 \rightarrow \mathbb{C}P^1$, uniquely specified by requiring that $\psi(\infty) = \infty$, and that $\psi(z) = z + o(1)$ around $\infty \in D_2$. The image of $\partial_{in}A_1$ under ψ (equivalently, the image of $\partial_{out}A_2$) is a smoothly embedded curve.

As a consequence of the Riemann mapping theorem for simply connected domains with smooth boundary, the image of $\partial_{out}A_1$ under ψ is also smooth, and so is the image of $\partial_{in}A_2$:



The space $A := A_1 \cup_\phi A_2$ can be therefore identified with the subset of $\mathbb{C}P^1$ that lies between these two curves.

Finally, by Lemma 3, the sheaf (10) agrees with the set of functions on A that are continuous, smooth on the boundaries, and holomorphic in the interior. \square

Conformal welding allows us to write down a very concrete model for $A\overline{\text{nn}}(S^1)$. Let $\mathbb{D}_- := \{z \in \mathbb{C} \cup \{\infty\} : |z| \geq 1\}$ and $\mathbb{D}_+ := \{z \in \mathbb{C} : |z| \leq 1\}$. Given a partially thin $A \in A\overline{\text{nn}}(S^1)$, there is a unique biholomorphic map

$$\psi : \mathbb{D}_- \cup A \cup \mathbb{D}_+ \xrightarrow{\cong} \mathbb{C}P^1$$

sending $\infty \in \mathbb{D}_-$ to $\infty \in \mathbb{C}P^1$, with $\psi'(\infty) = 1$ and $\psi''(\infty) = 0$. Letting $\psi_\pm = \pi|_{\mathbb{D}_\pm}$, this identifies $A\overline{\text{nn}}(S^1)$ with the space of pairs of embeddings

$$A\overline{\text{nn}}(S^1) \cong \left\{ \begin{array}{l} \psi_- : \mathbb{D}_- \hookrightarrow \mathbb{C}P^1 \\ \psi_+ : \mathbb{D}_+ \hookrightarrow \mathbb{C}P^1 \end{array} \left| \begin{array}{l} \psi_+(z) = a_0 + a_1z + a_2z^2 + \dots \\ \psi_-(z) = z + b_1z^{-1} + b_2z^{-2} + \dots \\ \psi_-(\mathring{\mathbb{D}}_-) \cap \psi_+(\mathring{\mathbb{D}}_+) = \emptyset \end{array} \right. \right\}.$$

If we drop the condition that $\psi_-(\mathring{\mathbb{D}}_-)$ and $\psi_+(\mathring{\mathbb{D}}_+)$ be disjoint, then we obtain a manifold $\{\psi_\pm : \mathbb{D}_\pm \hookrightarrow \mathbb{C}P^1 \mid \psi_-(z) = z + \mathcal{O}(z^{-1})\}$ in which $\text{Ann}(S^1)$ sits as an open subspace.

To see that $A\overline{\text{nn}}(S^1)$ is a complex semigroup, we compute its tangent spaces and show that the map

$$T_{A_1}A\overline{\text{nn}}(S^1) \oplus T_{A_2}A\overline{\text{nn}}(S^1) \rightarrow T_{A_1 \cup A_2}A\overline{\text{nn}}(S^1)$$

induced by semigroup operation is complex linear.

Lemma. *The tangent spaces of the completed semigroup of annuli are given by*

$$T_A A\overline{\text{nn}} = \frac{\mathfrak{X}_{\mathbb{C}}(\partial_{\text{out}}A) \oplus \mathfrak{X}_{\mathbb{C}}(\partial_{\text{in}}A)}{\mathfrak{X}_{\text{hol}}(A)} = \frac{\mathfrak{X}_{\mathbb{C}}(S^1) \oplus \mathfrak{X}_{\mathbb{C}}(S^1)}{\mathfrak{X}_{\text{hol}}(A)}$$

where $\mathfrak{X}_{\text{hol}}(A)$ denotes the set of vector fields on A which are continuous, holomorphic in the interior, and smooth on each boundary component (as in (6)).

Note that for $A = \mathbf{1} \in \text{Ann}(S)$, the lemma gives us

$$T_{\mathbf{1}}A\overline{\text{nn}}(S^1) = \frac{\mathfrak{X}_{\mathbb{C}}(S^1) \oplus \mathfrak{X}_{\mathbb{C}}(S^1)}{\mathfrak{X}_{\mathbb{C}}(S^1)} = \mathfrak{X}_{\mathbb{C}}(S^1),$$

justifying the claim that $A\overline{\text{nn}}(S^1)$ is a complexification of $\text{Diff}(S^1)$.

Remark: The left translation map $\ell_A : T_{\mathbf{1}}A\overline{\text{nn}}(S) \rightarrow T_A A\overline{\text{nn}}(S)$ is *not an isomorphism* (unless A is completely thin). It sends a vector $v \in \mathfrak{X}_{\mathbb{C}}(S^1)$ to the class of $(0, v) \in \mathfrak{X}_{\mathbb{C}}(S^1) \oplus \mathfrak{X}_{\mathbb{C}}(S^1)$.

Proof of lemma. Let $M := \{\psi_\pm : \mathbb{D}_\pm \hookrightarrow \mathbb{C}P^1 \mid \psi_\pm \text{ holomorphic, } \psi_-(z) = z + \mathcal{O}(z^{-1})\}$, let $N := \{\gamma_\pm : S^1 \rightarrow \mathbb{C} \mid \gamma_\pm \text{ have winding number } 1\}$, and let $\text{Emb}(A)$ denote the space of holomorphic embeddings of A into the complex plane (such that $\partial_{\text{in}}(A)$ is the boundary

of the bounded component of $\mathbb{C} \setminus A$). All three spaces are manifolds, and there are two obvious maps $s : M \rightarrow N$ and $s' : \text{Emb}(A) \rightarrow N$ given by restriction to the boundary.

There is also a retraction $r : N \rightarrow \text{Emb}(A)$ of s' constructed as follows. A pair $(\gamma_+, \gamma_-) \in N$ defines a pair of discs with parametrised boundary $D_\pm \subset \mathbb{C}P^1$, which we may use to form the genus zero surface $D_- \cup A \cup D_+$. The uniformizing map $D_- \cup A \cup D_+ \rightarrow \mathbb{C}P^1$ that sends $\infty \in D_-$ to $\infty \in \mathbb{C}$ and respects the second order jets restricts to an embedding $\sigma : A \hookrightarrow \mathbb{C}P^1$. We define $r(\gamma_+, \gamma_-)$ to be that embedding.

Note that the composite $r \circ s$ is constant, and that the fiber of r over that point is exactly M . Using the model of $\overline{\text{Ann}}$ given by

$$\overline{\text{Ann}} = \{ \psi_\pm \in M \mid \psi_-(\mathbb{D}_-) \cap \psi_+(\mathbb{D}_+) = \emptyset \},$$

we may identify the tangent space $T_A \overline{\text{Ann}}$ with the usual tangent of the manifold M .

At the level of tangent spaces, the diagram

$$M \xrightarrow{s} N \begin{array}{c} \xleftarrow{s'} \\ \xrightarrow{r} \end{array} \text{Emb}(A)$$

induces a split short exact sequence

$$0 \rightarrow T_A \overline{\text{Ann}} \rightarrow \Gamma(S^1 \sqcup S^1, (\gamma_- \sqcup \gamma_+)^* TCP^1) \rightarrow \Gamma_{\text{hol}}(A, \sigma^* TCP^1) \rightarrow 0.$$

The result follows since $\Gamma(S^1 \sqcup S^1, (\gamma_- \sqcup \gamma_+)^* TCP^1) = \Gamma(S^1, T_{\mathbb{C}} S^1)^{\oplus 2} = \mathfrak{X}_{\mathbb{C}}(S^1) \oplus \mathfrak{X}_{\mathbb{C}}(S^1)$, and $\Gamma_{\text{hol}}(A, \sigma^* TCP^1) = \mathfrak{X}_{\text{hol}}(A)$. \square

Proposition 5 *The map*

$$T_{A_1} \overline{\text{Ann}}(S^1) \oplus T_{A_2} \overline{\text{Ann}}(S^1) \rightarrow T_{A_1 \cup A_2} \overline{\text{Ann}}(S^1)$$

induced by the composition of annuli is complex linear.

Proof. Given $A_1, A_2 \in \overline{\text{Ann}}$, letting $\psi : \mathbb{D}_- \cup A_1 \cup A_2 \cup \mathbb{D}_+ \rightarrow \mathbb{C}P^1$ be the unique isomorphism such that $\psi|_{\mathbb{D}_-} = 1/z + O(z)$, one can write down a map

$$\begin{aligned} \overline{\text{Ann}}(S^1) \times \overline{\text{Ann}}(S^1) &\rightarrow \{ (\gamma_1, \gamma_2, \gamma_3) \mid \gamma_i : S \hookrightarrow \mathbb{C}P^1 \} \\ (A_1, A_2) &\mapsto (\psi|_{\partial_{\text{out}} A_1}, \psi|_{\partial_{\text{in}} A_1} = \psi|_{\partial_{\text{out}} A_2}, \psi|_{\partial_{\text{in}} A_2}). \end{aligned}$$

That map admits a retraction which sends a triple $(\gamma_1, \gamma_2, \gamma_3)$ to the annuli bound by $\gamma_1(S^1)$ and $\gamma_2(S^1)$, and $\gamma_2(S^1)$ and $\gamma_3(S^1)$, respectively.

At the level of tangent spaces, the existence of that retraction means that the vector space

$$\{ ((v_1^{\text{out}}, v_1^{\text{in}}), (v_2^{\text{out}}, v_2^{\text{in}})) \in (\mathfrak{X}_{\mathbb{C}} S^1 \oplus \mathfrak{X}_{\mathbb{C}} S^1) \oplus (\mathfrak{X}_{\mathbb{C}} S^1 \oplus \mathfrak{X}_{\mathbb{C}} S^1) \mid v_1^{\text{in}} = v_2^{\text{out}} \}$$

surjects onto

$$T_{A_1} \overline{\text{Ann}}(S^1) \oplus T_{A_2} \overline{\text{Ann}}(S^1) = \frac{\mathfrak{X}_{\mathbb{C}} S^1 \oplus \mathfrak{X}_{\mathbb{C}} S^1}{\mathfrak{X}_{\text{hol}}(A_1)} \oplus \frac{\mathfrak{X}_{\mathbb{C}} S^1 \oplus \mathfrak{X}_{\mathbb{C}} S^1}{\mathfrak{X}_{\text{hol}}(A_2)}.$$

So we get a commutative diagram

$$\begin{array}{ccc}
\{((v_1^{out}, v_1^{in}), (v_2^{out}, v_2^{in})) \in (\mathfrak{X}_{\mathbb{C}}S^1 \oplus \mathfrak{X}_{\mathbb{C}}S^1) \oplus (\mathfrak{X}_{\mathbb{C}}S^1 \oplus \mathfrak{X}_{\mathbb{C}}S^1) \mid v_1^{in} = v_2^{out}\} & \longrightarrow & \mathfrak{X}_{\mathbb{C}}S^1 \oplus \mathfrak{X}_{\mathbb{C}}S^1 \\
\downarrow & & \downarrow \\
((v_1^{out}, v_1^{in}), (v_2^{out}, v_2^{in})) \mapsto (v_1^{out}, v_2^{in}) & & \\
T_{A_1} \overline{\text{Ann}}(S^1) \oplus T_{A_2} \overline{\text{Ann}}(S^1) & \longrightarrow & T_{A_1 \cup A_2} \overline{\text{Ann}}(S^1)
\end{array}$$

The top horizontal map and the two vertical maps are visibly complex linear. Therefore so is the bottom map. \square

Let us now explain why the Lie algebra⁹ of $\overline{\text{Ann}}(S^1)$ is the Lie algebra of complexified vector fields on S^1 , with the opposite of the usual bracket of vector fields. We already saw

$$T_1 \overline{\text{Ann}}(S^1) \cong \mathfrak{X}_{\mathbb{C}}(S^1) \quad (11)$$

as vector spaces. The inclusion $\text{Diff}(S^1) \hookrightarrow \overline{\text{Ann}}(S^1)$ induces an inclusion of Lie algebras $\mathfrak{X}(S^1) \hookrightarrow T_1 \overline{\text{Ann}}(S^1)$. The latter being a complex Lie algebra, its bracket is completely determined by what happens on the real subspace $\mathfrak{X}(S^1)$. The isomorphism (11) is therefore one of Lie algebras.

Let S be a circle (an oriented manifold diffeomorphic to S^1). By combining Theorem 7 and Proposition 9, we'll see that $\text{Diff}(S)$ admits a universal central extension whose center is canonically isomorphic to $i\mathbb{R} \oplus \mathbb{Z}$:

$$0 \rightarrow i\mathbb{R} \oplus \mathbb{Z} \rightarrow i\mathbb{R} \oplus \mathbb{Z} \text{Diff}(S) \rightarrow \text{Diff}(S) \rightarrow 0.$$

Complexifying, we'll also get a central extension:

$$0 \rightarrow \mathbb{C} \oplus \mathbb{Z} \rightarrow \mathbb{C} \oplus \mathbb{Z} \overline{\text{Ann}}(S) \rightarrow \overline{\text{Ann}}(S) \rightarrow 0$$

which we conjecture is universal.

Given a complex number $c \in \mathbb{C}$ (this will later be the *central charge* of the CFT), we can form the associated central extension

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{C} \oplus \mathbb{Z} & \longrightarrow & \mathbb{C} \oplus \mathbb{Z} \overline{\text{Ann}}(S) & \longrightarrow & \overline{\text{Ann}}(S) \longrightarrow 0 \\
& & \begin{array}{c} (z, n) \\ \downarrow \\ (e^{cz}, n) \end{array} & & \downarrow & & \parallel \\
0 & \longrightarrow & \mathbb{C}^\times \oplus \mathbb{Z} & \longrightarrow & \mathbb{C}^\times \oplus \mathbb{Z} \overline{\text{Ann}}_c(S) & \longrightarrow & \overline{\text{Ann}}(S) \longrightarrow 0
\end{array} \quad (12)$$

where $\mathbb{C}^\times \oplus \mathbb{Z} \overline{\text{Ann}}_c(S)$ is defined as the pushout. Assuming $c \in \mathbb{R}$ (the central charge of a rational CFTs is always a rational number), we can also form the central extension

$$\begin{array}{ccccccc}
0 & \longrightarrow & i\mathbb{R} \oplus \mathbb{Z} & \longrightarrow & i\mathbb{R} \oplus \mathbb{Z} \text{Diff}(S) & \longrightarrow & \text{Diff}(S) \longrightarrow 0 \\
& & \begin{array}{c} (z, n) \\ \downarrow \\ (e^{cz}, n) \end{array} & & \downarrow & & \parallel \\
0 & \longrightarrow & U(1) \oplus \mathbb{Z} & \longrightarrow & U(1) \oplus \mathbb{Z} \text{Diff}_c(S) & \longrightarrow & \text{Diff}(S) \longrightarrow 0
\end{array}$$

which sits as a subgroup $U(1) \oplus \mathbb{Z} \text{Diff}_c(S) \subset \mathbb{C}^\times \oplus \mathbb{Z} \overline{\text{Ann}}_c(S)$.

⁹This is assuming the existence of a general procedure which takes a gadget like $\overline{\text{Ann}}(S^1)$ and assigns a Lie algebra to it. I currently don't know how such a general procedure should go.

Full CFT versus chiral CFT

These notes are mainly about functorial chiral CFT. But functorial full CFT have a more intuitive definition. So we start with those.

A **functorial full CFT** is a symmetric monoidal functor out of a certain central extension $\widetilde{Cob}^{\text{conf}}$ of the complex cobordism category (central extension by \mathbb{R}_+) into the category of topological vector spaces¹⁰. Given two closed oriented 1-manifolds S_1 and S_2 , an element of $\text{Hom}_{\widetilde{Cob}^{\text{conf}}}(S_1, S_2)$ is represented by a triple (Σ, g, r) , where Σ is a complex cobordism from S_1 to S_2 , g is a Riemannian metric on Σ within the given conformal class, and $r \in \mathbb{R}_+$ is a positive real number. Those triples are considered up to the equivalence relation generated by:

(i) if $\phi : \Sigma \rightarrow \Sigma'$ is an isomorphism (compatible with the inclusions of S_1 and S_2), then $(\Sigma, g, r) \equiv (\Sigma', \phi_*(g), r)$.

(ii) if $f : \Sigma \rightarrow \mathbb{R}$ is any function, then we set $(\Sigma, e^f g, r) \equiv (\Sigma, g, r \cdot e^{S(\Sigma, g, e^f g)})$, where

$$S(\Sigma, g, e^f g) = \frac{1}{24\pi} \int_{\Sigma} \left(\frac{1}{4} \|df\|^2 + fK \right) + \frac{1}{24\pi} \int_{\partial\Sigma} f k$$

is the so-called Liouville action functional. Here, K is the scalar curvature of the Riemannian metric g , and k is the geodesic curvature of the boundary of Σ in the metric g .

If $\tilde{\Sigma} = [(\Sigma, g, r)]$ is a morphism in $\widetilde{Cob}^{\text{conf}}$, we write $r' \cdot \tilde{\Sigma} := [(\Sigma, g, r' \cdot r)]$.

Definition: A **functorial full CFT** of central charge $c \in \mathbb{R}$ is a symmetric monoidal functor

$$Z : \widetilde{Cob}^{\text{conf}} \rightarrow \text{TopVec}$$

that satisfies $Z(r \cdot \tilde{\Sigma}) = r^c \cdot Z(\tilde{\Sigma})$.

A **functorial chiral CFT** is a rather different beast. Instead of taking its values in the category of topological vector spaces, it takes its values in concrete linear categories (so one category level higher!) Also, unlike for full CFT, the complex cobordism category requires no central extension.

Recall that a **concrete linear category** is a pair (\mathcal{C}, U) consisting of a linear category \mathcal{C} together with a faithful functor U from \mathcal{C} to the category of topological vector spaces, and that a **concrete functor** $(\mathcal{C}_1, U_1) \rightarrow (\mathcal{C}_2, U_2)$ between concrete linear categories is a pair consisting of a linear functor $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ and a linear natural transformation

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{F} & \mathcal{C}_2 \\ U_1 \searrow & \cong \nearrow & U_2 \\ & \text{TopVec} & \end{array}$$

¹⁰The meaning of ‘topological vector space’ is left intentionally flexible. In particular, the choice of tensor product may depend on the type of topological vector spaces one decides to work with.

Example: If G is a group and $H < G$ a subgroup, then the induction functor $\text{Ind}_H^G : \text{Rep}(H) \rightarrow \text{Rep}(G) : V \mapsto \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ is a concrete functor, as it comes equipped with a distinguished linear map $V \rightarrow \text{Ind}_H^G(V) : v \mapsto 1 \otimes v$.

In addition to the data of a symmetric monoidal functor $\text{Cob}^{\text{conf}} \rightarrow \text{LinCat}^{\text{concrete}}$ a functorial chiral CFT comes with an extra piece of structure which ensures that the functors (just the functors, not the concrete functors!) associated to annuli are trivial. Finally, there is a certain holomorphicity condition.

The word *chiral* in ‘chiral CFT’ refers to that last holomorphicity condition.

Recall from (12) that we have a central extension

$$0 \rightarrow \mathbb{C}^\times \oplus \mathbb{Z} \rightarrow \mathbb{C}^\times \oplus \mathbb{Z} \text{Ann}_c(S) \rightarrow \text{Ann}(S) \rightarrow 0$$

which depends on the central charge c . To avoid the bulky notation, let us write $\tilde{\text{Ann}}_c(S)$ in place of $\mathbb{C}^\times \oplus \mathbb{Z} \text{Ann}_c(S)$.

Definition: A functorial chiral CFT of central charge $c \in \mathbb{C}$ consists of:

- (1a) For every closed 1-manifold S , a linear category $\mathcal{C}(S)$ isomorphic to $\text{Vec}_{\text{fd}}^{\oplus r}$ for some $r \in \mathbb{N}$. The assignment $S \mapsto \mathcal{C}(S)$ is symmetric monoidal with respect to disjoint union of 1-manifolds, and tensor product of linear categories.
- (1b) For every closed 1-manifold S , a faithful functor $U : \mathcal{C}(S) \rightarrow \text{TopVec}$. The assignment $S \mapsto U$ is a symmetric monoidal transformation from $S \mapsto \mathcal{C}(S)$ to the constant 2-functor $S \mapsto \text{TopVec}$.
- (2a) For every complex cobordism Σ , a linear functor $F_\Sigma : \mathcal{C}(\partial_{\text{in}}\Sigma) \rightarrow \mathcal{C}(\partial_{\text{out}}\Sigma)$. These functors are compatible with the operations of disjoint union, identity cobordisms, and composition of cobordisms.
- (2b) For every complex cobordism Σ , and every object $\lambda \in \mathcal{C}(\partial_{\text{in}}\Sigma)$, a linear map $Z_\Sigma : U(\lambda) \rightarrow U(F_\Sigma(\lambda))$. The maps Z_Σ are compatible with the operations of disjoint union, identity cobordisms, and composition of cobordisms.
- (3a) For every $\tilde{A} \in \tilde{\text{Ann}}_c(S)$, a trivialization $T_{\tilde{A}} : F_{\tilde{A}} \rightarrow \text{id}_{\mathcal{C}(S)}$. The $T_{\tilde{A}}$ are compatible with identities and composition, and the central \mathbb{C}^\times acts in a standard way.
- (3b) For every $\lambda \in \mathcal{C}(S)$, the map which sends $\tilde{\Sigma}$ to the composite $U(T_{\tilde{\Sigma}}) \circ Z_\Sigma : U(V) \rightarrow U(F_\Sigma(V)) \rightarrow U(V)$ is continuous on $\tilde{\text{Ann}}_c(S)$ and holomorphic in its interior¹¹.

¹¹This is phrased in a somewhat imprecise way. See p. 24 for a more accurate description of the holomorphicity condition.

Remark. If we remove the holomorphicity condition in (3b), then one obtains the definition of ‘*general CFT*’ mentioned in the introduction. And if one furthermore insists that the categories $\mathcal{C}(S)$ are all trivial (i.e., isomorphic to Vec_{fd}), then one gets the notion of (heterotic) *full CFT*.

We summarise the above definition of functorial chiral CFT in Table 1.

The items in the first column of that table [items (1a), (2a), (3a)] correspond to the notion of a *modular functor*.¹² In that column, everything is finite dimensional; everything is topological.

The items in the second column [items (1b), (2b), (3b)] correspond to the notion of a *twisted field theory* (the modular functor is the twist). If we were to remove the twist, then we would be left with a single topological vector space for every 1-manifold S , and a single linear map for every complex cobordism.

DEFINITION (SKETCH): FUNCTORIAL CHIRAL CFT

(1a) For every 1-manifold S , a category $\mathcal{C}(S)$.	(1b) A ‘forgetful’ functor $U : \mathcal{C}(S) \rightarrow \text{TopVec}$.
(2a) For every cpx cobordism Σ , a functor $F_\Sigma : \mathcal{C}(\partial_{in}\Sigma) \rightarrow \mathcal{C}(\partial_{out}\Sigma)$.	(2b) For every $\lambda \in \mathcal{C}(\partial_{in}\Sigma)$, a linear map $Z_\Sigma : U(\lambda) \rightarrow U(F_\Sigma(\lambda))$.
(3a) For every $\tilde{A} \in \tilde{\text{Ann}}_c(S)$, a trivialization $T_{\tilde{A}} : F_A \rightarrow \text{id}_{\mathcal{C}(S)}$.	(3b) For every $\lambda \in \mathcal{C}(S)$, the map $\tilde{\text{Ann}}_c(S) \rightarrow \text{End}(U(\lambda))$ $\tilde{A} \mapsto U(T_{\tilde{A}}) \circ Z_A$ is holomorphic.

Table 1.

The two items in the first row [items (1a), (1b)] correspond to the idea that, for every 1-manifold S , there is an associated algebra $\mathcal{A}(S)$. That algebra is called the *algebra of observables*, and can be defined as the algebra of endomorphisms of the functor U . In more down-to-earth terms, the algebra of observables is given by

$$\mathcal{A}(S) = \bigoplus_{\substack{\lambda \in \mathcal{C}(S) \\ \lambda \text{ is simple}}} \text{End}(U(\lambda))$$

¹²...at least conjecturally (I say ‘conjecturally’ because I don’t think that this particular definition has ever been compared to the other definitions of modular functor.)

where the sum ranges of a set of representatives of the isomorphism classes of simple objects of $\mathcal{C}(S)$. Provided we appropriately restrict the class of representations that we allow, we can recover $\mathcal{C}(S)$ as the category of representations of $\mathcal{A}(S)$:

$$\mathcal{C}(S) = \text{Rep}(\mathcal{A}(S)).$$

The two items in the second row [items (2a), (2b)] correspond to the idea that, for every complex cobordism Σ , there is an associated $\mathcal{A}(\partial_{out}\Sigma)$ - $\mathcal{A}(\partial_{in}\Sigma)$ -bimodule H_Σ , equipped with a distinguished ‘vacuum vector’ $\Omega_\Sigma \in H_\Sigma$. One recovers F_Σ as the functor $H_\Sigma \otimes_{\mathcal{A}(\partial_{in}\Sigma)} -$, and Z_Σ as the operation of tensoring with Ω_Σ :

$$F_\Sigma = H_\Sigma \otimes - \quad Z_\Sigma = \Omega_\Sigma \otimes -.$$

One may define the bimodule H_Σ and its distinguished vector $\Omega_\Sigma \in H_\Sigma$ in terms of the functor F_Σ and the natural transformation Z_Σ as follows. First of all,

$$H_\Sigma = \bigoplus_{\substack{\lambda \in \mathcal{C}_{in}, \mu \in \mathcal{C}_{out} \\ \lambda, \mu \text{ simple}}} \text{Hom}(\mu, F(\lambda)) \otimes \text{Hom}(U(\lambda), U(\mu)),$$

where, as before, the sum ranges of a set of representatives of the isomorphism classes of simple objects. To construct Ω_Σ , note that for every $\lambda \in \mathcal{C}_{in}$, the map

$$Z_\lambda : U(\lambda) \rightarrow U(F(\lambda)) = \bigoplus_{\mu \in \mathcal{C}_{out}} \text{Hom}(\mu, F(\lambda)) \otimes U(\mu)$$

provides a distinguished vector in $\text{Hom}(U(\lambda), \bigoplus_{\mu \in \mathcal{C}_{out}} \text{Hom}(\mu, F(\lambda)) \otimes U(\mu)) =$

$$\bigoplus_{\mu \in \mathcal{C}_{out}} \text{Hom}(\mu, F(\lambda)) \otimes \text{Hom}(U(\lambda), U(\mu)).$$

The vacuum vector $\Omega_\Sigma \in H_\Sigma$ is the direct sum of all these, indexed over all the simples of \mathcal{C}_{in} .

Finally, the two items in the third row [items (3a), (3b)] correspond to the idea that H_Σ depends *topologically* on Σ (this means, in particular, that if Σ_1 and Σ_2 are diffeomorphic cobordisms, then H_{Σ_1} and H_{Σ_2} are isomorphic bimodules), while $\Omega_\Sigma \in H_\Sigma$ depends *holomorphically* on Σ .

Definition: A map $\iota : (\mathcal{C}_1, U_1, F_1, Z_1, T_1) \rightarrow (\mathcal{C}_2, U_2, F_2, Z_2, T_2)$ of functorial chiral CFTs is an *infinitesimal equivalence* if it’s an equivalence at the level of \mathcal{C} , F , Z , and for every 1-manifold S and $\lambda \in \mathcal{C}_1(S)$ the comparison map $U_1(\lambda) \rightarrow U_2(\iota(\lambda))$ is a dense inclusion.

Remark. Recall that the rationality condition is built into the above definition of functorial chiral CFT. If one wishes to adapt the definition so as to also include *non-rational* theories, then one should: A. Drop the condition that the linear categories $\mathcal{C}(S)$ be isomorphic to $\text{Vec}_{\text{fd}}^{\text{gr}}$, B. Only require there to be functors F_Σ associated to open-ended complex cobordisms Σ . Here, a cobordism Σ is said to be *open-ended* if every connected component has non-empty outgoing boundary (equivalently, $\pi_0(\partial_{out}\Sigma) \rightarrow \pi_0(\Sigma)$).

We do not know how to formulate the condition of infinitesimal equivalence for non-rational functorial chiral CFTs.

(1b) For every closed 1-manifold S , a faithful functor $U : \mathcal{C}(S) \rightarrow \text{TopVec}$ which equips $\mathcal{C}(S)$ with the structure of a concrete linear category.¹⁴

If $\mathcal{C}(S) \cong \text{Vec}_{\text{f.d.}}^{\oplus r}$, so that an object can be written as an r -tuple of finite dimensional vector spaces, then the functor U is always of the form $(V_1, \dots, V_r) \mapsto \bigoplus V_i \otimes W_i$, where the W_i are typically infinite dimensional.

The forgetful functor satisfies $U(\lambda \otimes \mu) = U(\lambda) \otimes U(\mu)$ and $U(1) = \mathbb{C}$, naturally in λ and μ , and compatibly with the associator, unitors, and braiding:

$$\begin{array}{ccccccc} U((\lambda \otimes \mu) \otimes \nu) & = & (U(\lambda) \otimes U(\mu)) \otimes U(\nu) & & U(1) \otimes U(\lambda) & = & U(1 \otimes \lambda) & & U(\lambda) \otimes U(1) & = & U(\lambda \otimes 1) & & U(\lambda \otimes \mu) & = & U(\lambda) \otimes U(\mu) \\ \downarrow & & \downarrow & & \parallel & & \downarrow & & \parallel & & \downarrow & & \downarrow & & \downarrow \\ U(\lambda \otimes (\mu \otimes \nu)) & = & U(\lambda) \otimes (U(\mu) \otimes U(\nu)) & & \mathbb{C} \otimes U(\lambda) & \longrightarrow & U(\lambda) & & U(\lambda) \otimes \mathbb{C} & \longrightarrow & U(\lambda) & & U(\mu \otimes \lambda) & = & U(\mu) \otimes U(\lambda) \end{array}$$

(2a) For every complex cobordism Σ from S_1 to S_2 , a linear functor $F_\Sigma : \mathcal{C}(S_1) \rightarrow \mathcal{C}(S_2)$.

If Σ and Σ' are complex cobordisms from S_1 to S_2 , then for every biholomorphic map $\phi : \Sigma \xrightarrow{\cong} \Sigma'$ such that $\phi|_{S_1} = \text{id}$ and $\phi|_{S_2} = \text{id}$, we have an invertible natural transformation $F_\Sigma \cong F_{\Sigma'}$, compatible with composition of maps.

We also have invertible natural transformations $F_{1_S} \cong \text{id}_{\mathcal{C}(S)}$, $F_{\Sigma_1 \cup \Sigma_2} \cong F_{\Sigma_1} \circ F_{\Sigma_2}$, and $F_{\Sigma_1 \sqcup \Sigma_2} \cong F_{\Sigma_1} \otimes F_{\Sigma_2}$. They are natural with respect to biholomorphic maps of complex cobordisms, and make the following diagrams commute:

$$\begin{array}{ccccccc} F_{1_S \cup \Sigma} & \longrightarrow & F_{1_S} \circ F_\Sigma & & F_{\Sigma \cup 1_S} & \longrightarrow & F_\Sigma \circ F_{1_S} & & F_{1_{S_1} \cup 1_{S_2}} & \longrightarrow & F_{1_{S_1}} \otimes F_{1_{S_2}} & & F_{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3} & \longrightarrow & F_{\Sigma_1} \circ F_{\Sigma_2 \cup \Sigma_3} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F_\Sigma & = & \text{id}_{\mathcal{C}(S)} \circ F_\Sigma & & F_\Sigma & = & F_\Sigma \circ \text{id}_{\mathcal{C}(S)} & & \text{id}_{\mathcal{C}(S_1 \cup S_2)} & \longrightarrow & \text{id}_{\mathcal{C}(S_1)} \otimes \text{id}_{\mathcal{C}(S_2)} & & F_{\Sigma_1 \cup \Sigma_2} \circ F_{\Sigma_3} & \longrightarrow & F_{\Sigma_1} \circ F_{\Sigma_2} \circ F_{\Sigma_3} \\ \\ F_{\Sigma \cup \emptyset} & \longrightarrow & F_\Sigma \circ \text{id}_{\text{Vec}} & & F_{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3} & \longrightarrow & F_{\Sigma_1} \otimes F_{\Sigma_2 \cup \Sigma_3} & & F_{(\Sigma_1 \cup \Sigma_2) \cup (\Sigma'_1 \cup \Sigma'_2)} & \longrightarrow & F_{\Sigma_1 \cup \Sigma_2} \otimes F_{\Sigma'_1 \cup \Sigma'_2} & & F_{\Sigma_1 \cup \Sigma_2} & \longrightarrow & F_{\Sigma_1} \otimes F_{\Sigma_2} \\ \searrow & & \swarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F_\Sigma & & & & F_{\Sigma_1 \cup \Sigma_2} \otimes F_{\Sigma_3} & \longrightarrow & F_{\Sigma_1} \otimes F_{\Sigma_2} \otimes F_{\Sigma_3} & & F_{\Sigma_1 \cup \Sigma'_1} \otimes F_{\Sigma_2 \cup \Sigma'_2} & \longrightarrow & (F_{\Sigma_1} \circ F_{\Sigma_2}) \otimes (F_{\Sigma'_1} \circ F_{\Sigma'_2}) & & F_{\Sigma_2 \cup \Sigma_1} & \longrightarrow & F_{\Sigma_2} \otimes F_{\Sigma_1} \end{array}$$

(The astute reader will have noticed that the above diagrams are a bit sloppy: the functors being compared don't always have the same domain/codomain. Fixing them is not difficult, but would make them very bulky.)

(2b) For every complex cobordism Σ from S_1 to S_2 and every object $\lambda \in \mathcal{C}(S_1)$, a continuous linear map $Z_\Sigma : U(\lambda) \rightarrow U(F_\Sigma(\lambda))$.

The maps Z_Σ are natural in λ . They're also natural in Σ , meaning that for every biholomorphic map $\phi : \Sigma' \rightarrow \Sigma$ fixing S_1 and S_2 , and every $\lambda \in \mathcal{C}(S_1)$, we have a commutative diagram

$$\begin{array}{ccc} U(\lambda) & \xrightarrow{Z_\Sigma} & U(F_\Sigma(\lambda)) \\ \parallel & & \parallel \\ U(\lambda) & \xrightarrow{Z_{\Sigma'}} & U(F_{\Sigma'}(\lambda)) \end{array}$$

We also have $Z_{1_S} = \text{id}_{U(\lambda)}$, $Z_{\Sigma_1 \cup \Sigma_2} = Z_{\Sigma_1} \circ Z_{\Sigma_2}$, and $Z_{\Sigma_1 \sqcup \Sigma_2} = Z_{\Sigma_1} \otimes Z_{\Sigma_2}$. (Some isomorphisms have been omitted for better readability. For example, the last equality

¹⁴Depending on the type of topological vector spaces one works with, one might want to modify the notion of tensor product accordingly.

should say that the following diagram is commutative:

$$\begin{array}{ccc}
 U(\lambda \otimes \mu) & \xrightarrow{Z_{\Sigma_1 \cup \Sigma_2}} & U(F_{\Sigma_1 \cup \Sigma_2}(\lambda \otimes \mu)) \\
 \parallel & & \parallel \\
 & & U(F_{\Sigma_1}(\lambda) \otimes F_{\Sigma_2}(\mu)) \\
 U(\lambda) \otimes U(\mu) & \xrightarrow{Z_{\Sigma_1} \otimes Z_{\Sigma_2}} & U(F_{\Sigma_1}(\lambda)) \otimes U(F_{\Sigma_2}(\mu))
 \end{array}$$

(3a) For every circle S , every annulus $A \in \text{Ann}(S)$, and every lift $\tilde{A} \in \mathbb{C}^\times \oplus \mathbb{Z} \text{Ann}_c(S)$, a trivialization $T_{\tilde{A}} : F_A(\lambda) \xrightarrow{\cong} \lambda$.

[Think: ‘the map $\Sigma \mapsto F_\Sigma$ is topological.’]

The maps $T_{\tilde{A}}$ are natural in λ . They also satisfy $T_{1_S} = \text{id}$ and $T_{\tilde{A}_1 \cup \tilde{A}_2} = T_{\tilde{A}_1} \circ T_{\tilde{A}_2}$ (omitting the isomorphism $F_{A_1 \cup A_2} \cong F_{A_1} \circ F_{A_2}$ for better readability). Moreover, the central \mathbb{C}^\times should act in the standard way: $T_{z\tilde{A}} = z \cdot T_{\tilde{A}}$ for every $z \in \mathbb{C}^\times$.

(3b) For every circle S and every object $\lambda \in \mathcal{C}(S)$, the map

$$\begin{array}{ccc}
 \mathbb{C}^\times \oplus \mathbb{Z} \text{Ann}_c(S) & \longrightarrow & \text{End}(U(\lambda)) \\
 \Psi & & \Psi \\
 \tilde{A} & \mapsto & \left(U(\lambda) \xrightarrow{Z_A} U(F_A(\lambda)) \xrightarrow{U(T_{\tilde{A}})} U(\lambda) \right)
 \end{array} \tag{13}$$

is holomorphic

[Think: ‘the map $A \mapsto Z_A$ is holomorphic.’]

Here, $\text{End}(U(\lambda))$ is equipped with the topology of pointwise convergence (the strong operator topology). The map $\mathbb{C}^\times \oplus \mathbb{Z} \text{Ann}_c(S) \rightarrow \text{End}(U(\lambda))$ required to be continuous, and holomorphic in the sense that for every finite dimensional complex manifold M and every holomorphic map $M \rightarrow \mathbb{C}^\times \oplus \mathbb{Z} \text{Ann}_c(S)$, the induced map $M \rightarrow \text{End}(U(\lambda))$ is holomorphic.

Note that, by a result of Grothendieck, if the $U(\lambda)$ are Banach spaces, a map $M \rightarrow \text{End}(U(\lambda))$ is holomorphic in the strong operator topology if and only if it is holomorphic in the norm topology.

Remark. When the $U(\lambda)$ are more general than Banach spaces, it might makes sense to insist that they be Fréchet spaces, i.e., inverse limits of Banach spaces

$$U(\lambda) = \varprojlim_{n \in \mathbb{N}} U_n(\lambda).$$

In that case it is probably a good idea to also insist that each U_n be a functor, and that

$$Z = \varprojlim Z_n \quad \text{with} \quad Z_n : U_n(\lambda) \rightarrow U_n(F(\lambda)),$$

so that each $(\mathcal{C}, F, T, U_n, Z_n)$ is a functorial chiral CFT in its own right.

Spin CFTs

Spin chiral CFTs are generalizations of chiral CFTs where all manifolds are equipped with **spin structures**, and all vector spaces are replaced by **super vector spaces**.

A *spin structure* on a complex cobordism Σ is a choice of a square root of its cotangent bundle. More precisely, it consists of a complex line bundle $\mathbb{S}_\Sigma \rightarrow \Sigma$, called the *spinor bundle*, together with an isomorphism $\mathbb{S}_\Sigma^{\otimes 2} \cong T^*\Sigma$. Here, the cotangent bundle is defined by $T_p^*\Sigma = \mathfrak{m}_p/\mathfrak{m}_p^2$, where \mathfrak{m}_p is the maximal ideal of functions $f \in \mathcal{O}_\Sigma$ that vanish at the point $p \in \Sigma$. The notion of spin structure extends verbatim to the case of 1-manifolds.

Remark. For a 1-manifold S (always assumed oriented), a square root of its cotangent bundle T^*S is equivalent to a square root of its complexified cotangent bundle $T_{\mathbb{C}}^*S$. The approach using $T_{\mathbb{C}}^*S$ makes it easier to understand how a spin structure on a complex cobordism induces a spin structure on its boundary.

Every spin manifold M admits a canonical *spin involution* $s_M : M \rightarrow M$. It acts as the identity on the underlying manifold, and acts by -1 on the spinor bundle \mathbb{S}_M .

A *super vector space* V is a $\mathbb{Z}/2$ -graded vector space $V = V_0 \oplus V_1$, where V_0 is called the *even* (or *bosonic*) part and V_1 is called the *odd* (or *fermionic*) part of V . The category super vector spaces has hom spaces

$$\mathrm{Hom}_{\mathrm{sVec}}(V, W) = \mathrm{Hom}(V_0, W_0) \oplus \mathrm{Hom}(V_1, W_1)$$

and tensor product

$$V \otimes W = \overbrace{(V_0 \otimes W_0 \oplus V_1 \otimes W_1)}^{\text{even part}} \oplus \overbrace{(V_0 \otimes W_1 \oplus V_1 \otimes W_0)}^{\text{odd part}}.$$

There is also an internal hom

$$\underline{\mathrm{Hom}}(V, W) = \overbrace{(\mathrm{Hom}(V_0, W_0) \oplus \mathrm{Hom}(V_1, W_1))}^{\text{even part}} \oplus \overbrace{(\mathrm{Hom}(V_0, W_1) \oplus \mathrm{Hom}(V_1, W_0))}^{\text{odd part}},$$

which is itself a super vector space. The symmetry of the tensor product $V \otimes W \xrightarrow{\cong} W \otimes V$ is given by $v \otimes w \mapsto w \otimes v$ when either v or w is even, and by $v \otimes w \mapsto -w \otimes v$ when both v and w are odd.

A *super linear category* is a linear category \mathcal{C} which is moreover enriched over sVec (i.e., there are of internal homs $\underline{\mathrm{Hom}}(\lambda, \mu) \in \mathrm{sVec}$ for any two objects $\lambda, \mu \in \mathcal{C}$) and tensored over sVec (i.e., for any $V \in \mathrm{sVec}$ and $\lambda \in \mathcal{C}$, one may form their tensor product $V \otimes \lambda \in \mathcal{C}$).

Every super vector space has a canonical *grading involution* $\gamma_V : V \rightarrow V$, which acts by 1 on the even part of V and acts by -1 on the odd part of V . Similarly, every super linear category \mathcal{C} admits a *grading involution* $\gamma_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ which acts as the identity on the set of objects, and is given by $\gamma_{\underline{\mathrm{Hom}}(\lambda, \mu)}$ on morphisms.

The spin variant of the notion of chiral CFTs is very similar to the definition presented in the previous section. It requires the following small number of adjustments.

As mentioned already, all 1-manifolds and all cobordisms should be equipped with spin structures, and all vector spaces and all linear categories should be replaced by their super variants. The main novel feature is that:

*The **spin involution** should always map to the **grading involution**.*

If $s : S \rightarrow S$ is the spin involution of a 1-manifold S , then the induced concrete functor

$$\begin{array}{ccc} \mathcal{C}(S) & \xrightarrow{F} & \mathcal{C}(S) \\ & \searrow Z \cong \swarrow & \\ & \text{TopsVec} & \end{array}$$

should be given by $F = \gamma$ and $Z = \gamma$. More precisely, there should be a specified identification $F \simeq \gamma_{\mathcal{C}(S)}$ such that for any $\lambda \in \mathcal{C}(S)$ the map $Z_\lambda : U(\lambda) \rightarrow U(F(\lambda)) \simeq U(\lambda)$ is given by $Z_\lambda = \gamma_{U(\lambda)}$.

We also require that for every spin cobordism Σ from S_1 to S_2 , the spin involution $s_\Sigma : \Sigma \cup s_{S_1} \cong s_{S_2} \cup \Sigma$ get sent to the commuting diagram

$$\begin{array}{ccc} \mathcal{C}(S_1) & \xrightarrow{F_\Sigma} & \mathcal{C}(S_2) \\ \gamma_{\mathcal{C}(S_1)} \downarrow & & \downarrow \gamma_{\mathcal{C}(S_2)} \\ \mathcal{C}(S_1) & \xrightarrow{F_\Sigma} & \mathcal{C}(S_2) \end{array}$$

(the natural transformation that fills the square is the identity natural transformation). Note that that the maps $Z_\Sigma : U(\lambda) \rightarrow U(F_\Sigma(\lambda))$ are even, ensuring the commutativity of the above diagram.

There is an equivalent way of stating all the above conditions in terms of the algebras of observables $\mathcal{A}(S)$ and the pointed bimodules H_Σ (see page 21). In that language, the spin involution $s_S : S \rightarrow S$ of a 1-manifold S should go to the grading involution $\gamma_{\mathcal{A}(S)} : \mathcal{A}(S) \rightarrow \mathcal{A}(S)$ of the corresponding algebra of observables, and the spin involution $s_\Sigma : \Sigma \rightarrow \Sigma$ of a complex cobordism should induce the grading involution $\gamma_{H_\Sigma} : H_\Sigma \rightarrow H_\Sigma$ of the corresponding bimodule. At last, the vacuum vector $\Omega_\Sigma \in H_\Sigma$ should be even.

Remark. Every non-spin functorial chiral CFT $(\mathcal{C}, H, F, Z, T)$ has an associated spin chiral CFT $({}^s\mathcal{C}, {}^sH, {}^sF, {}^sZ, {}^sT)$, where the category associated to a spin 1-manifold S is given by ${}^s\mathcal{C}(S) := \text{sVec} \otimes_{\text{Vec}} \mathcal{C}(S)$.

At first sight, this might seem to contradict the requirement that $s_S \mapsto \gamma_{\mathcal{C}(S)}$. But there is no contradiction as, for categories of the form ${}^s\mathcal{C} = \text{sVec} \otimes_{\text{Vec}} \mathcal{C}$, the grading involution is naturally equivalent to the identity functor. Indeed, the maps $\gamma_H \otimes \text{id}_\lambda : H \otimes \lambda \rightarrow H \otimes \lambda$ for $H \otimes \lambda \in \text{sVec} \otimes_{\text{Vec}} \mathcal{C}$ form a natural isomorphism between $\gamma_{{}^s\mathcal{C}}$ and $\text{id}_{{}^s\mathcal{C}}$.

Remark. The condition that the spin involution always map to the grading involution is called the *spin-statistics theorem*. It is possible to consider theories for which the the

spin-statistics theorem fails. However, I believe that it is only possible to formulate the condition of unitarity for those spin CFTs that do satisfy the spin-statistics theorem.

The Virasoro algebra

The goal of the next couple sections is to describe the central extensions of $\text{Diff}(S)$ and of $\text{Ann}(S)$ that were used in the definition of chiral CFT. We will construct central extensions

$$\begin{aligned} 0 \rightarrow i\mathbb{R} \oplus \mathbb{Z} \rightarrow {}^{i\mathbb{R} \oplus \mathbb{Z}}\text{Diff}(S) \rightarrow \text{Diff}(S) \rightarrow 0 \\ 0 \rightarrow \mathbb{C} \oplus \mathbb{Z} \rightarrow {}^{\mathbb{C} \oplus \mathbb{Z}}\text{Ann}(S) \rightarrow \text{Ann}(S) \rightarrow 0. \end{aligned} \quad (14)$$

We will prove that the first one is a universal central extension, and we conjecture that the second one is universal too.

At the level of Lie algebras (things are always easier at the Lie algebra level), the corresponding claim is that there exist central extensions

$$\begin{aligned} 0 \rightarrow i\mathbb{R} \rightarrow {}^{i\mathbb{R}}\mathfrak{X}(S) \rightarrow \mathfrak{X}(S) \rightarrow 0 \\ 0 \rightarrow \mathbb{C} \rightarrow {}^{\mathbb{C}}\mathfrak{X}_{\mathbb{C}}(S) \rightarrow \mathfrak{X}_{\mathbb{C}}(S) \rightarrow 0 \end{aligned}$$

and that these are universal central extensions. If one takes S to be the standard circle S^1 , one can be more specific, and write down the cocycle that describes these central extensions. This is the *Virasoro cocycle*:

$$\omega_{Vir}\left(f(z)\frac{\partial}{\partial z}, g(z)\frac{\partial}{\partial z}\right) = \frac{1}{12} \int_{S^1} \frac{\partial^3 f}{\partial z^3}(z) g(z) \frac{dz}{2\pi i} \quad (15)$$

equivalently:
$$\omega_{Vir}(\ell_m, \ell_n) = \frac{1}{12}(m^3 - m)\delta_{m+n,0}.$$

The Witt algebra \mathbb{W} is the algebraic span of the ℓ_n 's (the Lie algebra of algebraic vector fields on \mathbb{C}^\times), and its universal central extension is called the *Virasoro algebra*. It is standard convention to denote the basis vectors of the Witt algebra ℓ_n , and the corresponding basis vectors of the Virasoro by upper case letters L_n :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & Vir & \longrightarrow & \mathbb{W} \longrightarrow 0 \\ & & & & \Psi & & \Psi \\ & & & & L_n & \mapsto & \ell_n \end{array}$$

The commutation relations of the Virasoro algebra are given by

$$\boxed{[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}.}$$

Remark. The Virasoro commutation relations are usually written in a way that includes the central charge:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \quad (16)$$

Let Vir_c be the Lie algebra defined by (16). Provided c and c' are non-zero, there is an isomorphism $Vir_c \cong Vir_{c'}$ that fits in a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & Vir_c & \longrightarrow & \mathbb{W} \longrightarrow 0 \\ & & \downarrow \scriptstyle z \\ & & \downarrow \scriptstyle z \cdot c'/c & & \downarrow \scriptstyle \cong & & \parallel \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & Vir_{c'} & \longrightarrow & \mathbb{W} \longrightarrow 0 \end{array}$$

The Lie algebras Vir_c and $Vir_{c'}$ are however distinct as central extensions of \mathbb{W} by \mathbb{C} (i.e, there's no way to arrange for the left vertical map to be the identity map $\text{id}_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C}$).

Also, the notion of a *representation* of Vir_c is distinct from that of a representation of $Vir_{c'}$, because one always includes the requirement that the central element $1 \in \mathbb{C}$ acts by the identity operator.

In order to classify central extensions of the Witt algebra, we will review some notions of Lie algebra cohomology.

Let \mathfrak{g} be a Lie algebra, and let A be a vector space.

A **2-cocycle** is a bilinear map $\omega : \mathfrak{g} \times \mathfrak{g} \rightarrow A$ which is antisymmetric, and satisfies

$$\sum^3 \omega([X, Y], Z) = 0.$$

Given a 2-cocycle, one can form a central extension $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus A$, with Lie bracket

$$[(X, a), (Y, b)]_{\tilde{\mathfrak{g}}} := ([X, Y]_{\mathfrak{g}}, \omega(X, Y))$$

which fits into a central extension $0 \rightarrow A \rightarrow \tilde{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0$ (an extension such that $A \subset Z(\tilde{\mathfrak{g}})$). If the cocycle can be written in the form

$$\omega(X, Y) = \mu([X, Y])$$

for some linear map $\mu : \mathfrak{g} \rightarrow A$ (typically not a Lie algebra homomorphism), then we say that ω is a trivial 2-cocycle, and write $\omega = d\mu$.

Theorem. *The second Lie algebra cohomology group*

$$H^2(\mathfrak{g}; A) := \frac{\{2\text{-cocycles}\}}{\{\text{trivial 2-cocycles}\}}$$

is canonically isomorphic to the set of isomorphism classes of central extensions of \mathfrak{g} by A , where two central extensions $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}'$ are called isomorphic if there exists a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & \tilde{\mathfrak{g}} & \longrightarrow & \mathfrak{g} \longrightarrow 0 \\ & & \downarrow \scriptstyle \text{id}_A & & \downarrow \scriptstyle \cong & & \downarrow \scriptstyle \text{id}_{\mathfrak{g}} \\ 0 & \longrightarrow & A & \longrightarrow & \tilde{\mathfrak{g}}' & \longrightarrow & \mathfrak{g} \longrightarrow 0 \end{array}$$

where the two outer vertical maps are identity maps.

Proof outline. \oplus We already saw how to construct a central extension from a 2-cocycle. Suppose now that $\omega_2 - \omega_1 = d\mu$. Then

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & \tilde{\mathfrak{g}}_1 = \mathfrak{g} \oplus A & \longrightarrow & \mathfrak{g} \longrightarrow 0 \\ & & \parallel & & \downarrow \scriptstyle \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} & & \parallel \\ 0 & \longrightarrow & A & \longrightarrow & \tilde{\mathfrak{g}}_2 = \mathfrak{g} \oplus A & \longrightarrow & \mathfrak{g} \longrightarrow 0 \end{array}$$

is an isomorphism. So the map $\{2\text{-cocycles}\} \rightarrow \{\text{central extensions}\}$ descends to a map $H^2(\mathfrak{g}; A) \rightarrow \{\text{iso classes of central extensions}\}$.

⊖ Given a central extension of \mathfrak{g} by A , pick a splitting

$$0 \longrightarrow A \longrightarrow \tilde{\mathfrak{g}} \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{r} \end{array} \mathfrak{g} \longrightarrow 0$$

(usually not a Lie algebra homomorphism) and let $\omega(X, Y) := [s(X), s(Y)] - s([X, Y])$. Given another splitting, we can write it as $s' = s + \mu$ for some $\mu : \mathfrak{g} \rightarrow A$. The corresponding cocycles satisfy $\omega' = \omega - d\mu$. So they're equal in $H^2(\mathfrak{g}; A)$. \square

We also have:

Proposition 6 Let $0 \rightarrow A_i \rightarrow \tilde{\mathfrak{g}}_i \rightarrow \mathfrak{g} \rightarrow 0$ be central extensions that fit into a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & \tilde{\mathfrak{g}}_1 & \longrightarrow & \mathfrak{g} \longrightarrow 0 \\ & & f \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & A_2 & \longrightarrow & \tilde{\mathfrak{g}}'_2 & \longrightarrow & \mathfrak{g} \longrightarrow 0 \end{array}$$

and let $[\omega_i] \in H^2(\mathfrak{g}_i, A_i)$ be the corresponding cohomology classes. Then $[\omega_2]$ is the image of $[\omega_1]$ under the map $H^2(\mathfrak{g}, A_1) \rightarrow H^2(\mathfrak{g}, A_2)$ induced by $f : A_1 \rightarrow A_2$.

Theorem 7 The second cohomology of the Witt algebra is one dimensional $H^2(\mathbb{W}, \mathbb{C}) \cong \mathbb{C}$, and the Virasoro cocycle $[\omega_{Vir}]$ is a generator.

Let us check that ω_{Vir} is indeed a cocycle. For the purpose of this computation, we rewrite (15) in the following abbreviated (and less precise) form:

$$\omega_{Vir}(f, g) = \oint f'''g = \oint f'g''$$

We then easily compute: $\sum^3 \oint (fg' - f'g)h'' = \sum^3 \oint (f'g' + fg'' - f''g - f'g')h'' = 0$.

The following lemma will be surprisingly useful:

Lemma 8 Let \mathfrak{g} be a Lie algebra, and let $X \in \mathfrak{g}$ be such that $\text{ad}(X)$ exponentiates to a 1-parameter family of automorphisms of \mathfrak{g} [For us: $\mathfrak{g} = \mathbb{W}$, $X = il_0$, and $\text{ad}(il_0)$ exponentiates to an action of S^1 on \mathbb{W}]. For $\xi \in \mathfrak{g}$, let $\xi_t := \exp(t \cdot \text{ad}(X))(\xi)$, so that $\frac{d}{dt}\xi_t = [X, \xi_t]$. Then, for any 2-cocycle ω , we have

$$[\omega] = [\omega_t] \in H^2(\mathfrak{g}),$$

where $\omega_t(\xi, \eta) := \omega(\xi_t, \eta_t)$.

Proof.

$$\begin{aligned}
\omega(\xi_T, \eta_T) - \omega(\xi, \eta) &= \int_0^T \left(\frac{d}{dt} \omega(\xi_t, \eta_t) \right) dt \\
&\stackrel{\boxed{\frac{d}{dt} \xi_t = [X, \xi_t]}}{=} \int_0^T \left(\omega([X, \xi_t], \eta_t) + \omega(\xi_t, [X, \eta_t]) \right) dt \\
&\stackrel{\boxed{\text{cocycle identity}}}{=} \int_0^T \omega(X, [\xi_t, \eta_t]) dt \\
&\stackrel{\boxed{\xi \mapsto \xi_t \text{ is an automorphism}}}{=} \int_0^T \omega(X, [\xi, \eta]_t) dt = \mu([\xi, \eta])
\end{aligned}$$

where $\mu(\xi) := \int_0^T \omega(X, \xi_t) dt$. □

Suppose (as is the case in our example of interest), that $\text{ad}(X)$ exponentiates to an action of S^1 on \mathfrak{g} by Lie algebra automorphisms. Then letting $\text{avg}_{S^1}(\omega) := \int_{S^1} \omega_t dt$, we have

$$[\text{avg}_{S^1}(\omega)] = \left[\int_{S^1} \omega_t dt \right] = \int_{S^1} [\omega_t] dt = \int_{S^1} [\omega] dt = [\omega] \quad \text{in } H^2(\mathfrak{g}, A)$$

for any 2-cocycle ω . Given a linear map $\mu : \mathfrak{g} \rightarrow A$, let $\mu_t(\xi) := \mu(\xi_t)$, and let us define $\text{avg}_{S^1}(\mu) := \int_{S^1} \mu_t dt$. If a 2-cocycle ω is trivial, i.e., if there exists μ such that $\omega = d\mu$, then there also exists an S^1 -invariant μ with that same property: indeed, letting $\mu' := \text{avg}_{S^1}(\mu)$ we have

$$d\mu' = d(\text{avg}_{S^1}(\mu)) = \text{avg}_{S^1}(d(\mu)) = \text{avg}_{S^1}(\omega) = \omega.$$

From the above discussion, we deduce that

$$H^2(\mathfrak{g}; A) = \frac{\{ S^1\text{-invariant 2-cocycles} \}}{\{ d\mu \mid \mu : \mathfrak{g} \rightarrow A, \mu \text{ is } S^1\text{-invariant} \}}$$

Remark. The same argument works with any compact Lie group H in place of S^1 . Let \mathfrak{g} be a (typically infinite dimensional) Lie algebra, and let $\mathfrak{h} \subset \mathfrak{g}$ be a finite dimensional subalgebra such that the adjoint action of \mathfrak{h} on \mathfrak{g} exponentiates to the action of a compact Lie group H on \mathfrak{g} . Then $H^2(\mathfrak{g}; A) = \{ H\text{-invariant 2-cocycles} \} / \{ d\mu \mid \mu \text{ is } H\text{-invariant} \}$.

Armed with the above description of $H^2(\mathfrak{g}; A)$ we can prove the theorem:

Proof of Theorem 7. We'll show that the space of S^1 -invariant 2-cocycles $\omega : \mathbb{W} \times \mathbb{W} \rightarrow \mathbb{C}$ is two dimensional, spanned by the cocycles

$$\omega_1(\ell_m, \ell_n) := m^3 \cdot \delta_{m+n,0} \quad \text{and} \quad \omega_2(\ell_m, \ell_n) := m \cdot \delta_{m+n,0},$$

and that the space of 2-cocycles which are of the form $d\mu$ for some S^1 -invariant $\mu : \mathbb{W} \rightarrow \mathbb{C}$ is one dimensional, spanned by ω_2 . It will follow that $\dim(H^2(\mathbb{W}, \mathbb{C})) = 2 - 1 = 1$.

First of all, the space of S^1 -invariant linear functionals $\mathbb{W} \rightarrow \mathbb{C}$ is one dimensional, spanned by $\mu : \ell_n \mapsto \delta_{n,0}$. An easy computation yields $d\mu = 2\omega_2$.

Let now ω be an S^1 -invariant 2-cocycle. Let $c_{m,n} = \omega(\ell_m, \ell_n)$. S^1 -invariance implies that $c_{m,n} = 0$ when $m + n \neq 0$. So let's write $c_n = \omega(\ell_n, \ell_{-n})$. We have $c_{-n} = -c_n$

by antisymmetry. The cocycle identity $\sum^3 \omega([\ell_m, \ell_n], \ell_p) = 0$ is only interesting when $m + n + p = 0$ (otherwise it's trivially satisfied). At the level of the c_n 's, it reads

$$(m - n)c_{m+n} + (n - p)c_{n+p} + (p - m)c_{p+m} = 0.$$

Plugging in $p = -m - n$, we get

$$(m - n)c_{m+n} + (2n + m)c_{-m} - (2m + n)c_{-n} = 0.$$

Equivalently,

$$(m - n)c_{m+n} = (2n + m)c_m - (2m + n)c_n.$$

The case $n = 1$ of the above equation reads:

$$(m - 1)c_{m+1} = (2 + m)c_m - (2m + 1)c_1$$

This is a recurrence relation that expresses c_{m+1} in terms of c_m and c_1 provided $m + 1 \geq 3$ (otherwise $m - 1$ might be zero).

The sequence $\{c_m\}_{m \geq 1}$ is therefore entirely determined by the values of c_1 and of c_2 . In particular, the space of S^1 -invariant 2-cocycles is at most two dimensional. We already know that that space is at least two dimensional. So it's two dimensional. \square

Theorem 7 means that the Virasoro algebra is a **universal central extension** of the Witt algebra (I should say *the* universal central extension, because universal central extensions are unique up to unique isomorphism). Namely, by the same computation as above, one checks that $H^2(\mathbb{W}, A) = A$ for any vector space A . More precisely, any 2-cocycle is equivalent to $a \cdot \omega_{Vir}$ for some $a \in A$. Effectively, what this produces is, for every central extension $0 \rightarrow A \rightarrow \tilde{\mathfrak{g}} \rightarrow \mathbb{W} \rightarrow 0$, a homomorphism of central extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & Vir & \longrightarrow & \mathbb{W} & \longrightarrow & 0 \\ \exists! \downarrow & & \downarrow & & \downarrow & & \parallel & & \\ \forall \downarrow & & 0 & \longrightarrow & A & \longrightarrow & \tilde{\mathfrak{g}} & \longrightarrow & \mathbb{W} & \longrightarrow & 0 \end{array}$$

The left vertical map $\mathbb{C} \rightarrow A$ is uniquely characterized (by Proposition 6) by the fact that it sends $1 \in H^2(\mathbb{W}, \mathbb{C}) = \mathbb{C}$ to the element $a \in H^2(\mathbb{W}, A) = A$ that classifies the central extension $\tilde{\mathfrak{g}}$. The middle vertical map is also unique, because Vir is spanned by commutators of lifts of elements of \mathbb{W} . That's exactly what it means, by definition, that the Virasoro algebra is the universal central extension of the Witt algebra.

Remark. *The same argument shows that the continuous cohomology $H_{cts}^2(\mathfrak{X}_{\mathbb{C}}(S^1), \mathbb{C})$ (defined in the same way as usual Lie algebra cohomology, except that we now also require the cocycles to be continuous) is one dimensional, generated by ω_{Vir} .*

The same proof can also be adapted to show that $H_{cts}^2(\mathfrak{X}(S^1), i\mathbb{R}) = \mathbb{R}$, generated by ω_{Vir} . (That last statement also follows from the fact that cohomology commutes with complexification: $H^2(\mathfrak{g}, A) \otimes_{\mathbb{R}} \mathbb{C} = H^2(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}, A \otimes_{\mathbb{R}} \mathbb{C})$.)

Little Fact. Inside the two dimensional space of S^1 -invariant 2-cocycles, there is a one dimensional space of $PSU(1, 1)$ -invariant ones, spanned by ω_{Vir} . That's a good reason to prefer ω_{Vir} as opposed to, say, $(\ell_m, \ell_n) \mapsto m^3 \cdot \delta_{m+n, 0}$.

Unfortunately, *the Virasoro cocycle is not coordinate independent*. What this means in practice is that, given a circle S (a manifold diffeomorphic to S^1), there is *no canonical* 2-cocycle on $\mathfrak{X}(S)$ (or on $\mathfrak{X}_{\mathbb{C}}(S)$). But the concept of universal central extension still makes sense. And the good thing is that, if it exists, a universal central extension is unique up to unique isomorphism. So, even though we don't have a cocycle, we can still talk about *the* universal central extension of $\mathfrak{X}(S)$ (or of $\mathfrak{X}_{\mathbb{C}}(S)$) for any circle S .

Central extensions of (semi-)groups

We now address the question of, given a Lie (semi-)group G with Lie algebra \mathfrak{g} , and a central extension $0 \rightarrow A \rightarrow \tilde{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0$, how to build a corresponding central extension of G ?

If G is a Lie group, the Lie algebra \mathfrak{g} can be naturally identified with the Lie algebra of left-invariant vector fields on G , equipped with the usual Lie bracket of vector fields. The chain complex which computes Lie algebra cohomology can then be naturally identified with the complex of left-invariant differential forms on G , equipped with the usual de Rham differential. Recall that, given a manifold M and a 2-form $\alpha \in \Omega^2(M)$, its de Rham differential is given by

$$d\alpha(X, Y, Z) = \sum_{i=1}^3 X \cdot \alpha(Y, Z) - \sum_{i=1}^3 \alpha([X, Y], Z)$$

where $[X, Y]$ is the Lie bracket of vector fields.

Given an antisymmetric bilinear form $\omega : \mathfrak{g} \times \mathfrak{g} \rightarrow A$, let us write $\underline{\omega} \in \Omega^2(G)$ for the corresponding left-invariant form on G . The 2-form $\underline{\omega}$ is closed if and only if ω is a 2-cocycle in the sense introduced before:

$$\begin{aligned} d\underline{\omega} = 0 &\Leftrightarrow d\underline{\omega}(X, Y, Z) = 0, \quad \forall \text{ left invariant } X, Y, Z \in \mathfrak{X}(G), \\ &\Leftrightarrow \sum_{i=1}^3 X \cdot \underbrace{\omega(Y, Z)}_{\text{constant}} - \sum_{i=1}^3 \omega([X, Y], Z) = 0 \\ &\Leftrightarrow \sum_{i=1}^3 \omega([X, Y], Z) = 0, \quad \forall X, Y, Z \in \mathfrak{g}. \end{aligned}$$

Here, we've used the fact that, in order to check whether $\underline{\omega}$ is closed, it is enough to evaluate $d\underline{\omega}$ against left-invariant vector fields.

Proposition 9 1. *Given a simply connected Lie group G with Lie algebra \mathfrak{g} , and a 2-cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \rightarrow A$, let*

$$\tilde{G}_\omega := \left\{ (\gamma, a) \mid \begin{array}{l} \gamma : [0, 1] \rightarrow G, \\ \gamma(0) = e, a \in A \end{array} \right\} / \left((\gamma, a) \sim (\gamma', a + \int_h \omega) \text{ when } \begin{array}{l} \gamma'(1) = \gamma(1) \text{ and } h \text{ is a homotopy from } \gamma \text{ to } \gamma', \end{array} \right.$$

with group operation given by $(\gamma_1, a_1)(\gamma_2, a_2) = (\gamma_1 \cdot \gamma_1(1)\gamma_2, a_1 + a_2)$. Then

$$\tilde{G}_\omega \rightarrow G : (\gamma, a) \mapsto \gamma(1)$$

is a central extension of G by $\underline{A} := A/\{\text{periods of } \underline{\omega}\}$.

2. If $\omega' = \omega + d\mu$, then $\tilde{G}_\omega \cong \tilde{G}_{\omega'}$, with isomorphism given by $(\gamma, a) \mapsto (\gamma, a + \int_\gamma \mu)$.
3. If G is merely connected then, provided \mathfrak{g} has trivial abelianization, \tilde{G}_ω is a central extension of G by $\underline{A} \times \pi_1(G)$. (If $\mathfrak{g}_{ab} \neq 0$, the kernel of $\tilde{G}_\omega \rightarrow G$ might fail to be abelian.)
4. If $\mathfrak{g}_{ab} = 0$ and the central extension associated to ω is universal, then, provided the set of periods of ω is discrete inside A , the central extension

$$1 \longrightarrow \underline{A} \times \pi_1(G) \longrightarrow \tilde{G}_\omega \longrightarrow G \longrightarrow 1$$

is a universal central extension in the category of Lie groups.

Proof. 1. Since G is simply connected, any element of $K := \ker(\tilde{G}_\omega \rightarrow G)$ can be represented by a pair $(*, a)$, where $*$ denotes the constant path. By definition, we then have $(*, a) \sim (*, a + \int_h \underline{\omega})$ for every homotopy from the constant path to itself (also known as a based map $h : S^2 \rightarrow G$). The elements $\int_h \underline{\omega} \in A$ are, by definition, the periods of $\underline{\omega}$.

2. The map $(\gamma, a) \mapsto (\gamma, a + \int_\gamma \underline{\mu})$ is well-defined by an application of Stokes' theorem, and is visibly an isomorphism.

3. The projection map $K \rightarrow \pi_1(G) : [(\gamma, a)] \mapsto [\gamma]$ fits into a diagram

$$\begin{array}{ccccc} \underline{A} & \longrightarrow & K & \longrightarrow & \pi_1(G) \\ \parallel & & \downarrow & & \downarrow \\ \underline{A} & \longrightarrow & \tilde{G}_\omega & \longrightarrow & \tilde{G} \\ & & \downarrow & & \downarrow \\ & & G & \xlongequal{\quad} & G \end{array}$$

where all rows and columns are group extensions. The conjugation action $\tilde{G} \curvearrowright \pi_1(G)$ is trivial (since \tilde{G} is connected), so $\pi_1(G)$ is central in \tilde{G} . So, given $k \in K$ and $g \in \tilde{G}_\omega$, the commutator $[k, g] \in G_\omega$ maps to $1 \in \tilde{G}$. That commutator therefore lands in \underline{A} . The map $[k, -] : \tilde{G}_\omega \rightarrow \underline{A}$ is a homomorphism:

$$(kg_1k^{-1}g_1^{-1})(kg_2k^{-1}g_2^{-1}) = kg_1k^{-1}(kg_2k^{-1}g_2^{-1})g_1^{-1} = k(g_1g_2)k^{-1}(g_1g_2)^{-1} \quad \checkmark$$

and descends to a homomorphism $\tilde{G} \rightarrow \underline{A}$. But \tilde{G} is connected and $\mathfrak{g}_{ab} = 0$, so there are no non-trivial homomorphisms from \tilde{G} to an abelian group. Therefore K is central in \tilde{G}_ω and $K \rightarrow \tilde{G}_\omega \rightarrow G$ is a central extension.

It remains to show that $K \cong \underline{A} \times \pi_1(G)$, i.e., that the sequence $\underline{A} \rightarrow K \rightarrow \pi_1(G)$ splits. This follows from the general structure theory of abelian groups, using that \underline{A} is divisible and hence an injective abelian group.

4. If $A \rightarrow \tilde{\mathfrak{g}}_\omega \rightarrow \mathfrak{g}$ is universal, then for any central extension $B \rightarrow \tilde{G} \rightarrow G$ with associated Lie algebra central extension $\mathfrak{b} \rightarrow \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$, there is a unique map

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & \tilde{\mathfrak{g}}_\omega & \longrightarrow & \mathfrak{g} \longrightarrow 0 \\ & & f \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathfrak{b} & \longrightarrow & \tilde{\mathfrak{g}} & \longrightarrow & \mathfrak{g} \longrightarrow 0. \end{array}$$

Since \tilde{G}_ω is connected, there is at most one homomorphism $\tilde{G}_\omega \rightarrow \tilde{G}$ that integrates the map $\tilde{\mathfrak{g}}_\omega \rightarrow \tilde{\mathfrak{g}}$. So all we need to do is construct such a homomorphism.

The canonical splitting $\mathfrak{g} \rightarrow \tilde{\mathfrak{g}}_\omega$ induces a splitting $s : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$. Letting $F : A \rightarrow B$ be the homomorphism which integrates f , the map $\tilde{G}_\omega \rightarrow \tilde{G}$ is given by

$$[(\gamma, a)] \mapsto \delta(1) \cdot F(a)$$

where $\delta : [0, 1] \rightarrow \tilde{G}$ is the unique solution of $\delta(t)^{-1} \frac{d}{dt} \delta(t) = s(\gamma(t)^{-1} \frac{d}{dt} \gamma(t))$. [Here, $\gamma(t)^{-1} \frac{d}{dt} \gamma(t)$ denotes the left-translate of $\frac{d}{dt} \gamma(t) \in T_{\gamma(t)} G$ back to $T_e G = \mathfrak{g}$.] \square

Remark. The splitting of $0 \rightarrow \underline{A} \rightarrow K \rightarrow \pi_1(G) \rightarrow 0$ is not canonical, so the center of \tilde{G}_ω is only non-canonically isomorphic to $\underline{A} \times \pi_1(G)$.

The above proposition takes care of the central extension of $\text{Diff}(S^1)$ (the top row in (14)). But the case of $\text{Ann}(S^1)$ is more tricky because the various tangent spaces of $\text{Ann}(S^1)$ are no longer all isomorphic (or rather, left translation is not an isomorphism). So we can't talk about the left-invariant 2-form associated to a Lie algebra 2-cocycle. To go around this difficulty, we use a little trick. Given an annulus $A \in \text{Ann}(S^1)$, let

$$\text{Ann}^{\leq A} := \{A_1 \in \text{Ann}(S^1) \mid \exists A_2 : A_1 A_2 = A\} \cong \{\gamma : S^1 \hookrightarrow A \mid \gamma \text{ "wraps around } A"\}.$$

When thinking in terms of maps $\gamma : S^1 \rightarrow A$, the tangent space of $\text{Ann}^{\leq A}$ is easy to compute, and we see that the map $\text{Ann}^{\leq A_2} \rightarrow \text{Ann}^{\leq A}$ given by $B \mapsto A_1 B$ induces an isomorphism of tangent spaces

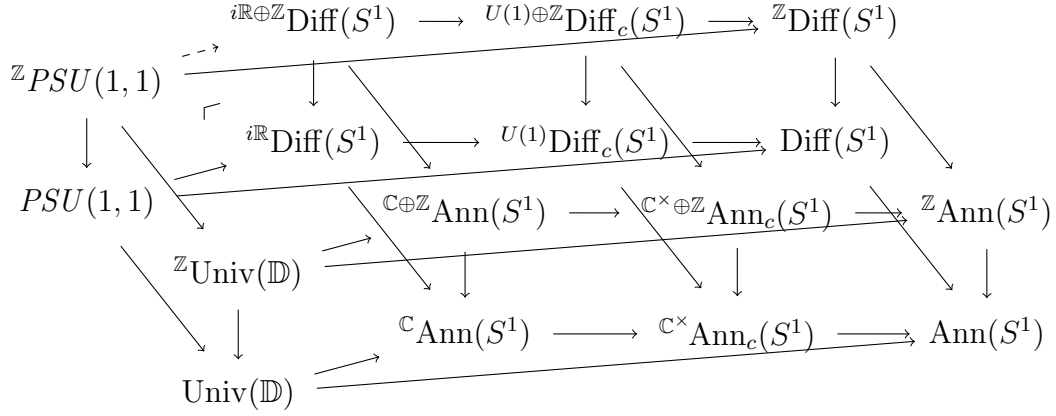
$$T_1(\text{Ann}^{\leq A_2}) = \mathfrak{X}_{\mathbb{C}}(S^1) \xrightarrow{\cong} T_{A_1}(\text{Ann}^{\leq A}).$$

So ω_{Vir} makes sense as a 2-form on $\text{Ann}^{\leq A}$, and we can define

$$\mathbb{C} \times \mathbb{Z} \text{Ann}(S^1) := \left\{ (A, \gamma, a) \left| \begin{array}{l} A \in \text{Ann}(S^1), a \in \mathbb{C} \\ \gamma : [0, 1] \rightarrow \text{Ann}^{\leq A} \\ \gamma(0) = 1, \gamma(1) = A \end{array} \right. \right\} / \left(\begin{array}{l} (\gamma, a) \sim (\gamma', a + \int_h \omega_{Vir}), \\ h \text{ a homotopy from } \gamma \text{ to } \gamma' \end{array} \right)$$

very much like what we did in Proposition 9.

Here's a big chart with all the groups and semigroups related to the Virasoro algebra:



The dotted map exists because the universal cover of $PSU(1,1)$ is also its universal central extension. This allows us to identify a canonical copy of \mathbb{Z} inside the center of $i\mathbb{R} \oplus \mathbb{Z} \text{Diff}(S^1)$, and to define $i\mathbb{R} \text{Diff}(S^1)$ as the quotient by that \mathbb{Z} . Similarly, $\mathbb{C} \text{Ann}(S^1)$ is the quotient of $\mathbb{C} \oplus \mathbb{Z} \text{Ann}(S^1)$ by that same copy of \mathbb{Z} .

The vacuum sector and its symmetries

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$, and $S^1 := \partial\mathbb{D}$. Given a functorial chiral CFT, let us define the *unit object*

$$\mathbf{1}_{\mathcal{C}} \in \mathcal{C}(S^1)$$

to be the image of $1_{\text{Vec}} = \mathbb{C} \in \text{Vec}_{\text{f.d.}} = \mathcal{C}(\emptyset)$ under the functor $F_{\mathbb{D}} : \mathcal{C}(\partial_{\text{in}}\mathbb{D}) = \mathcal{C}(\emptyset) \rightarrow \mathcal{C}(\partial_{\text{out}}\mathbb{D}) = \mathcal{C}(S^1)$. The *vacuum sector* H_0 of the CFT is the underlying vector space of $\mathbf{1}_{\mathcal{C}}$:

$$\hookrightarrow H_0 := U(\mathbf{1}_{\mathcal{C}}) = \mathbf{U}(F_{\mathbb{D}}(\mathbb{C})).$$

The vacuum sector comes with a *vacuum vector*

$$\hookrightarrow \Omega := Z_{\mathbb{D}}(1) \in H_0$$

defined as the image of $1 \in \mathbb{C}$ under the map $Z_{\mathbb{D}} : \mathbb{C} = U(\mathbb{C}) \rightarrow U(F_{\mathbb{D}}(\mathbb{C})) = U(\mathbf{1}_{\mathcal{C}})$. More generally, given a complex cobordism Σ with empty incoming boundary, we get a vector space $H_{\Sigma} := U(F_{\Sigma}(\mathbb{C}))$, and a vacuum vector

$$\Omega_{\Sigma} := Z_{\Sigma}(1) \in H_{\Sigma}.$$

Remark. In examples of interest, the unit object $\mathbf{1}_{\mathcal{C}} \in \mathcal{C}(S^1)$ is always simple, equivalently, the vacuum sector H_0 is an irreducible $\mathcal{A}(S^1)$ -module, but this property is not guaranteed by the axioms. A functorial chiral CFT with that property is called *irreducible*.

Given a finite collection $(\mathcal{C}_i, U_i, F_i, Z_i, T_i)$ of irreducible CFTs of same central charge,

their direct sum is defined on connected manifolds by $S \mapsto (\bigoplus \mathcal{C}_i(S), \bigoplus U_i)$, $\Sigma \mapsto (\bigoplus F_{i,\Sigma}, Z_{i,\Sigma})$, $\tilde{A} \mapsto \bigoplus T_{i,\tilde{A}}$, and is defined on disconnected manifolds to be the tensor product rule of what the theory assigns to each connected component. I suspect (but haven't checked) that every functorial chiral CFT is a direct sum of irreducible ones, and that the direct sum decomposition is canonical. In that sense, the study of functorial chiral CFTs completely reduces to the study of irreducible ones.

Given an object $\lambda \in \mathcal{C}(S^1)$, we sometimes write $H_\lambda := U(\lambda)$. If λ is irreducible and $\lambda \not\cong 1$, we call this a *charged sector* of the CFT.

The vacuum sector depends functorially on the disc \mathbb{D} ; the automorphism group $\text{Aut}(\mathbb{D}) = PSU(1,1)$ therefore acts on H_0 . And since the construction of $\Omega \in H_0$ also depends functorially on \mathbb{D} , it is invariant under the action of that group:

$$\Omega \in H_0^{PSU(1,1)}.$$

There are also other (semi)groups which act on H_0 . For example, the much bigger semigroup $\mathbb{C}^\times \oplus \mathbb{Z} \text{Ann}_c(S)$ acts on H_0 , as described in (13). That action does not fix the vacuum vector. There is also a somewhat smaller semigroup which acts on H_0 in a way which does fix the vacuum vector:

Definition 10 *The semigroup of univalent maps of the disc is given by*

$$\text{Univ}(\mathbb{D}) := \{\psi : \mathbb{D} \rightarrow \mathbb{D} \mid \psi \text{ is an embeddings}\}.$$

The inclusion $\text{Univ}(\mathbb{D}) \hookrightarrow \text{Ann}(S^1)$ which sends $\psi \in \text{Univ}(\mathbb{D})$ to the annulus $A_\psi := (\mathbb{D} \setminus \psi(\mathring{\mathbb{D}}), \varphi_{in} = \psi|_{\partial\mathbb{D}}, \varphi_{out} = \text{id})$ is a semigroup homomorphism: $A_{\psi_1 \circ \psi_2} = A_{\psi_1} \cup A_{\psi_2}$.

The action of ψ on H_0 is given by

$$H_0 = U(F_{\mathbb{D}}(\mathbb{C})) \xrightarrow{Z_{A_\psi}} U(F_{A_\psi} F_{\mathbb{D}}(\mathbb{C})) \cong U(F_{A_\psi \cup \mathbb{D}}(\mathbb{C})) \cong U(F_{\mathbb{D}}(\mathbb{C})) = H_0 \quad (17)$$

and it indeed fixes the vacuum vector:

$$\Omega = Z_{\mathbb{D}}(1) \xrightarrow{Z_{A_\psi}} Z_{A_\psi} Z_{\mathbb{D}}(1) \mapsto Z_{A_\psi \cup \mathbb{D}}(1) \mapsto Z_{\mathbb{D}}(1) = \Omega.$$

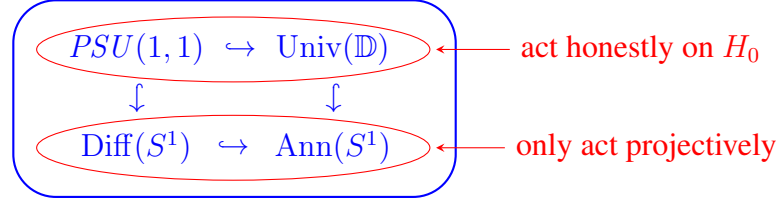
We prove the next lemma under the assumption that the CFT is irreducible (the statement should also hold true without that assumption):

Lemma 11 *Let $\psi : \mathbb{D} \rightarrow \mathbb{D}$ be a univalent map, let A_ψ be its image in $\text{Ann}(S)$, and let $\tilde{A} \in \mathbb{C}^\times \times \mathbb{Z} \text{Ann}_c(S)$ be an arbitrary lift. Then the actions of \tilde{A} and ψ on H_0 given by (13) and (17) agree up to scalar.*

Proof. Write S_ψ for the isomorphism $F_{A_\psi} F_{\mathbb{D}}(\mathbb{C}) \xrightarrow{\cong} F_{A_\psi \cup \mathbb{D}}(\mathbb{C}) \xrightarrow{\cong} F_{\mathbb{D}}(\mathbb{C})$. By definition, the actions of \tilde{A} and ψ are given by $U(T_{\tilde{A}}) \circ Z_A$ and $U(S_\psi) \circ Z_A$, respectively. Since

$\mathbf{1}_{\mathcal{C}} = F_{\mathbb{D}}(\mathbb{C}) \in \mathcal{C}(S^1)$ is a simple object, there exists a constant $a \in \mathbb{C}^{\times}$ such that $T_{\tilde{A}} = a \cdot S_{\psi}$. It follows that $U(T_{\tilde{A}}) \circ Z_A = a \cdot U(S_{\psi}) \circ Z_A$. \square

Summarizing, we have the following four (semi)groups which all act compatibly on the vacuum sector of a CFT. The ones in the top row act honestly (i.e., without central extension), whereas the ones in the bottom row only act projectively:



Let ${}^{\mathbb{Z}}PSU(1,1)$ denote the universal cover of $PSU(1,1)$, which is also its universal central extension. Similarly, let us write ${}^{\mathbb{Z}}Univ(\mathbb{D})$ for the universal cover of $Univ(\mathbb{D})$. The inclusion $Univ(\mathbb{D}) \hookrightarrow Ann(S)$ induces a map of central extensions

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & {}^{\mathbb{Z}}Univ(\mathbb{D}) & \longrightarrow & Univ(\mathbb{D}) \longrightarrow 0 \\
& & \downarrow \scriptstyle \begin{array}{c} n \\ \downarrow \\ (0,n) \end{array} & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{C}^{\times} \oplus \mathbb{Z} & \longrightarrow & \mathbb{C}^{\times} \oplus \mathbb{Z} Ann_c(S) & \longrightarrow & Ann(S) \longrightarrow 0
\end{array}$$

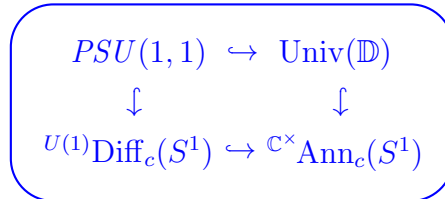
By Lemma 11, the restriction of the action of $\mathbb{C}^{\times} \oplus \mathbb{Z} Ann_c(S)$ on H_0 to the subsemigroup ${}^{\mathbb{Z}}Univ(\mathbb{D})$ descends to the quotient $Univ(\mathbb{D})$. So the action of $\mathbb{C}^{\times} \oplus \mathbb{Z} Ann_c(S)$ on H_0 descends to

$$\mathbb{C}^{\times} Ann_c(S) := (\mathbb{C}^{\times} \oplus \mathbb{Z} Ann_c(S)) / \mathbb{Z}.$$

Let also:

$$U(1)Diff_c(S) := (U(1) \oplus \mathbb{Z} Diff_c(S)) / \mathbb{Z}.$$

All in all, *the following (semi)groups act on the vacuum sector of any chiral CFT:*



Remark. *In the presence of spin structures, all these (semi)groups get replaced by their double covers. In particular, the group $PSU(1,1)$ of Möbius transformations gets replaced by its double cover $SU(1,1)$.*

The above diagrams should be contrasted with the case of charged sectors, where it's only the following (semi)groups which act:

$$\begin{array}{ccc} \mathbb{Z}PSU(1,1) & \hookrightarrow & \mathbb{Z}\text{Univ}(\mathbb{D}) \\ \downarrow & & \downarrow \\ U(1) \oplus \mathbb{Z}\text{Diff}_c(S^1) & \hookrightarrow & \mathbb{C}^\times \oplus \mathbb{Z}\text{Ann}_c(S^1) \end{array}$$

Let $\lambda \in \mathcal{C}(S^1)$ be a simple object, and let $H_\lambda = U(\lambda)$ be the corresponding charged sector. For \tilde{A} in the kernel of the map $\mathbb{C}^\times \oplus \mathbb{Z}\text{Ann}(S) \rightarrow \text{Ann}(S)$,

$$\tilde{A} \in \ker(\mathbb{C}^\times \oplus \mathbb{Z}\text{Ann}(S) \rightarrow \text{Ann}(S)) \cong \mathbb{C} \oplus \mathbb{Z},$$

since F_A and Z_A are trivial, the action (13) of \tilde{A} on H_λ simplifies to $\tilde{A} \mapsto U(T_{\tilde{A}})$, where moreover $T_{\tilde{A}} : \lambda \rightarrow \lambda$ is just a scalar. To recapitulate, by (13), for any simple object $\lambda \in \mathcal{C}(S^1)$, the semigroup

$$\mathbb{C} \oplus \mathbb{Z}\text{Ann}(S)$$

acts on the corresponding charged sector H_λ . The central \mathbb{C} acts via the character $z \mapsto e^{cz}$, where c is the **central charge**. This is an invariant of the chiral CFT and does not depend on the choice of sector. The central \mathbb{Z} acts via some character $n \mapsto (\theta_\lambda)^n$, where θ_λ is called the **conformal spin** of the sector (in a rational CFT, the conformal spins are always roots of unity). This number does depend on the sector (for example, the conformal spin of the vacuum sector is always trivial). Let L_0 be the infinitesimal generator of rotations (we'll be more specific about this later). This operator always has spectrum bounded from below, its smallest eigenvalue is denoted h_λ and called the **minimal energy**. It satisfies $e^{2\pi i h_\lambda} = \theta_\lambda$.

The fusion product

Let $\mathcal{C} := \mathcal{C}(S^1)$. When $\Sigma = \text{img}$ is a pair of pants, with $\partial_{in} = S^1 \sqcup S^1$ and $\partial_{out} = S^1$, we may identify the linear functor $F_\Sigma : \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C}$ with a bilinear functor

$$\begin{aligned} \boxtimes &:= F_{\text{img}} : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \\ &(\lambda, \mu) \mapsto \lambda \boxtimes \mu \end{aligned}$$

is called the *fusion product*. It comes equipped with a bilinear map

$$Z_{\text{img}} : U(\lambda) \times U(\mu) \rightarrow U(\lambda \boxtimes \mu)$$

between the underlying vector spaces (in much the same way as the tensor product $M \otimes_R N$ of two modules M and N over a commutative ring R comes equipped with a bilinear map $M \times N \rightarrow M \otimes_R N$).

The bilinear map Z_Σ genuinely depends on the complex structure on the pair of pants Σ . But the fusion product F_Σ is essentially independent of that complex structure.

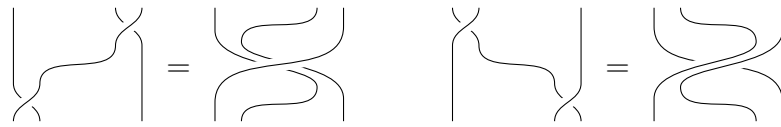
For simplicity, let us use pairs of pants Σ embedded in \mathbb{C} , where the boundary circles are round, and parametrized by $z \mapsto az + b$ with $a, b \in \mathbb{R}$:

$$\Sigma = \text{Diagram of a pair of pants with two inner holes and one outer boundary} \quad (18)$$

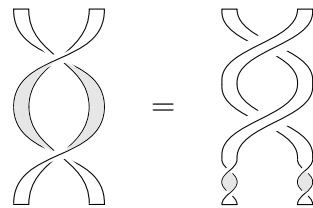
Note that if $A = \text{Diagram of an annulus}$ is an annulus with boundary parametrized by $z \mapsto az + b$ with a and b real, then F_A is canonically trivialized. The trivialization is given by $T_{\tilde{A}}$, where $\tilde{A} \in \text{Univ}(\mathbb{D})^{\mathbb{Z}}$ is the canonical lift of $A \in \text{Univ}(\mathbb{D})$ to an element of the universal cover (using that a and b are real). By composing and un-composing a pair of pants (18) with such annuli, one can reach any other pair of pants of the form (18) in a way which is unique ‘up to homotopy’¹⁵. So the fusion product is well-defined, canonically up to canonical isomorphism.

The fusion product is visibly associative and unital, and it endows $\mathcal{C}(S^1)$ with the structure of a monoidal category. But it is more. It’s also braided and balanced.

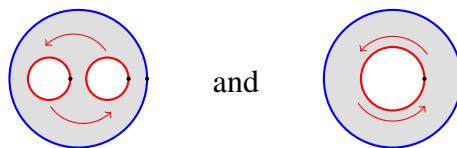
Definition: A monoidal category (\mathcal{C}, \otimes) is *braided* if it’s equipped with a family of natural isomorphisms $\beta_{\lambda, \mu} : \lambda \otimes \mu \rightarrow \mu \otimes \lambda$ that satisfy the two hexagon axioms:



Definition: A braided monoidal category $(\mathcal{C}, \otimes, \beta)$ is *balanced* if it’s equipped with a family of natural isomorphisms $\theta_\lambda : \lambda \rightarrow \lambda$ that satisfy $\theta_{\lambda \otimes \mu} = \beta_{\mu, \lambda} \circ \beta_{\lambda, \mu} \circ (\theta_\lambda \otimes \theta_\mu)$. The isomorphism θ_λ is called the *twist*, and is denoted graphically by the full twist of a ribbon. In that graphical notation, the above axiom becomes:



In terms of circles with holes, the braiding β and the twist θ correspond to the motions



¹⁵ \triangle There is a gap in the argument here.

In order to deal with such motions, it's important to relax the condition that the boundary parametrizations be of the form $z \mapsto az + b$ with $a, b \in \mathbb{R}$, and also allow $a, b \in \mathbb{C}$. (The little black dots in the above picture are indicators of where $1 \in S^1$ goes under the boundary parametrizations.)

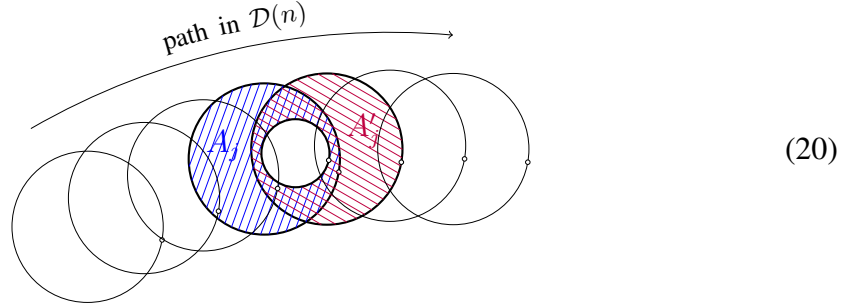
Let $\mathcal{C} := \mathcal{C}(S^1)$, and let's introduce the following moduli space:

$$\mathcal{D}(n) := \left\{ \begin{array}{l} n \text{ non-overlapping round circles in } \mathbb{D} \text{ with} \\ \partial \text{ parametrized by } z \mapsto az + b \text{ with } a, b \in \mathbb{C}. \end{array} \right\} \quad (19)$$

For each disc configuration $P \in \mathcal{D}(n)$, we get a functor $F_P : \mathcal{C}^n \rightarrow \mathcal{C}$, compatibly with composition. Let us write $P \prec P'$ if every circle of P is contained in the corresponding circle of P' .

Claim: for each homotopy class of path $\gamma : [0, 1] \rightarrow \mathcal{D}(n)$ from P_1 to P_2 , there is an associated invertible natural transformation $T_\gamma : F_{P_1} \rightarrow F_{P_2}$.

The construction of T_γ goes as follows. Subdivide $[0, 1]$ into small intervals $[t_i, t_{i+1}]$ and let $P_i := \gamma(t_i)$. If the subdivision is fine enough, the circles of P_i and of P_{i+1} will have large overlaps. Pick $P'_i \in \mathcal{D}(n)$ such that $P_i \succ P'_i \prec P_{i+1}$, and write $P'_i = P_i \cup (A_1 \sqcup \dots \sqcup A_n)$ and $P'_i = P_{i+1} \cup (A'_1 \sqcup \dots \sqcup A'_n)$ for suitable annuli A_j and A'_j . Provided we pick P'_i close enough to P_i and to P_{i+1} , the annuli $A_j, A'_j \in \text{Univ}(\mathbb{D})$ come with preferred lifts $\tilde{A}_j, \tilde{A}'_j$ to the universal cover of $\text{Univ}(\mathbb{D})$. We go from F_{P_i} to $F_{P'_i} = F_{P_i} \circ F_{A_1 \sqcup \dots \sqcup A_n}$ by composing with the trivializations $T_{\tilde{A}_j}$, and we then go back to $F_{P_{i+1}}$ by composing with the trivializations $T_{\tilde{A}'_j}$.



By finely triangulating the domain of a homotopy $h : [0, 1]^2 \rightarrow \mathcal{D}(n)$ between two paths γ_0 and γ_1 from P_1 to P_2 and playing a similar game as above, we can see that $T_\gamma : F_{P_1} \rightarrow F_{P_2}$ only depends on the homotopy class of γ .

The above arguments show that \mathcal{C} is not only monoidal, but also braided, and balanced. But it's even more:

Definition: A monoidal category (\mathcal{C}, \otimes) is called *rigid* if every object $\lambda \in \mathcal{C}$ has a left dual and a right dual. Here, a left dual is an object $\lambda^\vee \in \mathcal{C}$ together with maps $\text{ev} : \lambda^\vee \otimes \lambda \rightarrow 1$ and $\text{coev} : 1 \rightarrow \lambda \otimes \lambda^\vee$ satisfying $(1_\lambda \otimes \text{ev}) \circ (\text{coev} \otimes 1_\lambda) = 1_\lambda$ and $(\text{ev} \otimes 1_{\lambda^\vee}) \circ (1_{\lambda^\vee} \otimes \text{coev}) = 1_{\lambda^\vee}$. Right duals are defined similarly. Even though this is not obvious from the definition, being rigid is just a property (it's not extra structure). In other words, if an object has a dual (say a left dual), then any two duals are canonically isomorphic.

Definition: A braided tensor category is called *ribbon* if it is balanced, rigid, and for every object $\lambda \in \mathcal{C}$, we have $\text{ev} \circ (\theta_{\lambda^\vee} \otimes 1_\lambda) = \text{ev} \circ (1_{\lambda^\vee} \otimes \theta_\lambda)$.

Definition: A braided tensor category is called *modular* if it is ribbon and the S -matrix $[\langle \otimes \rangle]_{\lambda\mu}$ is invertible.

The category $\mathcal{C} = \mathcal{C}(S^1)$ that a chiral CFT assigns to a circle is always modular, but this is hard to prove¹⁶. Showing that \mathcal{C} is rigid is already very hard. It has been conjectured that the mere fact that \mathcal{C} is part of a modular functor (the left column of the table on page 20) should imply that \mathcal{C} is modular. But this remains an open question.

Remark. The fact that the circles were round in the definition of $\mathcal{D}(n)$ is not so important. What is important is that the parametrizations of the incoming circles extend to holomorphic maps on \mathbb{D} , so as to have A_j, A'_j in $\text{Univ}(\mathbb{D})$ in (20).

Loop groups

There is another class of infinite dimensional Lie groups which are very important in conformal field theory, and to which Proposition 9 readily applies: loop groups.

Fix:

- A finite dimensional, compact, simple, simply connected Lie group G , called the *gauge group*.
- A positive integer $k \in \mathbb{N}$, called the *level*.

The *loop group* of G is the group of smooth maps from S^1 to G :

$$LG := \text{Map}_{C^\infty}(S^1, G),$$

and its Lie algebra $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ is called the *loop algebra*. Let $\omega : L\mathfrak{g} \times L\mathfrak{g} \rightarrow i\mathbb{R}$ be the cocycle given by

$$\omega(f, g) := \frac{1}{2\pi i} \int_{S^1} \langle f, dg \rangle, \quad (21)$$

where $\langle \cdot, \cdot \rangle$ is *minus*¹⁷ the *basic inner product* on \mathfrak{g} . For $\mathfrak{g} = \mathfrak{su}(2)$ (also for $\mathfrak{su}(n)$), this is given by $\langle X, Y \rangle = -\text{tr}(XY)$. For other simple Lie algebras, it is the smallest G -invariant inner product whose restriction to any $\mathfrak{su}(2) \subset \mathfrak{g}$ is a positive integer multiple of the above inner product of $\mathfrak{su}(2)$.

Here, $\langle f, dg \rangle \in \Omega^1(S^1)$ is a somewhat peculiar notation. It denotes the image of $f dg \in \Omega^1(S^1; \mathfrak{g} \otimes \mathfrak{g})$ under the map $\Omega^1(S^1; \mathfrak{g} \otimes \mathfrak{g}) \rightarrow \Omega^1(S^1; \mathbb{R})$ induced by $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{R}$.

¹⁶There is a proof by Y.-Z. Huang in the language of VOAs, and a proof by Kawahigashi–Longo–Mueger in the language of conformal nets.

¹⁷The ‘basic inner product’ is an invariant bilinear form on $\mathfrak{g}_{\mathbb{C}}$ whose restriction to \mathfrak{g} is negative definite. We prefer to write the formula (21) in terms of an inner product on \mathfrak{g} which is positive definite.

Let's quickly check that ω is a 2-cocycle:

$$\oint \langle [f, g], dh \rangle \stackrel{\text{integration by parts}}{=} - \oint \langle [df, g], h \rangle - \oint \langle [f, dg], h \rangle = - \oint \langle df, [g, h] \rangle + \oint \langle dg, [f, h] \rangle \quad \checkmark$$

$\langle \cdot, \cdot \rangle$ is G -invariant $\Leftrightarrow \langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle \quad \forall X, Y, Z \in \mathfrak{g}$

This 2-cocycle induces a central extension $\widetilde{L\mathfrak{g}}$ of the loop algebra $L\mathfrak{g}$. The complexification $\widetilde{L\mathfrak{g}}_{\mathbb{C}}$ of this central extension is (the C^∞ completion of) the *affine Lie algebra*. That same Lie algebra goes by various names: it also called the *current algebra*, and also the *affine Kac-Moody algebra*¹⁸.

Theorem 12 *The second continuous cohomology $H_{cts}^2(L\mathfrak{g}, \mathbb{R})$ is one dimensional, and the cocycle $(f, g) \mapsto \int_{S^1} \langle f, dg \rangle$ represents a generator.*

As a corollary, we learn that $\widetilde{L\mathfrak{g}}$ is a universal central extension of $L\mathfrak{g}$.

Proof. By the same argument as in the proof of Theorem 7,

$$H_{cts}^2(L\mathfrak{g}; \mathbb{R}) = \frac{\{ G\text{-invariant 2-cocycles} \}}{\{ d\mu \mid \mu : L\mathfrak{g} \rightarrow \mathbb{R}, \mu \text{ is } G\text{-invariant} \}}$$

where G acts of $L\mathfrak{g}$ by its adjoint action on \mathfrak{g} . The space of G -invariant linear functionals $\mu : L\mathfrak{g} \rightarrow \mathbb{R}$ is trivial, so all we need to show is that the space of G -invariant 2-cocycles is one dimensional. We already know that it's at least one dimensional. So we need to show that it's at most one dimensional. At this point, it becomes convenient to complexify. Given $X \in \mathfrak{g}_{\mathbb{C}}$, let us introduce the notation X_n for $Xz^n \in L\mathfrak{g}_{\mathbb{C}}$. The Lie bracket of such elements is given by $[X_m, Y_n] = [X, Y]_{m+n}$.

Let ω be a G -invariant 2-cocycle. By continuity, ω is entirely determined by its restriction to the X_n 's. Write $c_{m,n}$ for the map $X, Y \mapsto \omega(X_m, Y_n)$. Since ω is G -invariant, so is $c_{m,n}$. The space of G -invariant bilinear forms $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \rightarrow \mathbb{C}$ is one dimensional, spanned by (minus) the basic inner product. So the $c_{m,n}$ are multiples of the basic inner product. In particular, they satisfy $c_{m,n}(X, Y) = c_{m,n}(Y, X)$. By the antisymmetry of ω , we then have

$$c_{m,n}(X, Y) = \omega(X_m, Y_n) = -\omega(Y_n, X_m) = -c_{n,m}(Y, X) = -c_{n,m}(X, Y).$$

So $c_{m,n} = -c_{n,m}$.

Since ω is a 2-cocycle, the $c_{m,n}$ satisfy

$$\begin{aligned} 0 &= c_{m+n,p}([X, Y], Z) + c_{p+m,n}([Z, X], Y) + c_{n+p,m}([Y, Z], X) \\ &\stackrel{\text{because the } c_{m,n} \text{ are } G\text{-invariant}}{=} c_{m+n,p}([X, Y], Z) + c_{p+m,n}([X, Y], Z) + c_{n+p,m}([X, Y], Z) \\ &\stackrel{\text{commutators span } \mathfrak{g}_{\mathbb{C}}}{\Rightarrow} c_{m+n,p} + c_{p+m,n} + c_{n+p,m} = 0. \end{aligned}$$

Setting $m = n = 0$, we get

$$c_{0,p} + 2c_{p,0} = 0 \quad \Rightarrow \quad c_{0,p} = 0, \quad \forall p.$$

¹⁸To be precise, the term 'affine Kac-Moody algebra' usually refers to the semi-direct product $\widetilde{L\mathfrak{g}}_{\mathbb{C}} \rtimes \mathbb{C}$, and 'current algebra' sometimes refers to the non-centrally-extended algebra.

Setting $n = 1$ and $p = r - (m + 1)$, we get

$$c_{m+1,r-(m+1)} = c_{1,r-1} + c_{m,r-m} \Rightarrow c_{m,r-m} = m \cdot c_{1,r-1}, \quad \forall m.$$

Setting $m = r$ in the last equation, we get

$$0 = c_{r,r-r} = r \cdot c_{1,r-1} \Rightarrow c_{1,r-1} = 0, \quad \forall r \neq 0.$$

So $c_{m,n} = 0$ when $n \neq -m$, and $c_{m,-m} = m \cdot c_{1,-1}$. The cocycle ω is therefore entirely determined by the value of $c_{1,-1}$, and the space of G -invariant 2-cocycles is at most one dimensional. \square

We now wish to apply Proposition 9 to the cocycle (21).

Unlike $\text{Diff}(S^1)$, whose fundamental group was non-trivial but whose higher homotopy groups were all trivial, the loop group LG is simply connected but has lots of non-trivial higher homotopy groups. We care about $\pi_2(LG)$. As a manifold, LG is diffeomorphic to $G \times \Omega G$, where ΩG denotes the *based* loop group of G . So

$$\pi_2(LG) = \pi_2(G \times \Omega G) = \underbrace{\pi_2(G)}_{=\pi_1(\Omega G)=0} \times \underbrace{\pi_3(G)}_{=\pi_2(\Omega G)=\mathbb{Z}} = \mathbb{Z}.$$

[The computations $\pi_1(\Omega G) = 0$ and $\pi_2(\Omega G) = \mathbb{Z}$ are rather non-trivial. They can be done by applying Morse theory to ΩG , with respect to a suitable Morse function. (A suitable Morse function can be obtained by taking the energy functional $\gamma \mapsto \int_{S^1} \|\gamma^{-1}\gamma'\|^2$, which is not Morse, and deforming it a bit.) One the other hand, they're fairly easy for $G = SU(n)$ by using the fiber sequences $SU(n-1) \rightarrow SU(n) \rightarrow S^{2n-1}$ and the fact that $SU(2) = S^3$.]

To go further, we need to compute the group of periods of $\underline{\omega}$. The answer turns out to be that the **periods of $\underline{\omega}$** are equal to $2\pi i\mathbb{Z} \subset i\mathbb{R}$ (we'll do that computation in a moment). So, by Proposition 9, we get a central extension of LG by $i\mathbb{R}/2\pi i\mathbb{Z} = U(1)$, which is also its universal central extension. We call it the *level 1 central extension* of the loop group, and denote it \widetilde{LG} . The level k central extension is then obtained as a pushout:

$$\begin{array}{ccccccc} 0 & \longrightarrow & U(1) & \longrightarrow & \widetilde{LG} & \longrightarrow & LG \longrightarrow 0 \\ & & \downarrow \scriptstyle z \\ & & U(1) & \longrightarrow & \widetilde{LG}_k & \longrightarrow & LG \longrightarrow 0 \end{array}$$

The cocycle which is adapted to the canonical basis element of the Lie algebra of $U(1) \subset \widetilde{LG}_k$ is given by

$$\omega_k(f, g) := \frac{k}{2\pi i} \int_{S^1} \langle f, dg \rangle$$

where $\langle \cdot, \cdot \rangle$ is minus the basic inner product on \mathfrak{g} . We write $\widetilde{L\mathfrak{g}}_k$ for the corresponding central extension of $L\mathfrak{g}$. (It is isomorphic to $\widetilde{L\mathfrak{g}}$ as a mere Lie algebra, but not as a central extension of $L\mathfrak{g}$ by $i\mathbb{R}$.)

Let us now compute the periods of $\underline{\omega}$. For any simple group G , there is a homomorphism $SU(2) \rightarrow G$ that represents a generator of $\pi_3(G)$ (recall that $SU(2) \cong S^3$). Moreover, the restriction of minus the basic inner product of G is minus the basic inner product of $SU(2)$. So it's enough to deal with the case $G = SU(2)$. Let's identify $G = SU(2)$ with the group $\{q \in \mathbb{H} : |q| = 1\}$ of unit quaternions via (the \mathbb{R} -linear extension of)

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k \leftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ corresponds to the space of imaginary quaternions, and minus the basic inner product is given by $\langle i, i \rangle = \langle j, j \rangle = \langle k, k \rangle = 2$.

Trying to integrate $\underline{\omega}$ over a generator of $\pi_2(LG)$ is a nightmare of a computation. But integrating it over *twice* a generator turns out to be feasible. Let us model S^2 as the space $\{q = ai + bj + ck : |q|^2 = 1\}$ of unit imaginary quaternions, and let us take S^1 to be $[0, 2\pi]/\sim$. The following map represents twice a generator of $\pi_2(LG)$:

$$h : S^2 \rightarrow LG$$

$$h(q) := (\theta \mapsto \cos(\theta) + q \sin(\theta)).$$

By symmetry considerations, the 2-form $h^*\underline{\omega}$ is a constant multiple of the standard volume form of S^2 . So it's enough to consider what happens at the point $i \in S^2$. The tangent space at i is spanned by j and k , and one easily computes $\frac{\partial h}{\partial j}(\theta) = j \sin(\theta)$ and $\frac{\partial h}{\partial k}(\theta) = k \sin(\theta)$. Translating back to the origin, we get

$$h^{-1} \frac{\partial h}{\partial j}(\theta) = (\cos \theta - i \sin \theta) j \sin \theta = \frac{1}{2} (j \sin 2\theta + k(\cos 2\theta - 1))$$

$$h^{-1} \frac{\partial h}{\partial k}(\theta) = (\cos \theta - i \sin \theta) k \sin \theta = \frac{1}{2} (k \sin 2\theta + j(1 - \cos 2\theta))$$

So *twice* the smallest period of $2\pi i \underline{\omega}$ is given by

$$\begin{aligned} & \text{vol}(S^2) \cdot \int_0^{2\pi} \left\langle h^{-1} \frac{\partial h}{\partial j}, \frac{d}{d\theta} \left(h^{-1} \frac{\partial h}{\partial k} \right) \right\rangle d\theta \\ &= 4\pi \int_0^{2\pi} \left\langle \frac{1}{2} (j \sin 2\theta + k(\cos 2\theta - 1)), \frac{d}{d\theta} \frac{1}{2} (k \sin 2\theta + j(1 - \cos 2\theta)) \right\rangle \\ &= 2\pi \int_0^{2\pi} \left\langle j \sin 2\theta + k(\cos 2\theta - 1), k \cos 2\theta + j \sin 2\theta \right\rangle \\ &= 2\pi \int_0^{2\pi} \sin^2(2\theta) \langle j, j \rangle + \cos^2(2\theta) \langle k, k \rangle = 2\pi \int_0^{2\pi} 2 = 8\pi^2. \end{aligned}$$

So the periods of $2\pi i \underline{\omega}$ are $4\pi^2 \mathbb{Z}$, and the periods of $\underline{\omega}$ are $2\pi i \mathbb{Z} \subset i\mathbb{R}$.

Canonical anticommutation relations

Let $S^1 \subset \mathbb{C}$ denote the standard unit circle, equipped with its standard spin structure \mathbb{S} inherited from the trivial spin structure on \mathbb{C} . We write $f(z)\sqrt{dz}$ for a section of \mathbb{S} (where \sqrt{dz} is a formal symbol), and the isomorphism $\mathbb{S}^{\otimes 2} \cong T^*S$ is given by $f(z)\sqrt{dz} \otimes g(z)\sqrt{dz} \mapsto f(z)g(z)dz$. Let $\Gamma(\mathbb{S}) = \Gamma(S^1, \mathbb{S})$ denote the space of spinors fields on the

circle.

Definition: The algebra $CAR(S^1)$ of **C**anonical **A**nticommutation **R**elations is given by:

Generators: There is one generator $c(f)$ for every section $f \in \Gamma(\mathbb{S})$ and the symbol $c(f)$ depends linearly on f , namely, $c(f + g) = c(f) + c(g)$ and $c(\lambda f) = \lambda c(f)$ for $\lambda \in \mathbb{C}$.

For any sections $f, g \in \Gamma(\mathbb{S})$, we have:

Relations:
$$[c(f), c(g)]_+ = \frac{1}{2\pi i} \int_{S^1} fg$$
 where $[\cdot, \cdot]_+$ is the anticommutator $[A, B]_+ := AB + BA$. Here, fg is viewed as a 1-form via the isomorphism $\mathbb{S}^{\otimes 2} \cong T^*S$.

**-structure:* If we let $f \mapsto \bar{f}$ be the antilinear involution on $\Gamma(\mathbb{S})$ given by $\overline{z^n \sqrt{dz}} := z^{-n-1} \sqrt{dz}$, then we set $c(f)^* := c(\bar{f})$

The way to remember the formula for $f \mapsto \bar{f}$ is to view \sqrt{dz} as some kind of substitute for $z^{\frac{1}{2}}$. The formula $z^n \sqrt{dz} \mapsto z^{-n-1} \sqrt{dz}$ then becomes $z^{n+\frac{1}{2}} \mapsto z^{-(n+\frac{1}{2})}$, which agrees with our intuition about bar for z on the unit circle.

The operation $f \mapsto \bar{f}$ on sections of \mathbb{S} also admits a geometric description, explained in the lemma below. That alternative description makes sense for arbitrary spin 1-manifolds, and thus allows us to define the *-structure on $CAR(S)$, for S an arbitrary spin 1-manifold (not just the standard unit circle).

Lemma. Let $\Gamma(\mathbb{S}) := \Gamma(S^1, \mathbb{S})$. The sections $f \in \Gamma(\mathbb{S})$ that satisfy $\bar{f} = f$ are those whose square pairs positively with every normal outgoing vectors field; the sections $f \in \Gamma(\mathbb{S})$ that satisfy $\bar{f} = -f$ are those whose square pairs positively to every normal ingoing vectors field:

$$\begin{array}{ccc} \bar{f} = f & \Leftrightarrow & f^2 \left(\begin{array}{c} \text{outgoing} \\ \text{vectors} \\ \text{field} \end{array} \right) \geq 0 \quad \forall v \text{ normal outgoing} \\ \text{that's a 1-form under the} & \swarrow & \searrow \\ \text{isomorphism } \mathbb{S}^{\otimes 2} \cong T^* & & \text{that's a function on } S^1 \\ \bar{f} = -f & \Leftrightarrow & f^2 \left(\begin{array}{c} \text{ingoing} \\ \text{vectors} \\ \text{field} \end{array} \right) \geq 0 \quad \forall v \text{ normal ingoing} \end{array}$$

Proof: The condition of pairing positively with normal outgoing vectors defines a ray bundle (a ray is half of a line) inside $T^*\mathbb{D}|_{S^1}$. Its preimage under the squaring map $f \mapsto f^2 : \mathbb{S} \rightarrow T^*\mathbb{D}$ is a real line bundle $\mathbb{S}^+ \subset \mathbb{S}$. Similarly, the condition of pairing positively with normal ingoing vectors defines a ray bundle inside $T^*\mathbb{D}|_{S^1}$ (the negative of the previous ray bundle) whose preimage under the squaring map is a real line bundle $\mathbb{S}^- \subset \mathbb{S}$. Since $(\mathbb{S}^-)^2 = -(\mathbb{S}^+)^2$, we have $\mathbb{S}^- = i\mathbb{S}^+$, and in particular $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$.

Now, $z^n \sqrt{dz} + z^{-n-1} \sqrt{dz}$ and $iz^n \sqrt{dz} - iz^{-n-1} \sqrt{dz}$ form a basis of $\{f \in \Gamma(\mathbb{S}) \mid \bar{f} = f\}$. The standard normal outgoing vector field is $z \partial_z$, and we check:

$$\begin{aligned}(z^n \sqrt{dz} + z^{-n-1} \sqrt{dz})^2 (z \partial_z) &= (z^{2n+1} + 2 + z^{-(2n+1)}) = |1 + z^{2n+1}|^2 \geq 0 \\ (iz^n \sqrt{dz} - iz^{-n-1} \sqrt{dz})^2 (z \partial_z) &= (-z^{2n+1} + 2 - z^{-(2n+1)}) = |1 - z^{2n+1}|^2 \geq 0.\end{aligned}$$

It follows that $\{\bar{f} = f\} \subseteq \Gamma(\mathbb{S}^+)$. Similarly, $iz^n \sqrt{dz} + iz^{-n-1} \sqrt{dz}$ and $z^n \sqrt{dz} - z^{-n-1} \sqrt{dz}$ form a basis of $\{f \in \Gamma(\mathbb{S}) \mid \bar{f} = -f\}$, and since

$$\begin{aligned}(iz^n \sqrt{dz} + iz^{-n-1} \sqrt{dz})^2 (-z \partial_z) &= |1 + z^{2n+1}|^2 \geq 0 \\ (z^n \sqrt{dz} - z^{-n-1} \sqrt{dz})^2 (-z \partial_z) &= |1 - z^{2n+1}|^2 \geq 0,\end{aligned}$$

we have $\{\bar{f} = -f\} \subseteq \Gamma(\mathbb{S}^-)$.

Since $\Gamma(\mathbb{S}) = \{\bar{f} = f\} \oplus \{\bar{f} = -f\}$, we conclude that $\{\bar{f} = f\} = \Gamma(\mathbb{S}^+)$ and $\{\bar{f} = -f\} = \Gamma(\mathbb{S}^-)$. \square

The algebra of canonical anticommutation relations introduced above is the observables of a spin chiral CFT known as the *Majorana Free Fermion*. There is another, closely related spin CFT known as the *Dirac Free Fermion*. They are given by:

	Majorana	Dirac
generators:	$c(f)$ for $f \in \Gamma(\mathbb{S})$	$a(f)$ and $a^\dagger(f)$ for $f \in \Gamma(\mathbb{S})$
relations:	$[c(f), c(g)]_+ = \frac{1}{2\pi i} \int fg$	$[a(f), a(g)]_+ = [a^\dagger(f), a^\dagger(g)]_+ = 0, [a(f), a^\dagger(g)]_+ = \frac{1}{2\pi i} \int fg$
-operation:	$c(f)^ = c(\bar{f})$	$a(f)^* = a^\dagger(\bar{f})$

These two CFTs are related by $(\text{Dirac Free Fermion}) \cong (\text{Majorana Free Fermion})^{\otimes 2}$, and the isomorphism is given by

$$\begin{aligned}a(f) &\longmapsto \frac{1}{\sqrt{2}}(c(f) \otimes 1 + 1 \otimes ic(f)) \\ a^\dagger(f) &\longmapsto \frac{1}{\sqrt{2}}(c(f) \otimes 1 - 1 \otimes ic(f)) \\ \frac{1}{\sqrt{2}}(a(f) + a^\dagger(f)) &\longleftarrow c(f) \otimes 1 \\ \frac{1}{i\sqrt{2}}(a(f) - a^\dagger(f)) &\longleftarrow 1 \otimes c(f).\end{aligned}$$

Examples of chiral CFTs

In this section, we introduce three classes of chiral CFTs:

- the unitary *chiral minimal models*
(there's one such model for every $c = 1 - \frac{6}{m(m+1)}$, for $m = 2, 3, 4, \dots$)
- the *chiral WZW models*
(there's one such model for every choice of gauge group G and level k), and
- the chiral *Majorana free fermion*.

At first, we will describe the linear categories that those models assign to 1-manifolds. In the case of a chiral **minimal model**, the categories $\mathcal{C}(S)$ are given by

$$\mathcal{C}(S) = \text{Rep}(Vir_c(S))$$

where $Vir_c(S)$ denotes the appropriate (universal) central extension of $\mathfrak{X}_{\mathbb{C}}(S)$ by \mathbb{C} . (And we insist that the central \mathbb{C} acts in the standard way.)

In the chiral **WZW model**, these categories are (roughly) given by

$$\mathcal{C}(S) = \text{Rep}(\widetilde{L_S G_k})$$

where $\widetilde{L_S G_k}$ denotes the appropriate central extension of $L_S G := \text{Map}_{C^\infty}(S, G)$ by $U(1)$. (And we insist that the central $U(1)$ acts in the standard way.)

And for the **Majorana free fermion**, this is the representation category of the algebra of canonical anticommutation relations:

$$\mathcal{C}(S) = \text{Rep}(CAR(S)).$$

The above descriptions are not very precise, because we haven't said anything about the class of representations that we're allowing. And without any specifications, those categories are huge. So we *do* need to say a bit more. Let us introduce a couple of technical conditions:

Definition: A representation of Vir_c has *positive energy* if the associated operator L_0 has discrete spectrum, the spectrum is bounded from below, and all the (generalized) eigenspaces are finite dimensional.

In our case of interest, we always have $e^{2\pi i L_0} = \theta_\lambda$ (where θ_λ is the conformal spin). So L_0 is in fact diagonalizable, and there is no need to talk about generalized eigenspaces.

Remark. The operator L_0 is obviously coordinate dependent. However, assuming the action of Vir_c integrates to an action of $U(1) \oplus \mathbb{Z} \text{Diff}_c(S^1)$, the property of being positive energy is independent of the choice of coordinate, because one can conjugate any coordinate into any other coordinate by an element of $\text{Diff}(S^1)$.

[I'm guessing that every positive energy representation of Vir_c integrates to a representation of $U(1) \oplus \mathbb{Z} \text{Diff}_c(S^1)$, and even to a representation of $\mathbb{C}^\times \oplus \mathbb{Z} \text{Ann}_c(S^1)$. But this probably depends on the type of topological vector spaces that one is working with.]

Definition: An irreducible representation of $\widetilde{L_S \mathfrak{g}_k}$ has *positive energy* if it extends to a representation of $\widetilde{L_S \mathfrak{g}_k} \rtimes Vir_c$ for some c , and the Virasoro action has positive energy. A positive energy representation of $\widetilde{L_S \mathfrak{g}_k}$ is a finite direct sum of irreducible positive energy representations of $\widetilde{L_S \mathfrak{g}_k}$.

Remark 13 When working with Hilbert spaces, one should be aware that the actions of $\widetilde{L_S \mathfrak{g}_k}$ and of $Vir_c(S)$ are by *unbounded* operators.

[A representation of $\widetilde{L}\mathfrak{g}_k$ is called integrable if it integrates to a representation of \widetilde{LG}_k . Every integrable positive energy representation of $\widetilde{L}\mathfrak{g}_k$ is infinitesimally equivalent to a unitary representation, and every unitary positive energy representation on a Hilbert space is integrable.]

Definition: Let S be a connected spin 1-manifold. An irreducible representation of $CAR(S)$ has positive energy if it extends to a representation of $CAR(S) \rtimes^{U(1) \oplus \mathbb{Z}} \text{Diff}_c(S)$ for some c (which is necessarily given by $c = 1/2$), and the $\text{Diff}_c(S)$ action has positive energy. A positive energy representation is a finite direct sum of irreducible ones.

Given the above definitions, we can go back and re-define the above representation categories with a little bit more attention to detail. In place of $\text{Rep}(\text{Vir}_c(S))$, we should have written

$$\begin{aligned} \text{Rep}_{\text{en.}}^{\text{unitary}}(\text{Vir}_c(S)) &:= \{\text{positive energy unitary representations of } \text{Vir}_c(S)\} \\ &= \{\text{positive energy unitary representations of } U(1) \oplus \mathbb{Z} \text{Diff}_c(S)\}. \end{aligned}$$

Similarly, in place of $\text{Rep}(\widetilde{L}_S G_k)$, we should have written

$$\begin{aligned} \text{Rep}_{\text{en.}}(\widetilde{L}_S G_k) &:= \{\text{positive energy unitary representations of } \widetilde{L}_S G_k\} \\ &= \{\text{positive energy unitary representations of } \widetilde{L}_S \mathfrak{g}_k\}, \end{aligned}$$

where $\widetilde{L}_S \mathfrak{g}_k$ is the central extension of $L_S \mathfrak{g} := C^\infty(S, \mathfrak{g})$ by $i\mathbb{R}$ defined by the cocycle $\omega_k(f, g) := \frac{k}{2\pi i} \int_S \langle f, dg \rangle$. And finally, for the Majorana free fermion we should have

$$\text{Rep}_{\text{en.}}^{\text{pos}}(CAR(S)) := \{\text{positive energy unitary representations of } CAR(S)\}.$$

Note that in order to formulate the notion of unitarity, one uses the fact that $CAR(S)$ is a $*$ -algebra.

In the case of the WZW and of the free fermion χ CFTs, we can also describe the concrete functor F_Σ associated to a complex cobordism, at least when $\partial\Sigma \neq \emptyset$ (more precisely, when each connected component of Σ has non-empty boundary). In the case of the minimal models, we don't yet have a concrete enough description of the Lie algebra $\text{Vir}_c(S)$ for us to be able to write something along the same lines as below. Let Σ be a connected complex cobordism with $\partial\Sigma \neq \emptyset$, and let $\mathcal{C}_{in/out} := \mathcal{C}(\partial_{in/out}\Sigma)$, so that

$$F_\Sigma : \mathcal{C}_{in} \longrightarrow \mathcal{C}_{out}.$$

Definition 14 Given an object $(V, \rho_V) \in \mathcal{C}_{in}$, its image $(W, \rho_W) \in \mathcal{C}_{out}$ under the functor F_Σ comes equipped with a linear map $Z_\Sigma : V \rightarrow W$ satisfying:

$$\begin{array}{c} \boxed{\text{WZW model}} \\ \downarrow \\ \forall f \in \begin{cases} \mathcal{O}(\Sigma; \mathfrak{g}_\mathbb{C}) \\ \Gamma_{\text{hol}}(\Sigma, \mathbb{S}_\Sigma) \end{cases} \quad Z_\Sigma \circ \rho_V(f_{in}) = \rho_W(f_{out}) \circ Z_\Sigma \\ \uparrow \\ \boxed{\text{free fermion}} \end{array} \quad (22)$$

where $f_{in/out} := f|_{\partial_{in/out}\Sigma}$.

Moreover, (W, ρ_W) and Z_Σ should be universal in the sense that for any $(\tilde{W}, \rho_{\tilde{W}}) \in \mathcal{C}_{out}$ and for any linear map $\tilde{Z} : V \rightarrow \tilde{W}$ satisfying the same relations as above, there should exist a unique morphism $\kappa : W \rightarrow \tilde{W}$ in \mathcal{C}_{out} that makes the following diagram commute:

$$\begin{array}{ccc} & Z_\Sigma & \rightarrow W \\ V & \searrow & \downarrow \kappa \\ & \tilde{Z} & \rightarrow \tilde{W} \end{array}$$

The above universal property defines a functor $F_\Sigma : \mathcal{C}_{in} \rightarrow \mathcal{C}_{out}$. But, unfortunately, does not guarantee that it has any good formal properties. The following problem has been solved by James Tener in his PhD in the case of the free fermion. but remains open in the case of the WZW models:

Open problem: Given composable cobordisms Σ_1 and Σ_2 , prove that the natural map $F_{\Sigma_1 \cup \Sigma_2}(\lambda) \rightarrow F_{\Sigma_1} \circ F_{\Sigma_2}(\lambda)$ is an isomorphism.

Now, why is this difficult?...

Well... for a universal construction to be well behaved, one needs the category in which it takes place to be “big enough”. And, from that point of view, the positive energy condition is very awkward. So what we’d like is to be able to perform the universal construction in a bigger category (one which doesn’t include the the positive energy condition), and have the result automatically satisfy the positive energy condition...

If our cobordism is an annulus $A \in \text{Ann}(S)$, then the trivialization

$$T_{\tilde{A}} : F_A \longrightarrow \text{id}_{\mathcal{C}(S)}$$

associated to a lift $\tilde{A} \in \mathbb{C}^\times \oplus \mathbb{Z} \text{Ann}_c(S)$ is constructed as follows. Let $\mathcal{A}(S)$ denote either $\widetilde{L_S \mathfrak{g}_k}$ or $CAR(S)$, depending on which chiral CFT we’re treating. For every $\lambda \in \mathcal{C}(S)$, by the positive energy condition, the action $\rho : \mathcal{A}(S) \rightarrow \text{End}(U(\lambda))$ extends to an action, again denoted ρ , of $\mathcal{A}(S) \rtimes^{U(1) \oplus \mathbb{Z}} \text{Diff}_c(S)$. By definition, this means that we have actions of $\mathcal{A}(S)$ and of $^{U(1) \oplus \mathbb{Z}} \text{Diff}_c(S)$ on $U(\lambda)$, satisfying the following covariance relation:

$$\rho(\varphi f) = \rho(\varphi) \rho(f) \rho(\varphi^{-1}) \quad \forall f \in \begin{cases} \mathcal{C}^\infty(S, \mathfrak{g}) \\ \Gamma(\mathbb{S}) \end{cases}$$

Here, $f \mapsto \varphi f$ denotes the action of (the image of) φ in $\text{Diff}(S)$ on $\mathcal{C}^\infty(S, \mathfrak{g}) \subset \widetilde{L_S \mathfrak{g}_k}$ or on $\Gamma(\mathbb{S}) \subset CAR(S)$.

Since the action of $^{U(1) \oplus \mathbb{Z}} \text{Diff}_c(S)$ has positive energy, it extends to a holomorphic representation of $\mathbb{C}^\times \oplus \mathbb{Z} \text{Ann}_c(S)$ on $U(\lambda)$.¹⁹ We construct the morphisms $T_{\tilde{A}} : F_A(\lambda) \rightarrow \lambda$

¹⁹This is stated as a theorem in [Y. Neretin. *Holomorphic continuations of representations of the group of diffeomorphisms of the circle*; translation in Math. USSR-Sb. 67 (1990), no. 1, 75–97], but the paper does not include a proof of holomorphicity.

making the following diagram commute

$$\begin{array}{ccc}
 & Z_A & U(F_A(\lambda)) \\
 U(\lambda) & \xrightarrow{\quad} & \downarrow U(T_{\tilde{A}}) \\
 & \rho(\tilde{A}) & U(\lambda)
 \end{array}$$

by applying the universal property in Definition 14 to the object $\lambda \in \mathcal{C}(S)$ and to the map $\rho(\tilde{A}) : U(\lambda) \rightarrow U(\lambda)$. In order to invoke the universal property, we need to check that the relation (22) holds, namely

$$\rho(\tilde{A})\rho(f_{in}) = \rho(f_{out})\rho(\tilde{A}) \quad \forall f \in \left\langle \begin{array}{l} \mathcal{O}(A; \mathfrak{g}_C) \\ \Gamma_{hol}(A, \mathbb{S}_\Sigma) \end{array} \right\rangle. \quad (23)$$

Let $\text{Ann}^{\leq A} = \{A_1 \in \text{Ann}(S) \mid \exists A_2 : A_1 A_2 = A\} = \{\gamma : S \hookrightarrow A \mid \gamma \text{ wraps around } A\}$, and let $\mathbb{C}^\times \oplus \mathbb{Z} \text{Ann}_c^{\leq A}$ be its pre-image in $\mathbb{C}^\times \oplus \mathbb{Z} \text{Ann}_c(S)$.

Claim: The map

$$\begin{aligned}
 \Phi : \mathbb{C}^\times \oplus \mathbb{Z} \text{Ann}_c^{\leq A} &\longrightarrow \text{End}(U(\lambda)) \\
 \tilde{A}_1 &\mapsto \rho(\tilde{A}_1)\rho(f_\gamma)\rho(\tilde{A}_2)
 \end{aligned}$$

is constant. Here, \tilde{A}_2 is the unique solution of the equation $\tilde{A}_1 \tilde{A}_2 = \tilde{A}$, and f_γ is the restriction of f along the map $\gamma : S \rightarrow A$.

Equation (23) follows since $\rho(\tilde{A})\rho(f_{in}) = \Phi(\tilde{A})$ and $\rho(f_{out})\rho(\tilde{A}) = \Phi(1)$.

Proof. The map Φ is holomorphic (continuous, and holomorphic in the interior). If A_1 is completely thin, then

$$\Phi(\tilde{A}_1) = \rho(\tilde{A}_1)\rho(f_\gamma)\rho(\tilde{A}_1^{-1})\rho(\tilde{A}) = \rho(f)\rho(\tilde{A})$$

doesn't depend on \tilde{A}_1 . So Φ is constant on the subset of completely thin A_1 's. The map $\text{Diff}(S) = \{\text{thin } A_1\text{'s}\} \hookrightarrow \text{Ann}^{\leq A}$ is the inclusion of a real manifold into a complexification. We finish by noting that a holomorphic function which is constant on the real submanifold is necessarily constant on the complexification. \square

The positive energy condition

In the previous sections, we motivated the introduction of the positive energy condition by saying ‘‘otherwise, there’s too many representations’’. But there’s a much better reason to include that condition. That’s because, in a functorial CFT, that condition is *forced* on you:

Theorem 15 *In a functorial chiral CFT, the action of the Virasoro algebra on any sector always has positive energy.*

The proof will be based on the following result:

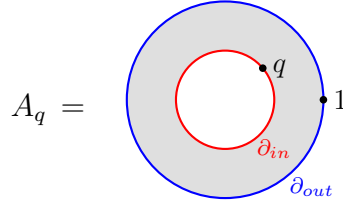
Proposition 16 *If Σ is a complex cobordism, then $Z_\Sigma : U(\lambda) \rightarrow U(F_\Sigma(\lambda))$ is a trace-class operator.*

We'll give the definition of trace-class a bit later. For the moment, we just need to know that, for a diagonal operator on a topological vector space, we have

$$\text{Diag}(\alpha_1, \alpha_2, \dots) \text{ is trace-class} \implies \lim_{n \rightarrow \infty} \alpha_n = 0.$$

When our vector space is a Hilbert space, we have the much stronger result that an operator of the form $\text{Diag}(\alpha_1, \dots)$ is trace-class iff $\sum |\alpha_n| < \infty$. But things don't work quite as nicely when dealing with more general types of vector spaces. *[The implication $\sum |\alpha_n| < \infty \implies \text{Diag}(\alpha_1, \dots)$ is trace-class only holds when the diagonal operator is defined w.r.t. an unconditional basis. The implication $\text{Diag}(\alpha_1, \dots)$ trace-class $\implies \sum |\alpha_n| < \infty$ almost never holds.]*

Proof of Theorem 15. For q a complex number, $|q| < 1$, let $A_q := \{z \in \mathbb{C} : |q| \leq |z| \leq 1\}$, with boundary parametrizations $\varphi_{in} : z \mapsto qz : S^1 \rightarrow \partial_{in}A$ and $\varphi_{out} = \text{id}_{S^1}$:



Pick a lift of $\tilde{A}_q \in {}^{\mathbb{Z}}\text{Univ}(\mathbb{D})$ of A_q to the universal cover of $\text{Univ}(\mathbb{D})$. Equivalently, pick a logarithm of q . The corresponding operator on $U(\lambda)$ is then given by

$$\rho(\tilde{A}_q) = U(T_{\tilde{A}_q}) \circ Z_{A_q} = q^{L_0} := e^{\log(q)L_0}.$$

The operator $U(T_{\tilde{A}_q}) \circ Z_{A_q}$ is trace-class by Proposition 16. In particular, its sequence of eigenvalues (counted with multiplicity) tends to zero. This is equivalent to the spectrum of L_0 being discrete, bounded from below, and all its eigenspaces being finite dimensional. \square

Before discussing the proof of Proposition 16, we recall some definitions from functional analysis. From now on, we assume that all our vector spaces are complete locally convex topological vector spaces.²⁰

Definition: An operator $f : V \rightarrow W$ is trace-class²¹ if it is in the image of the map

$$E : W \otimes_{\pi} V' \rightarrow \mathcal{L}(V, W).$$

²⁰A topology is called 'locally convex' if it is generated by a set of (semi-)norms.

²¹When working with Hilbert spaces, one typically uses the term 'trace class'. When working with more general topological vector spaces, one typically uses the word 'nuclear' for that same notion.

Here, V' is the continuous dual of V (the set of continuous linear maps $V \rightarrow \mathbb{C}$), and \otimes_π is the projective tensor product of topological vector spaces (defined by the universal property that continuous bi-linear maps out of the product are the same thing as continuous linear maps out of the tensor product). If $f : V \rightarrow V$ is trace-class, then its trace $\text{tr}(f) \in \mathbb{C}$ is the image of $E^{-1}(f)$ under the evaluation map $V \otimes V' \cong V' \otimes V \xrightarrow{ev} \mathbb{C}$.

Warning: When working with general topological vector spaces, the map E can fail to be injective; this can already happen with Banach spaces. When this happens, $\text{tr}(f)$ typically fails to be well defined. But everything is ok (i.e., the map E is injective) when the spaces have bases. (A subset $(b_n)_{n \in \mathbb{N}}$ of a topological vector space V is a basis if for every $v \in V$ there is a unique sequence of numbers (a_n) such that $v = \sum a_n b_n$.)

Lemma 17 A linear map $f : V \rightarrow W$ is trace-class if and only if there exists a space X , and linear maps $a : \mathbb{C} \rightarrow W \otimes_\pi X$, and $b : X \otimes_\pi V \rightarrow \mathbb{C}$, such that

$$f = \left(V \xrightarrow{a \otimes \text{id}} W \otimes X \otimes V \xrightarrow{\text{id} \otimes b} W \right).$$

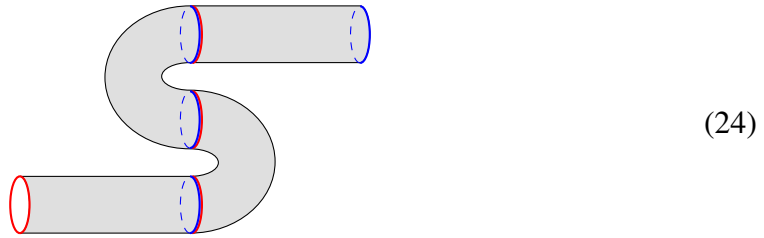
Proof. \Rightarrow : Take $X = V'$, $a = E^{-1}(f)$, and $b = ev : V' \otimes V \rightarrow \mathbb{C}$.

\Leftarrow : b induces a map $\tilde{b} : X \rightarrow V'$. Then f is the image of $(\text{id}_W \otimes \tilde{b})a \in W \otimes_\pi V'$. \square

As a corollary, trace-class maps form an ideal: if a map f is trace-class, then so is $f \circ g$, and so is $h \circ f$.

Remark. The statement of Lemma 17 also holds true when V , W , and X are Hilbert spaces, and the projective tensor product is replaced by the Hilbert space tensor product. But the proof is rather different (it relies on the fact that the composition of two Hilbert-Schmidt operators is always trace-class).

The proof of Proposition 16 will be based on the fact that every annulus can be decomposed as follows:



We first compute $F_{\text{red}}(\mathbb{C})$ and $F_{\text{blue}}(\mathbb{C})$. Let $1_{\text{vec}} = \mathbb{C}$ be the unit object of $\mathcal{C}(\emptyset) = \text{Vec}_{\text{f.d.}}$.

Proposition. There exists a canonical involution $\lambda \mapsto \bar{\lambda}$, called *charge conjugation*, on the set of isomorphism classes of simple objects of $\mathcal{C}(S^1)$, such that

$$F_{\text{red}}(\mathbb{C}) = \bigoplus_{\lambda} \lambda \otimes \bar{\lambda} \quad \text{and} \quad F_{\text{blue}}(\mu \otimes \nu) = \delta_{\mu, \bar{\nu}} \mathbb{C}.$$

Proof. Write

$$F_{\mathcal{C}}(\mathbb{C}) = \bigoplus_{\lambda, \mu} a_{\lambda, \mu} \lambda \otimes \mu \quad \text{and} \quad F_{\mathcal{D}}(\mu \otimes \nu) = b_{\mu, \nu} \mathbb{C}.$$

The triviality of $F_{\mathcal{C}}$ means that the matrices $a = (a_{\lambda, \mu})$ and $b = (b_{\mu, \nu})$ satisfy $ab = 1$. Similarly, we have $ba = 1$. Since the entries of a and of b all lie in \mathbb{N} , they are permutation matrices.

The composition of \mathcal{C} with (the cobordism associated to) the diffeomorphism that switches the two boundary circles is isomorphic (rel boundary) to that same cobordism \mathcal{C} . It follows that $(\text{switch}) \circ F_{\mathcal{C}} \simeq F_{(\text{switch})} \circ F_{\mathcal{C}} \simeq F_{(\text{switch}) \cup \mathcal{C}} \simeq F_{\mathcal{C}}$, and hence that $a_{\lambda, \mu} = a_{\mu, \lambda}$. Any permutation matrix which is symmetric is an involution. So a and b are involutions, and $a = b$. \square

Proof of Proposition 16. Since trace-class maps form an ideal, and since the tensor product of two trace-class maps is again trace-class, it's enough to show that Z_{Σ} is trace-class when $\Sigma = A$ is an annulus. Cutting A as in (24), we can decompose $Z_{\mathcal{C}}$ as:

$$\begin{array}{ccc} & H_{\lambda} \otimes H_{\bar{\lambda}} \otimes H_{\lambda} & \\ & \uparrow & \searrow \\ H_{\lambda} & \xrightarrow{Z_{\mathcal{C}}} \bigoplus_{\mu} H_{\mu} \otimes \underbrace{H_{\bar{\mu}} \otimes H_{\lambda}}_{\in \ker(F_{\mathcal{D}}) \text{ unless } \mu=\lambda} & \xrightarrow{Z_{\mathcal{D}}} H_{\lambda} \end{array}$$

We are then done by Lemma 17, and the fact that trace-class maps form an ideal. \square

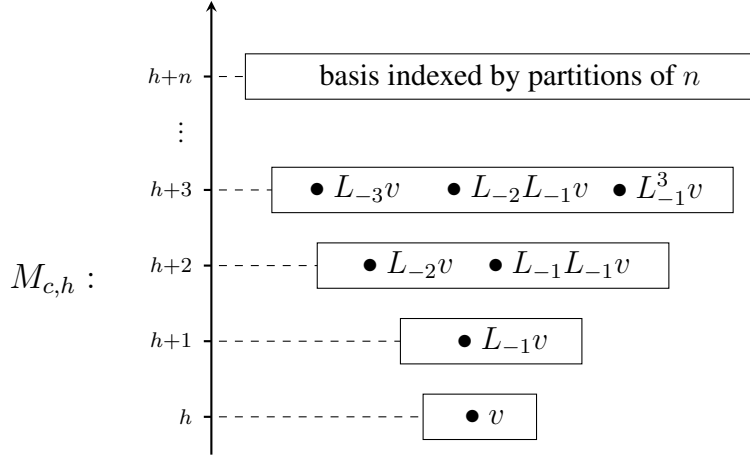
Unitary representations of the Virasoro algebra

Let us explain a bit what the representation theory of the Virasoro algebra looks like. First of all, for every value c of the central charge and h of the minimal energy, one can form the **Verma module**

$$M_{c,h} := \text{Ind}_{\text{Vir}_c^{\geq 0}}^{\text{Vir}_c} \mathbb{C}_{c,h}.$$

Here, $\text{Vir}_c^{\geq 0} := \text{Span}\{L_n\}_{n \geq 0} \oplus 1 \cdot \mathbb{C}$ acts on the one dimensional module $\mathbb{C}_{c,h}$ by $L_0 \mapsto h$, and $L_n \mapsto 0$ for $n > 0$ (and the central element $1 \in \text{Vir}_c^{\geq 0}$ acts by 1). Fix $v \in \mathbb{C}_{c,h}$. The Verma module $M_{c,h}$ is a graded by $\mathbb{N} + h$, with basis given by elements $L_{n_1} L_{n_2} \dots L_{n_k} v$,

for $n_1 \geq \dots \geq n_k > 0$:



The next step towards constructing unitary representations of the Virasoro algebra is to consider the **simple quotients**

$$L_{c,h} := M_{c,h}/J_{c,h},$$

where $J_{c,h} \subsetneq M_{c,h}$ is unique the maximal proper submodule of the Verma module. Equivalently, $J_{c,h}$ can be described as the set of all vectors (in degree $\geq h+1$) that cannot be brought back to a non-zero multiple of v by a sequence of L_m 's (and, by virtue of the Virasoro Lie algebra relations, it is enough to only consider L_m 's with $m > 0$):

$$J_{c,h} = \bigoplus_{\substack{i=h+n, \\ n \in \mathbb{N}_{>0}}} \{ \xi \in M_{c,h}(i) \mid L_{m_1} \dots L_{m_k} \xi = 0, \forall (m_1, \dots, m_k), m_i > 0, \sum m_i = n \}.$$

Here, $M_{c,h}(i)$ denotes the degree i part of the Verma module. The elements of $J_{c,h}$ are called *null-vectors*.

Example: If $h = 0$, then $L_{-1}v$ is a null-vector.

Proof. We compute $L_1 L_{-1}v = [L_1, L_{-1}]v = 2L_0v = 0$. □

Definition: An irreducible representation of the Virasoro algebra Vir_c on a Hilbert space²² is called *unitary* if L_0 is (unbounded) self-adjoint, and $L_n^* = L_{-n}$.

Alternatively, a representation is called unitary if its underlying vector space can be equipped with a positive definite inner product under which $L_n^* = L_{-n}$.

²²Unbounded operators are only densely defined. So, strictly speaking, a Hilbert space is not a representation. The algebraic direct sum of the L_0 -eigenspaces which carries a genuine representations of Vir_c .

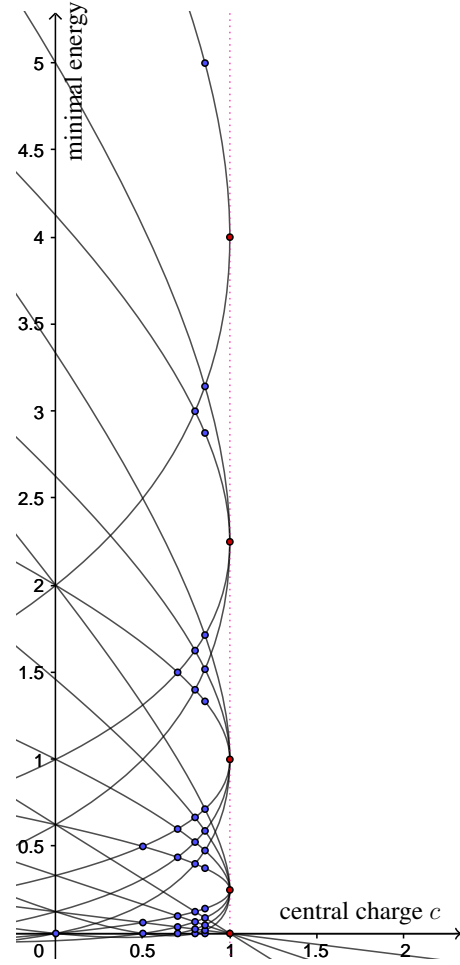
Lemma. *If $L_{c,h}$ is unitary, then $h \geq 0$.*

Proof. We compute $\langle L_{-1}v, L_{-1}v \rangle = \langle L_1 L_{-1}v, v \rangle = \langle 2L_0v, v \rangle = 2h$. \square

It turns out that for $c > 1$ there are no further restrictions. The Verma modules $M_{c,h}$ are simple for all $h > 0$, and the simple quotients $L_{c,h}$ are unitary for all $h \geq 0$. To summarise, for $c > 1$, the irreducible unitary representations of Vir_c are classified by their minimal energy, which can take any value in $\mathbb{R}_{\geq 0}$. The corresponding chiral CFT is not rational, and is called chiral Liouville theory.

If $c = 1$, the chiral CFT is still not rational. It satisfies $M_{1,h} = L_{1,h}$ iff $h \in \mathbb{R}_+ \setminus \{\frac{n^2}{4} | n \in \mathbb{N}\}$, and the simple quotients $L_{1,h}$ are unitary for all $h \geq 0$.

In the range $c < 1$, there exists a discrete set of values of c for which the Virasoro algebra admits unitary representations (outside of that set, Vir_c has no unitary representations). These are the numbers of the form $c = 1 - \frac{6}{m(m+1)}$ for $m = 2, 3, 4, \dots$

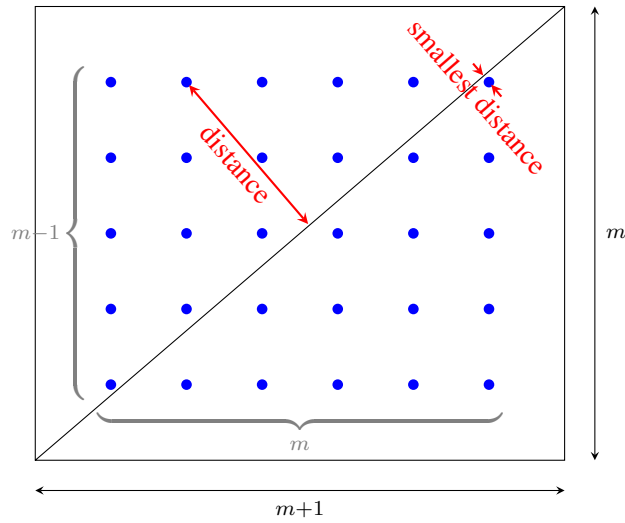


Discrete series unitary irreps of Vir_c .

$$c = 0, \frac{1}{2}, \frac{7}{10}, \frac{4}{5}, \frac{6}{7}, \frac{25}{28}, \frac{11}{12}, \frac{14}{15}, \frac{52}{55}, \frac{21}{22}, \frac{25}{26}, \frac{88}{91}, \dots$$

They correspond to the unitary minimal models. For such a value of the central charge, Vir_c has exactly $m(m-1)/2$ irreducible unitary representations. They are classified by their minimal energy, which can take any value of the form $h_{p,q} := \frac{[(m+1)p-mq]^2-1}{4m(m+1)}$, for $1 \leq p \leq m-1$ and $1 \leq q \leq m$. The curved lines in the above picture are hyperbolas. They are the locus of pairs (c, h) for which the Verma module $M_{c,h}$ admits null-vectors (i.e., such that $J_{c,h} \neq 0$). The blue dots are the unitary simple modules.

A good mnemonic for the above minimal energies $h_{p,q}$ is to note that they're equal to the square-distance to the diagonal minus the smallest square-distance to the diagonal in the following rectangular array of dots:



(normalized so that the smallest square-distance is $\frac{1}{4m(m+1)}$).

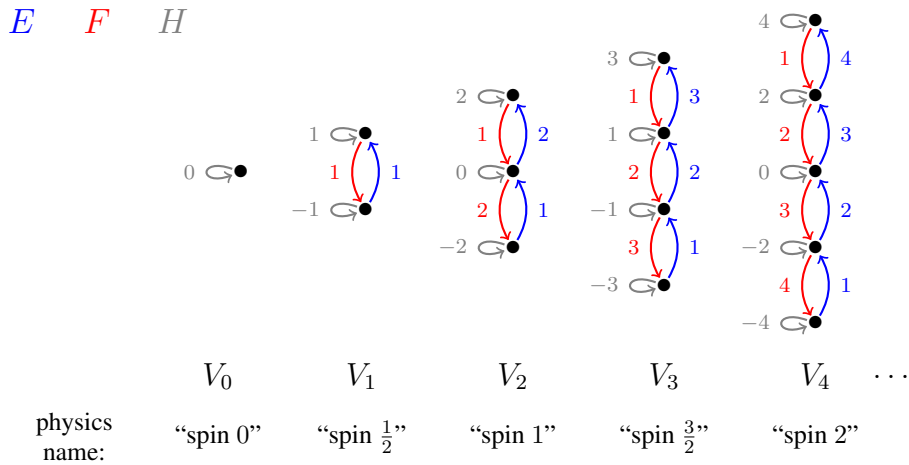
Structure of finite dimensional simple Lie algebras

In this section, we recall various facts from the theory of finite dimensional simple Lie algebras, before treating the more complicated topic of affine Lie algebras. All our Lie algebras will be over \mathbb{C} .

We start by analyzing $\mathfrak{sl}(2)$. It is the smallest simple Lie algebra, and has the property that every other simple Lie algebra is build out of copies of it. It has a basis given by the matrices $E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, subject to the commutation relations

$$[E, F] = H \quad [H, E] = 2E \quad [H, F] = -2F. \quad (25)$$

The irreducible finite dimensional $\mathfrak{sl}(2)$ -reps V_0, V_1, V_2, \dots are classified by their dimension $\dim(V_n) = n + 1$. The next picture is a graphical depiction of these irreps.



Each bullet represents a basis element, and the actions of E, F, H are indicated by the colored arrows. If we call v_i the basis element corresponding to the i th bullet (the top one being v_0) then, for example, a red arrow labelled a between the i th and the $(i + 1)$ st bullet indicates that $F(v_i) = av_{i+1}$.

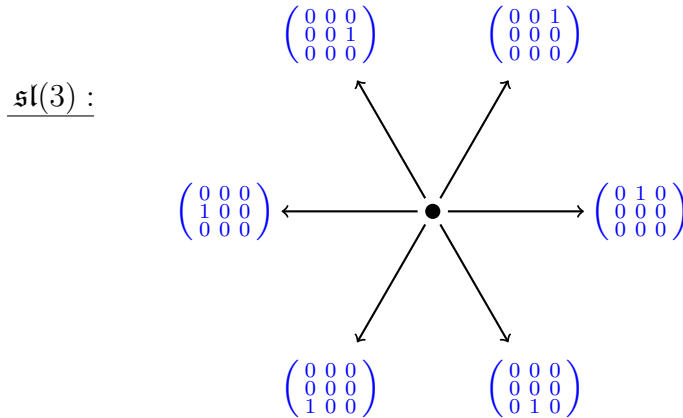
Let now \mathfrak{g} be an arbitrary finite dimensional simple Lie algebra over \mathbb{C} . A *Cartan subalgebra* is an abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which arises as the Lie algebra of a maximal torus T inside the simply connected Lie group G associated to \mathfrak{g} (such a torus is unique up to conjugacy). The dimension $r := \dim(\mathfrak{h})$ is called the *rank* of \mathfrak{g} .

As an \mathfrak{h} -representation, \mathfrak{g} decomposes as a direct sum of its Cartan subalgebra \mathfrak{h} and certain one-dimensional subspaces $\mathfrak{g}^\alpha \subset \mathfrak{g}$ called *root spaces*:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha. \quad (26)$$

The direct sum is indexed by a finite set $\Phi \subset \mathfrak{h}^*$ called the *root system* of \mathfrak{g} , and the elements of Φ are called the *roots* of \mathfrak{g} . The root spaces \mathfrak{g}^α are spanned by *root vectors* $E^\alpha \in \mathfrak{g}^\alpha$. The letter satisfy $[H, E^\alpha] = (\alpha, H)E^\alpha, \forall H \in \mathfrak{h}$. The root vectors furthermore satisfy $[E^\alpha, E^\beta] \in \mathfrak{g}^{\alpha+\beta}$ when $\alpha + \beta \in \Phi$, and $[E^\alpha, E^\beta] = 0$ otherwise.

We illustrate the notions of root system and root vectors in the case of the Lie algebra $\mathfrak{sl}(3)$ of traceless 3×3 matrices, with its Cartan subalgebra \mathfrak{h} of diagonal matrices:



The plane in which the above picture is drawn is \mathfrak{h}^* , the dual of the Cartan. The six roots form the vertices of a regular hexagon, and the root vectors are 3×3 matrices with a single non-zero off-diagonal entry.

For each root $\alpha \in \Phi$, the root vectors $E^\alpha, F^\alpha := E^{-\alpha}$, and their commutator $H^\alpha := [E^\alpha, F^\alpha] \in \mathfrak{h}$ span a subalgebra $\mathfrak{sl}(2)^\alpha := \text{Span}\{E^\alpha, F^\alpha, H^\alpha\} \subset \mathfrak{g}$ isomorphic to $\mathfrak{sl}(2)$. The elements $E^\alpha, F^\alpha, H^\alpha \in \mathfrak{g}$ can be normalised to satisfy the same commutation relations (25) as $E, F, H \in \mathfrak{sl}(2)$.

The Lie algebra \mathfrak{g} carries a unique invariant bilinear form, up to scalar. Given such a form, we may consider the induced bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{h}^* . The roots $\alpha \in \mathfrak{h}^*$ either all have the same square-length, in which case the Lie algebra is called *simply laced*, or there are two square-lengths of roots. In the latter case, the roots are divided into ‘long roots’ and ‘short roots’. The *basic inner product* is the invariant bilinear form on \mathfrak{g} normalised

so that either $\langle \alpha, \alpha \rangle = 2$ for every long root (when \mathfrak{g} is not simply laced), or $\langle \alpha, \alpha \rangle = 2$ for every root (when \mathfrak{g} is simply laced). If \mathfrak{g} is not simply laced, let d be the ratio between the square-length of a long root and that of a short root (this number is called the *lacity* of \mathfrak{g} , and is either 2 or 3). The element $H^\alpha = [E^\alpha, E^{-\alpha}] \in \mathfrak{h}$ can then be described as

$$H^\alpha = \begin{cases} \langle \alpha, - \rangle & \text{if } \alpha \text{ is a long root or } \mathfrak{g} \text{ is simply laced,} \\ d\langle \alpha, - \rangle & \text{if } \alpha \text{ is a short root} \end{cases} \quad (27)$$

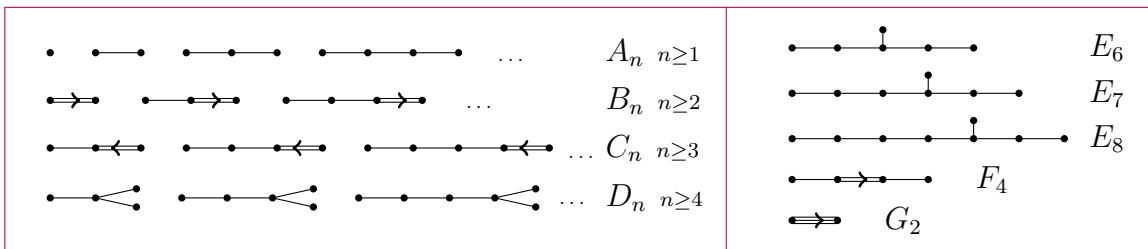
(here, we've identified an element of \mathfrak{h} with the corresponding linear functional on \mathfrak{h}^*).

For $\alpha \in \Phi$ a root, let us write

$$s_\alpha : \mathfrak{h}^* \rightarrow \mathfrak{h}^* \\ \xi \mapsto \xi - 2 \frac{\langle \xi, \alpha \rangle}{\|\alpha\|^2} \cdot \alpha$$


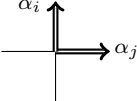

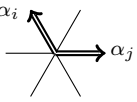

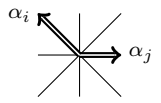

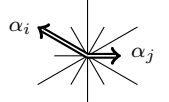
for the reflection across the hyperplane perpendicular to α . These reflections act by symmetries of the root system, and generate the *Weyl group* W . Let $\mathfrak{h}_{\mathbb{R}}^*$ be the \mathbb{R} -span of the roots (so that \mathfrak{h}^* is the complexification of $\mathfrak{h}_{\mathbb{R}}^*$). The hyperplanes $\{\xi : \langle \xi, \alpha \rangle = 0\}$ partition $\mathfrak{h}_{\mathbb{R}}^*$ into W many chambers; pick one and call it the *dominant Weyl chamber*. It has r many walls (where r is the rank of \mathfrak{g}). The roots $\alpha_1, \dots, \alpha_r \in \Phi$ that are perpendicular to these walls and inwards-pointing are called the *simple roots* of \mathfrak{g} .

The *Dynkin diagram* of \mathfrak{g} is a type of graph which encodes the geometry of the simple roots of \mathfrak{g} . Each node corresponds to a simple root, and the angles between simple roots as well as their relative lengths are encoded in the way the nodes are connected (except when two nodes are not connected by an edge in which case the relative lengths are not encoded). The Dynkin diagrams that arise from simple finite dimensional Lie algebra are called *of finite type*. There are four infinite families of such Dynkin diagrams, denoted A_n, B_n, C_n, D_n , and five exceptional cases: E_6, E_7, E_8, F_4, G_2 (the subscript denotes the rank of \mathfrak{g}):



The Dynkin diagrams of finite type

The type of edge between two vertices of the Dynkin diagram encodes the relative geometry of the corresponding simple roots:

Pairs of vertices in the Dynkin diagram	Geometry of simple roots
i j 	
i j 	
i j 	
i j 	

(28)

The Dynkin diagram is a complete invariant of a Lie algebra:

Simple finite dimensional Lie algebras are classified by their Dynkin diagram.

It is in fact possible to write down a presentation of the Lie algebra \mathfrak{g} directly from its Dynkin diagram: there are three sets of generators E^i, F^i, H^i , indexed by the vertices of the Dynkin diagram, with relations

$$\begin{aligned}
 [H^i, H^j] &= 0 \\
 [H^i, E^j] &= a_{ij} E^j \\
 [H^i, F^j] &= -a_{ij} F^j \\
 [E^i, F^j] &= \delta_{ij} H^i \\
 \underbrace{[E^i, \dots [E^i, E^j]]}_{|a_{ij}|+1} &= \underbrace{[F^i, \dots [F^i, F^j]]}_{|a_{ij}|+1} = 0,
 \end{aligned}$$

where the a_{ij} are prescribed as follows:

i j	a_{ij}
$i=j$	2
\bullet \bullet	0
\bullet — \bullet	-1
\bullet \Rightarrow \bullet	-1
\bullet \Leftrightarrow \bullet	-1
\bullet \Leftarrow \bullet	-2
\bullet $\Leftarrow\Leftarrow$ \bullet	-3

(29)

These relations are known as the *Serre relations*, and the matrix (a_{ij}) is known as the *Cartan matrix* associated to the Dynkin diagram. The subalgebra $\mathfrak{b} \subset \mathfrak{g}$ generated by just the E_i and the H_i is called the *Borel subalgebra* of \mathfrak{g} .

Recall that the Cartan subalgebra \mathfrak{h} is the Lie algebra of a maximal torus $T \subset G$ inside the simply connected Lie group associated to \mathfrak{g} . The set $\Lambda := \text{Hom}(T, \mathbb{C}^\times)$ is a lattice in \mathfrak{h}^* called the *weight lattice*. Every finite dimensional representation V of \mathfrak{g} is a direct sum

$$V = \bigoplus_{\lambda \in \Lambda} V(\lambda),$$

indexed by the weight lattice, of its *weight spaces* $V(\lambda) := \{v \in V : Hv = (\lambda, H)v, \forall H \in \mathfrak{h}\}$. The set of $\lambda \in \Lambda$ such that $V(\lambda) \neq 0$ is called the *set of weights* of V . For example, the set of weights of the adjoint representation is $\Phi \cup \{0\}$.

For any $\lambda \in \mathfrak{h}^*$, we may consider the *Verma module*

$$M_\lambda := \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda,$$

where the action of the Borel on the one-dimensional module \mathbb{C}_λ is given by $H \mapsto (\lambda, H)$ for $H \in \mathfrak{h}$, and $E_i \mapsto 0$. The image of $1 \in \mathbb{C}_\lambda$ is called the *highest weight vector* of M_λ . Unlike the Verma module, the *simple quotient*

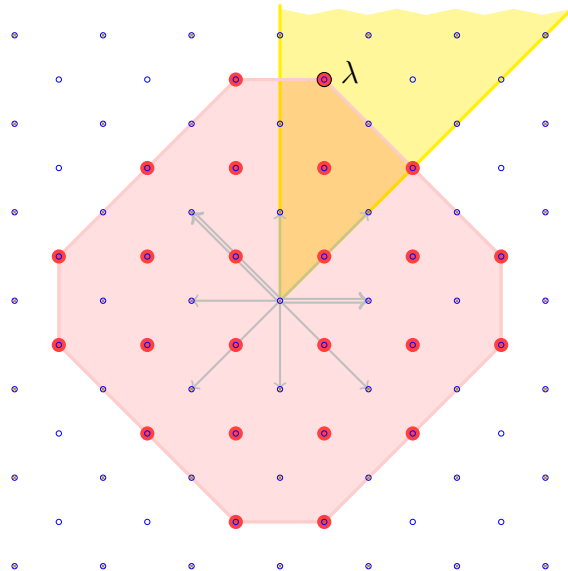
$$L_\lambda := M_\lambda / J_\lambda$$

maximal proper submodule of M_λ

is sometimes finite dimensional. Specifically, L_λ is finite dimensional if and only if λ belongs to the set $\Lambda_+ \subset \Lambda$ of *dominant weights*, defined as the intersection of the weight lattice with the dominant Weyl chamber.

The above construction establishes a one-to-one correspondence between the set of isomorphism classes of finite dimensional irreducible representations, and the set Λ_+ of dominant weights of \mathfrak{g} .

The weight λ is called the *highest weight* of L_λ . Let the *root lattice* $\Lambda_{\text{root}} \subset \Lambda$ be the sub-lattice of the weight lattice spanned by the roots. For $\lambda \in \Lambda_+$, the set of weights of L_λ can be then described as the intersection of the shifted root lattice $\Lambda_{\text{root}} + \lambda$ with the *weight polytope* P_λ , where the latter is the convex hull of the orbit of λ under the Weyl group. We illustrate all these notions in the case of an irrep of $\mathfrak{so}(5)$ (the simple Lie algebra of type B_2) associated to some $\lambda \in \Lambda_+$:



The red bullets mark the weights of L_λ . The dominant Weyl chamber is in yellow. The weight polytope P_λ is in pink. The roots are in gray, with the simple roots in bold. The tiny blue circles indicate the weight lattice, and the tiny brown crosses mark the root lattice.

Affine Lie algebras

For \mathfrak{g} a finite dimensional simple Lie algebra over \mathbb{C} , let $L\mathfrak{g} := \mathfrak{g}[t, t^{-1}]$ be the algebra of polynomial loops in \mathfrak{g} (algebraic functions on \mathbb{C}^\times with values in \mathfrak{g}). For X an element of \mathfrak{g} , we write X_n for $Xz^n \in L\mathfrak{g}$, so that the bracket of $L\mathfrak{g}$ is given by $[X_m, Y_n] = [X, Y]_{m+n}$. The *affine Lie algebra*

$$\widetilde{L\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$$

is the central extension of $L\mathfrak{g}$ associated to the cocycle $\omega(f, g) = \text{Res}_0 \langle df, g \rangle$.²³ Its bracket is given by

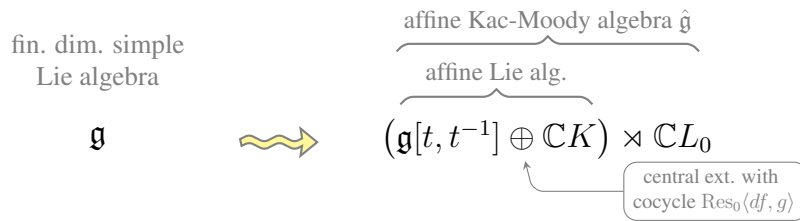
$$[X_m, Y_n] = [X, Y]_{m+n} + m\delta_{m+n,0} \langle X, Y \rangle K,$$

and the element K spans its center.

Starting from a Dynkin diagram, we have seen in (29) how to construct a simple Lie algebra \mathfrak{g} by means of the Serre relations. The Lie algebra is finite dimensional if and only if the Dynkin diagram is of finite type (i.e., one of $A_n, B_n, C_n, D_n, E_{6-8}, F_4, G_2$), but the Serre relations make sense for more general Dynkin diagrams too, leading to a class of infinite dimensional Lie algebras known as *Kac-Moody algebras*. A large portion of the theory of finite dimensional simple Lie algebras passes over essentially unchanged to the more general setup of Kac-Moody algebras.²⁴ This includes notions such as *roots*, *weights*, the *Weyl group*, and the classification of (certain) simple modules by *dominant weights* $\lambda \in \Lambda_+$, including the fact that the set of weights of L_λ is given by $P_\lambda \cap (\Lambda_{\text{root}} + \lambda)$.

The reflections that generate the Weyl group are given by the same formula $s_\alpha(\xi) = \xi - \frac{2\langle \xi, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$ as in the finite dimensional case, with the notable difference that the invariant inner product on $\mathfrak{h}_{\mathbb{R}}^*$ is no longer positive definite.

Surprisingly, *affine Lie algebras are (almost) instances of Kac-Moody algebras!* The reason for the ‘almost’ is that it’s not $\widetilde{L\mathfrak{g}}$, but rather the semi-direct product $\hat{\mathfrak{g}} := \widetilde{L\mathfrak{g}} \rtimes \mathbb{C}L_0$ which is a Kac-Moody algebra (where $[X_n, L_0] = nX_n$):²⁵



The Cartan subalgebra of $\hat{\mathfrak{g}}$ is given by $\hat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}L_0$, with invariant inner product given by $\langle \cdot, \cdot \rangle_{\hat{\mathfrak{h}}} := \langle \cdot, \cdot \rangle_{\mathfrak{h}} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The Weyl group of $\hat{\mathfrak{g}}$ is called the *affine Weyl group* and denoted \hat{W} . It acts on $\hat{\mathfrak{h}}^*$ and contains the finite Weyl group W as a subgroup.

²³Here, unlike in (21), $\langle \cdot, \cdot \rangle$ denotes the basic inner product on \mathfrak{g} .

²⁴More precisely, this holds for the so-called *symmetrizable Kac-Moody algebras*. This class includes all the affine Kac-Moody algebras. It excludes Dynkin diagrams such as $\hat{\Delta}$, for which it is not possible to consistently assign lengths to the simple roots while respecting the rules (28).

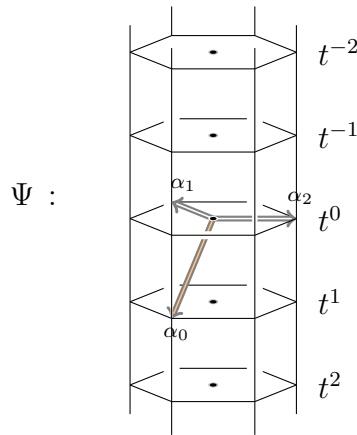
²⁵It is customary to alter slightly the Serre relations when the Cartan matrix has zero determinant. In that case, one adds an extra generator to the Cartan subalgebra to ensure that the simple roots are linearly independent as elements of \mathfrak{h}^* .

The Kac-Moody algebras obtained by the above construction are called *affine Kac-Moody algebras*.

To see that $\hat{\mathfrak{g}} = \widetilde{L\mathfrak{g}} \rtimes \mathbb{C}L_0$ is an instance of a Kac-Moody algebra, we first note that the weights of its adjoint representation are given by

$$\Psi := (\Phi \cup \{0\}) \times \mathbb{Z} \subset \{\xi \in \hat{\mathfrak{h}}^* : (\xi, K) = 0\} \subset \hat{\mathfrak{h}}^*.$$

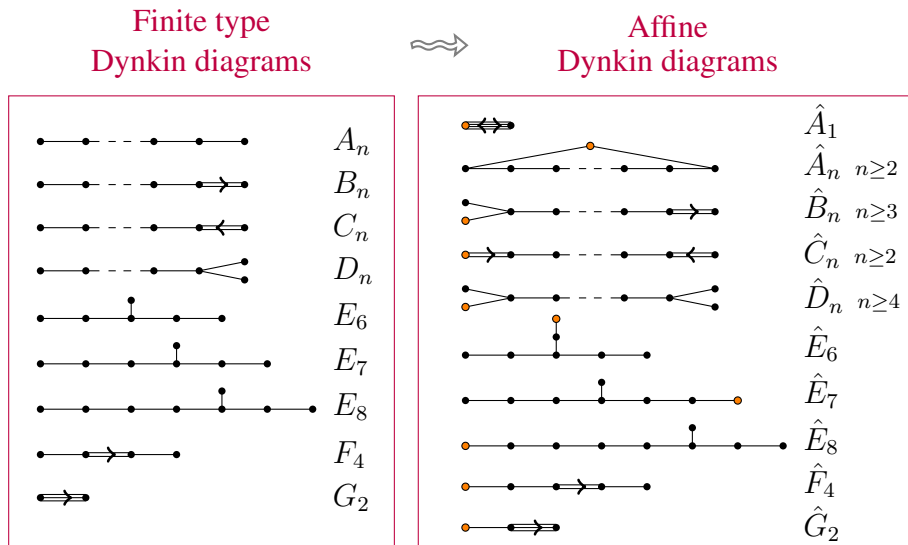
Let $\alpha_1, \dots, \alpha_r \in \Phi$ be the simple roots of \mathfrak{g} , which we identify with their images $(\alpha_i, 0) \in \Psi$, and let $\alpha_0 := (\alpha_{\min}, 1) \in \Psi$ be the so-called *affine root*. Here, $\alpha_{\min} \in \Phi$ denotes the *lowest root* of \mathfrak{g} , characterised by the property that $\alpha_{\min} - \alpha_i \notin \Phi$ for any simple root α_i . We illustrate the case of affine $\mathfrak{su}(3)$:



The crucial observation is that the standard generators E^i, F^i, H^i of \mathfrak{g} (viewed as elements of $\hat{\mathfrak{g}}$) along with

$$E^0 := (\text{root vector for } \alpha_{\min}) \otimes t, \quad F^0 := (\text{root vector for } -\alpha_{\min}) \otimes t^{-1}, \quad H^0 := [E^0, F^0]$$

(normalised so that $H^0 = \langle \alpha_{\min}, - \rangle + K \in \hat{\mathfrak{h}}$) satisfy the Serre relations for an extended Dynkin diagram known as the *affine Dynkin diagram* associated to \mathfrak{g} .



Starting from the Dynkin diagram of \mathfrak{g} , one constructs the associated affine Dynkin diagram by adding one extra vertex (corresponding to the affine root α_0). The angles and relative lengths of α_{\min} to the other simple roots tell us, using (28), how to connect the new vertex to the old ones²⁶. (Note that, for the purpose of computing angles and lengths, α_0 and α_{\min} are interchangeable, as the invariant inner product on $\mathfrak{h}_0^* := \{\xi \in \hat{\mathfrak{h}}^* : (\xi, K) = 0\} \subset \hat{\mathfrak{h}}^*$ is degenerate, with $\alpha_0 - \alpha_{\min}$ in its kernel.)

A representation of $\hat{\mathfrak{g}}$ is said to have *level k* if the central element $K \in \hat{\mathfrak{g}}$ acts by the scalar k . This can equivalently be formulated by saying that the set of weights of the representation is contained in the hyperplane $\hat{\mathfrak{h}}_k^* := \{\xi \in \hat{\mathfrak{h}}^* : (\xi, K) = k\} \subset \hat{\mathfrak{h}}^*$. The $\hat{\mathfrak{g}}$ -modules we will be interested in are the *level k integrable positive energy representations*. Here, a $\hat{\mathfrak{g}}$ -module V is called *integrable* if for each subalgebra $\mathfrak{sl}(2)^i := \text{Span}\{E^i, F^i, H^i\} \subset \hat{\mathfrak{g}}$, $i \in \{0, \dots, r\}$, V decomposes as a direct sum of finite dimensional $\mathfrak{sl}(2)^i$ -reps.²⁷ And an integrable representation has *positive energy* if L_0 is diagonalizable, its spectrum is discrete and bounded below, and its eigenspaces are finite dimensional (let us abbreviate these conditions by ‘ L_0 has positive energy’). The integrable positive energy representations of $\hat{\mathfrak{g}}$ are the analogs of the finite dimensional representations of a finite dimensional simple Lie algebra. As in the finite dimensional case, such representations are classified by their highest weight (and for the representation to be level k , the highest weight λ must satisfy $(\lambda, K) = k$). And the set of weights of the simple module L_λ with highest weight λ is given by $P_\lambda \cap (\Lambda_{\text{root}} + \lambda)$.

[We had previously defined a representation of $\widetilde{L\mathfrak{g}}$ to have positive energy if it extends to a representation of $\widetilde{L\mathfrak{g}} \rtimes \text{Vir}$ for which L_0 has positive energy²⁸. Alternatively, one may define a positive energy of $\widetilde{L\mathfrak{g}}$ to be one that extends to a representation of $\hat{\mathfrak{g}} = \widetilde{L\mathfrak{g}} \rtimes \mathbb{C}L_0$ for which L_0 has positive energy. In the next section, we’ll see that these two definitions are in fact equivalent, even though the first one seems a priori much stronger.]

Recall that the affine Weyl group \hat{W} is generated by the reflections s_0, \dots, s_r associated to the simple roots $\alpha_1, \dots, \alpha_r$ plus the affine root α_0 . The weight polytope P_λ (which is the convex hull of the \hat{W} -orbit of λ) is contained in the affine space

$$\hat{\mathfrak{h}}_{k,\mathbb{R}}^* := \text{Span}_{\mathbb{R}}\{\alpha_0, \dots, \alpha_r\} + \lambda.$$

Let us analyse the action of the affine Weyl group on that space. First note that $\hat{\mathfrak{h}}_{k,\mathbb{R}}^*$ is canonically isomorphic to $\mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}$. The subgroup W generated by s_1, \dots, s_r acts in the usual way on $\mathfrak{h}_{\mathbb{R}}^*$ and trivially on \mathbb{R} . The extra reflection $s_0 : \xi \mapsto \xi - \frac{2\langle \xi, \alpha_0 \rangle}{\langle \alpha_0, \alpha_0 \rangle} \alpha_0 = \xi - \langle \xi, \alpha_0 \rangle \alpha_0$ is then characterised by the fact that it fixes the hyperplane

$$\begin{aligned} \ker(H_0) \cap \hat{\mathfrak{h}}_{k,\mathbb{R}}^* &= \{\xi : \langle \alpha_{\min}, \xi \rangle + k = 0\} \\ &= \{\xi : \langle \alpha_{\min}, \xi \rangle = -k\} \\ &= \{\xi : \langle \alpha_{\max}, \xi \rangle = k\}, \end{aligned}$$

here, we view H_0 as a linear functional on $\hat{\mathfrak{h}}^*$

²⁶The case of affine $\mathfrak{su}(2)$ is exceptional as the two roots α_1 and α_{\min} form an angle of 180° , which does appear in (28). We solve this by inventing a new type of edge to denote that configuration, namely \longleftrightarrow .

²⁷Note that unitary $\hat{\mathfrak{g}}$ -modules are automatically integrable.

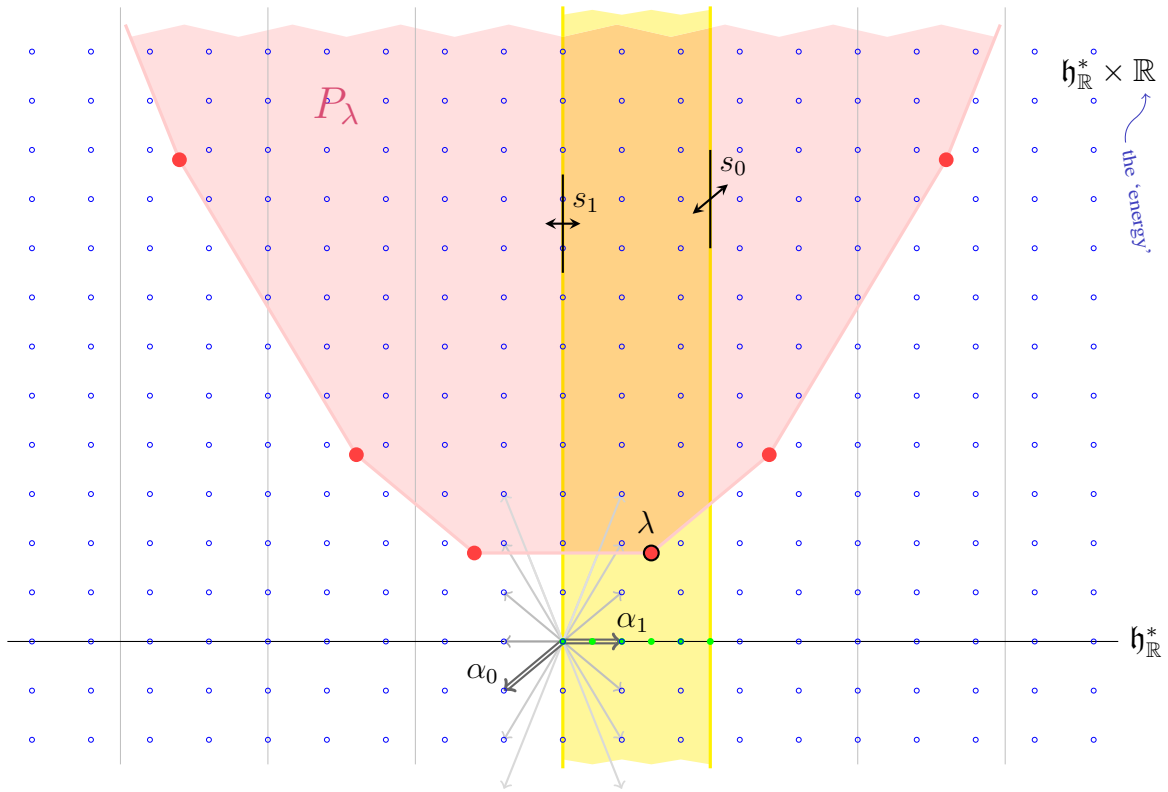
²⁸For non-integrable representations, the positive energy condition should not insist that L_0 be diagonalizable. It should only require its generalised eigenspaces to be finite dimensional.

and displaces every vector by a multiple of α_0 (where $\alpha_{\max} = -\alpha_{\min}$ denotes the highest root of \mathfrak{g}). The fundamental domain for the action of \hat{W} on $\hat{\mathfrak{h}}_{k,\mathbb{R}}^* \cong \mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}$ is delimited by the hyperplanes $\langle \xi, \alpha_i \rangle = 0$ and $\langle \xi, \alpha_{\max} \rangle = k$. It is isomorphic to $kA \times \mathbb{R}$, where kA denotes the *Weyl alcove* of \mathfrak{g} scaled by a factor of k :

Definition: The *Weyl alcove* is the r -dimensional simplex $A \subset \mathfrak{h}_{\mathbb{R}}^*$ bound by the walls of the Weyl chamber, and by the hyperplane $\langle \xi, \alpha_{\max} \rangle = 1$ (the hyperplane that bisects the segment $[0, \alpha_{\max}]$):

$$A := \{ \xi \in \mathfrak{h}^* : \langle \xi, \alpha_i \rangle \geq 0 \text{ for } i \in \{1, \dots, r\} \text{ and } \langle \xi, \alpha_{\max} \rangle \leq 1 \}.$$

We illustrate all the above in the case of a representation of $\hat{\mathfrak{sl}}(2)$ at level $k = 5$:



The reflections s_1 and s_0 generate the affine Weyl group of $\mathfrak{sl}(2)$. Its fundamental domain on $\hat{\mathfrak{h}}_{k,\mathbb{R}}^*$ is the yellow strip $[0, k] \times \mathbb{R}$ (an instance of $kA \times \mathbb{R}$). The \hat{W} -orbit of λ is in red. The tiny blue circles mark the root lattice (or rather, the image of the root lattice under the standard identification $\hat{\mathfrak{h}}_{0,\mathbb{R}}^* \cong \hat{\mathfrak{h}}_{k,\mathbb{R}}^*$). Some roots are drawn in gray. The green dots mark the set $kA \cap \Lambda$.

It is important to remember that a positive energy representation of $\widetilde{L\mathfrak{g}}$ is not a representation of $\hat{\mathfrak{g}} = \widetilde{L\mathfrak{g}} \rtimes \mathbb{C}L_0$. It is a representation of $\widetilde{L\mathfrak{g}}$ that extends to a representation of $\widetilde{L\mathfrak{g}} \rtimes \mathbb{C}L_0$ but the extension is *not part of the data*. Moreover, such extensions are never unique: a representation of $\widetilde{L\mathfrak{g}} \rtimes \mathbb{C}L_0$ can always be modified by a character of

$\mathbb{C}L_0$ without changing the way in which $\widetilde{L}\mathfrak{g}$ acts. This operation has the effect of shifting all the weights of the representation by a certain (arbitrary) amount in the \mathbb{C} direction of $\hat{\mathfrak{h}}_k^* \cong \mathfrak{h}^* \oplus \mathbb{C}$. The upshot is that **level k integrable positive energy representations of $\widetilde{L}\mathfrak{g}$ are *not* classified by dominant weights $\lambda \in \hat{\mathfrak{h}}_k^*$. They are instead classified by their images under the projection $\hat{\mathfrak{h}}_k^* \rightarrow \mathfrak{h}^*$. The set of possible such projections is $k\mathbf{A} \cap \Lambda \subset \mathfrak{h}^*$, the intersection of the weight lattice with the scaled Weyl alcove:**

Theorem. *The set of irreducible level k integrable positive energy representations of the affine Lie algebra $\widetilde{L}\mathfrak{g}$ is in canonical bijection with the finite set*

$$A_k := k\mathbf{A} \cap \Lambda = \{ \lambda \in \Lambda_+ : \langle \lambda, \alpha_{\max} \rangle \leq k \}.$$

The correspondence sends a representation to the highest weight of its lowest energy subspace (L_0 -eigenspace with lowest eigenvalue).

We record a couple of equivalent descriptions of the category of level k integrable positive energy representations of $\widetilde{L}\mathfrak{g}$:

$$\text{Rep}_{\text{pos}}^{\text{integrable}}(\widetilde{L}\mathfrak{g}_k) = \text{Rep}_{\text{pos}}^{\text{unitary}}(\widetilde{L}\mathfrak{g}_k) = \text{Rep}_{\text{pos}}(\widetilde{L}G_k)$$

(here G is the compact simply connected Lie group associated to \mathfrak{g}). We illustrate with some examples the set A_k of simple objects of that category:

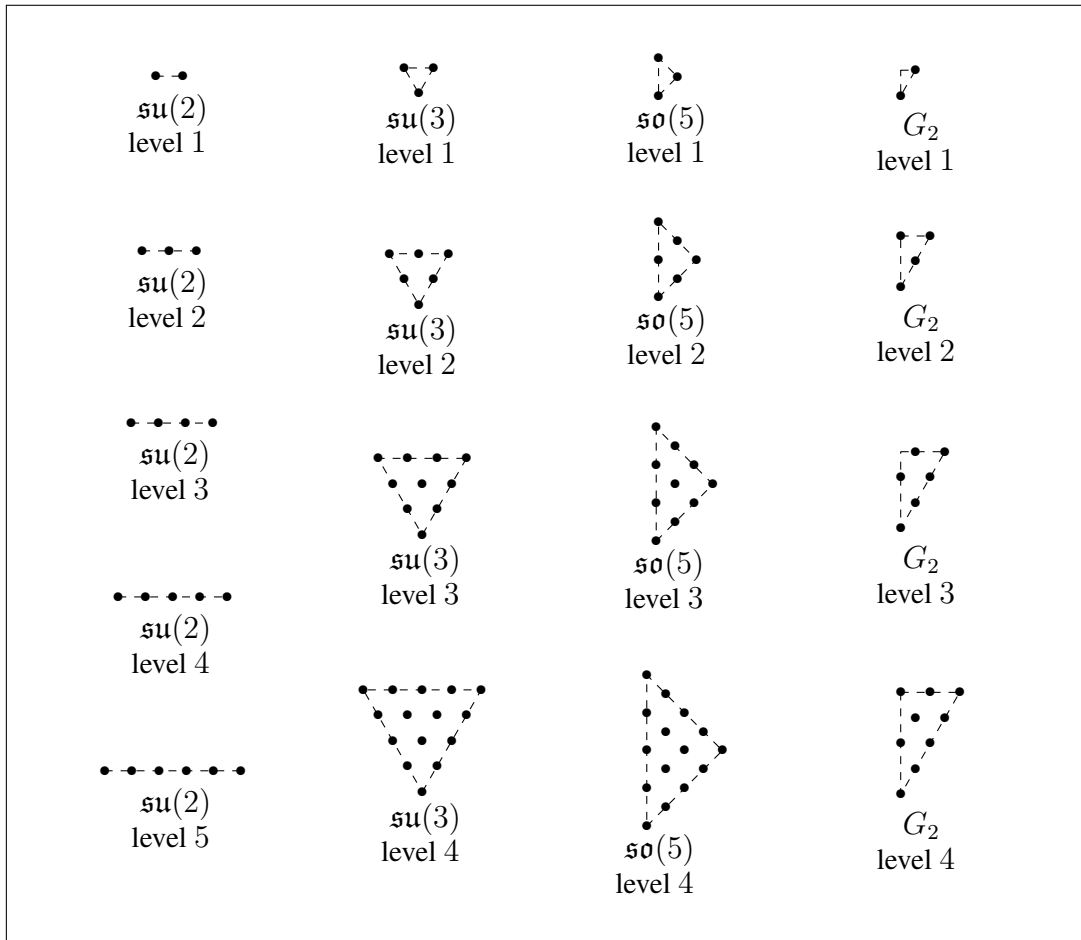


Table 2.

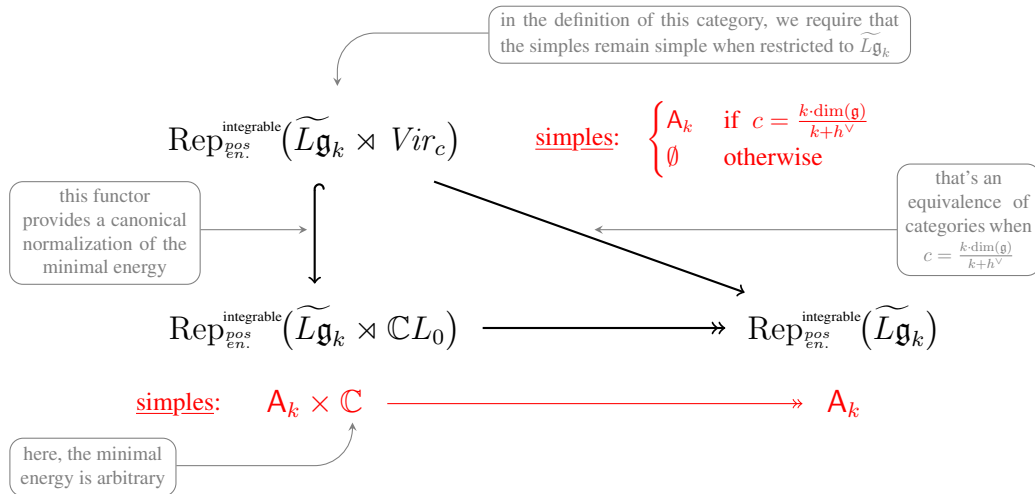
The Segal-Sugawara construction

Recall that a positive energy representation of the affine Lie algebra $\widetilde{L\mathfrak{g}}$ is one that extends to a representation of $\widetilde{L\mathfrak{g}} \rtimes \mathbb{C}L_0$ for which L_0 has positive energy. Let V be an irreducible such representation. Remarkably, the action of $\widetilde{L\mathfrak{g}}$ then automatically extends to the larger Lie algebra $\widetilde{L\mathfrak{g}} \rtimes Vir$ (and again L_0 has positive energy).

More precisely, if V has level k , then there exists a unique central charge $c \geq 0$, depending on k , such that the representation extends, uniquely, to a representation of $\widetilde{L\mathfrak{g}}_k \rtimes Vir_c$. The central charge is given by the formula $c = \frac{k \cdot \dim(\mathfrak{g})}{k+h^\vee}$, where h^\vee denotes the *dual Coxeter number* of \mathfrak{g} , defined on the next page.

Warning: A positive energy representation of $\widehat{\mathfrak{g}} = \widetilde{L\mathfrak{g}} \rtimes \mathbb{C}L_0$ usually *doesn't* extend to a representation of $\widetilde{L\mathfrak{g}} \rtimes Vir$: one typically needs to change the action of $\mathbb{C}L_0$ for it to extend to $\widetilde{L\mathfrak{g}} \rtimes Vir$.

We summarise the situation in the following diagram. All the arrows are given by restriction (and all the categories are semisimple):



We recall that the commutation relations of $\widetilde{L\mathfrak{g}}_k \rtimes Vir_c$ are given by:

$$\begin{aligned} [X_m, Y_n] &= [X, Y]_{m+n} + kn \langle X, Y \rangle \delta_{m+n,0} \\ [X_m, L_n] &= mX_{m+n} \quad \leftarrow \text{since } z^{n+1} \frac{\partial}{\partial z} z^m = mz^{m+n} \quad {}^{29} \\ [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \end{aligned} \quad (30)$$

where, as before, X_n stands for $Xz^n \in L\mathfrak{g} \subset \widetilde{L\mathfrak{g}}_k$. Note that, as a consequence of the relation $[X_m, L_0] = mX_m$, the operator X_m **lowers energy by m** (where ‘energy’ is synonym of L_0 -eigenvalue).

²⁹This uses the Lie bracket (8) (the opposite of the Lie bracket of vector fields), which is well-adapted to *right* actions.

Before going on, we must explain what dual Coxeter number is. This invariant of the Lie algebra \mathfrak{g} can be defined in a variety of ways. We present here a list, without any attempt at showing that these definitions are all equivalent:

Digression on the dual Coxeter number h^\vee

- One standard way of defining the dual Coxeter number is by declaring $2h^\vee$ to be the ratio between the Killing form and the basic inner product. Here, $\langle X, Y \rangle_{\text{Killing}} := \text{Tr}(\text{ad}(X) \circ \text{ad}(Y))$, and the basic inner product is characterised by the fact that the long roots (all roots when \mathfrak{g} is simply laced) have square-norm 2.
- Letting $r = \dim(\mathfrak{h})$ be the rank of \mathfrak{g} , the dual Coxeter number is also characterised by the formula

$$r \cdot h^\vee = \#\{\text{long roots}\} + \frac{1}{d} \#\{\text{short roots}\}.$$

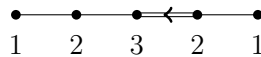
Here, d is the lacity of \mathfrak{g} (and all roots count as long when \mathfrak{g} is simply laced).

- Here's a method for computing h^\vee . Take the affine Dynkin diagram associated to \mathfrak{g} . If the Lie algebra is not simply laced, reverse the direction of all the arrows. Call the result D^\vee . That's the so-called *dual affine Dynkin diagram*. A labelling of its nodes by positive integers is called *harmonic* if $\forall v \in D^\vee$ we have

$$2 \times \text{label of } v = \sum \text{labels of neighbours of } v$$

with the extra rule that 'big neighbours count with multiplicity'. Then h^\vee is the sum of all the labels in the minimal harmonic labelling of D^\vee .

We illustrate this method for the Lie algebra F_4 . The minimal harmonic labelling of the dual affine Dynkin diagram looks as follows:

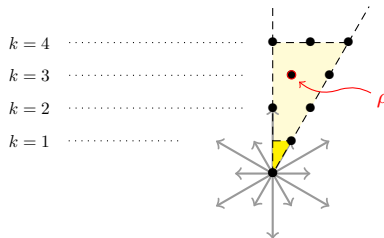


check harmonic condition:

- $2 \times 1 = 2 \checkmark$
- $2 \times 2 = 1 + 3 \checkmark$
- $2 \times 3 = 2 + 2 \checkmark$
- $2 \times 2 = 3 + 1 \checkmark$
- $2 \times 1 = 2 \checkmark$

giving us $h^\vee = 1 + 2 + 3 + 2 + 1 = 9$.

- Yet another way to characterize the dual Coxeter number is to say that h^\vee is the smallest level k such that $k\Lambda$ contains an element of the weight lattice in its interior. We illustrate this in the case of the Lie algebra G_2 :



By the way, that element in the interior is called the *Weyl vector*. It's always denoted ρ , and is given by the formula $\rho = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha$. (Here, $\Phi_+ := \Phi \cap \text{Span}_{\mathbb{N}}\{\alpha_1, \dots, \alpha_r\}$ is the set of positive roots.)

Let us now go back to the main goal of this section, which is to prove the following theorem. Its proof will occupy the rest of the section.

Theorem 18 For $V \in \text{Rep}_{\text{pos. en.}}^{\text{integrable}}(\widetilde{L\mathfrak{g}}_k)$ irreducible, there exists a unique $c \geq 0$, and a unique extension of V to a representation of $\widetilde{L\mathfrak{g}}_k \rtimes \text{Vir}_c$. The Virasoro generators are given by the *Segal-Sugawara formulas*:

$$L_n := \frac{1}{2(k+h^\vee)} \sum_{X \in \mathcal{B}} \left(\sum_{m < 0} X_m X_{n-m} + \sum_{m \geq 0} X_{n-m} X_m \right),$$

where \mathcal{B} is an orthonormal basis of \mathfrak{g} with respect to the basic inner product, and h^\vee is the dual Coxeter number. The central charge is given by

$$c = \frac{k \cdot \dim(\mathfrak{g})}{k + h^\vee}$$

and the minimal energy of V is

$$h = \frac{\|\lambda + \rho\|^2 - \|\rho\|^2}{2(k + h^\vee)},$$

where $\lambda \in A_k \subset \mathfrak{h}^*$ is the highest weight of the lowest energy subspace of V .

Remark. The Segal-Sugawara formula involves an infinite sum. However, since X_m lowers energy by m , on any given vector $v \in V$ there's only finitely many terms which act in a non-zero way. So the expression in fact makes sense, despite the infinite sum.

We first show that when (V, π) is an irreducible representation of $\widetilde{L\mathfrak{g}}_k$, then the action of the L_m and the value of the central charge c are uniquely determined, provided they exist. We prove this using the following two lemmas:

Lemma 19 The equation $[\pi(X_m), \pi(L_n)] = m\pi(X_{m+n})$ uniquely determines $\pi(L_n)$ up to the addition of a scalar.

Proof. Let $\pi(L_n)$ and $\pi'(L_n)$ be two solutions. Then $[\pi(X_m), \pi(L_n) - \pi'(L_n)] = 0$. So $\pi(L_n) - \pi'(L_n) : V \rightarrow V$ is a morphism of $\widetilde{L\mathfrak{g}}_k$ -representations. By Schur's lemma, $\pi'(L_n) = \pi(L_n) + \text{cst}$. \square

Lemma 20 If the operators $\pi(L_n)$ satisfy then, $[\pi(L_m), \pi(L_n)] = \pi([L_m, L_n]) + \text{cst}$.

Proof. $[\pi(X_r), [\pi(L_m), \pi(L_n)]] = \underbrace{[[\pi(X_r), \pi(L_m)], \pi(L_n)]}_{r \cdot \pi(X_{r+m})} - \underbrace{[[\pi(X_r), \pi(L_n)], \pi(L_m)]}_{r \cdot \pi(X_{r+n})} = (m-n)r\pi(X_{r+m+n})$. So $[\pi(L_m), \pi(L_n)]$ satisfies the same commutation relations as $\pi([L_m, L_n])$, and we're done by the previous lemma. \square

If we make a random pick for the $\pi(L_m)$'s, subject to the relations $[\pi(X_m), \pi(L_n)] = m\pi(X_{m+n})$, then we won't quite get a representation of the Witt algebra. (The constants which appear in the last lemma are the 2-cocycle that measures the failure of π being a representation.) What we get instead is a representation of a central extension $\widetilde{\mathbb{W}} \rightarrow \mathbb{W}$. Since Vir is the universal central extension of \mathbb{W} , we get a unique Lie algebra homomorphism $Vir \rightarrow \widetilde{\mathbb{W}}$ that commutes with the projection to \mathbb{W} . The central charge is where the central element of Vir goes.

The Segal-Sugawara operators are affine Lie algebra analogs of the *quadratic Casimir*

$$C := \sum_{X \in \mathcal{B}} XX,$$

which lives in the universal enveloping algebra $U\mathfrak{g}$ of \mathfrak{g} . Before tackling Segal-Sugawara, we'll need to understand the action of C on irreps of \mathfrak{g}

Note that, in the definition of the quadratic Casimir, in place of the orthonormal basis \mathcal{B} , we could have used any pair of bases $\{X_i\}$ and $\{X^i\}$ satisfying $\langle X_i, X^j \rangle = \delta_i^j$. Bases like that are called *dual bases* of \mathfrak{g} :

Lemma 21 *Let $\{X_i\}$ and $\{X^i\}$ be dual bases of \mathfrak{g} . Then $C = \sum_i X_i X^i$.*

Proof. We claim that if $(\{X_i\}, \{X^i\})$ and $(\{Y_j\}, \{Y^j\})$ are two pairs of dual bases, then $\sum X_i X^i = \sum Y_j Y^j$. Let $A = (a_j^i)$ and $B = (b_i^j)$ be the matrices specified by $Y_j = \sum_i a_j^i X_i$ and $Y^j = \sum_i b_i^j X^i$. Then A and B are each other's inverses. It follows that

$$\sum_j Y_j Y^j = \sum_{ijk} X_i a_j^i b_k^j X^k = \sum_{ik} \delta_k^i X_i X^k = \sum_i X_i X^i. \quad \square$$

The quadratic Casimir is independent of the choice of dual bases. It is therefore invariant under the action of $\text{Aut}(\mathfrak{g})$ on $U\mathfrak{g}$. In particular, it is invariant under the adjoint action of G on $U\mathfrak{g}$, hence invariant under the adjoint action of \mathfrak{g} on $U\mathfrak{g}$. This means:

$$\forall X \in \mathfrak{g} \quad [X, C] = 0 \quad \text{in } U\mathfrak{g}.$$

As $U\mathfrak{g}$ is generated by \mathfrak{g} , it follows that $C \in Z(U\mathfrak{g})$. Hence, by Schur's lemma, C acts as a scalar on any irrep of \mathfrak{g} . It will be important to compute those scalars.

To perform the computation, we make use of the freedom provided by Lemma 21 to choose a convenient set of dual bases of \mathfrak{g} . Let $E^\alpha \in \mathfrak{g}^\alpha$ be root vectors, normalized so that $E^\alpha, E^{-\alpha}, H^\alpha := [E^\alpha, E^{-\alpha}]$ satisfy the $\mathfrak{sl}(2)$ -relations (25). The basic inner product pairs \mathfrak{g}^α with $\mathfrak{g}^{-\alpha}$ and \mathfrak{h} with itself. By (27), we have

$$\begin{aligned} \langle E^\alpha, E^{-\alpha} \rangle &= \frac{1}{2} \langle [H^\alpha, E^\alpha], E^{-\alpha} \rangle \\ &= \frac{1}{2} \langle H^\alpha, [E^\alpha, E^{-\alpha}] \rangle \\ &= \frac{1}{2} \langle H^\alpha, H^\alpha \rangle = \begin{cases} \frac{1}{2} \langle \alpha, \alpha \rangle = 1 & \text{if } \alpha \text{ is a long root or } \mathfrak{g} \text{ is simply laced,} \\ \frac{1}{2} \langle d\alpha, d\alpha \rangle = d & \text{if } \alpha \text{ is a short root} \end{cases} \end{aligned}$$

where d is the lacity of \mathfrak{g} . We can thus write C as:

$$C = \sum_{\substack{\alpha \in \Phi, \\ \alpha \text{ long}}} E^\alpha E^{-\alpha} + \frac{1}{d} \sum_{\substack{\alpha \in \Phi, \\ \alpha \text{ short}}} E^\alpha E^{-\alpha} + \sum H_i H^i,$$

where $\{H_i\}$ and $\{H^i\}$ be dual bases of \mathfrak{h} (and all roots count as long when \mathfrak{g} is simply laced). Let us abbreviate this as

$$C = \sum'_{\alpha \in \Phi} E^\alpha E^{-\alpha} + \sum H_i H^i, \quad (31)$$

where the prime means ‘add a factor of $\frac{1}{d}$ if the root is short’.

Proposition. *Let L_λ be the irreducible representation of \mathfrak{g} of highest weights $\lambda \in \Lambda_+$. Then C acts on L_λ by the scalar*

$$\langle \lambda, \lambda + 2\rho \rangle = \|\lambda + \rho\|^2 - \|\rho\|^2.$$

Proof. We compute the scalar via its action on the highest weight vector $v \in L_\lambda$. Note that $\sum H_i H^i v$ is independent of the choice of dual bases of \mathfrak{h} . By picking $\{H_i\} = \{H^i\}$ an orthonormal basis, we readily see that $\sum H_i H^i v = \|\lambda\|^2 v$. Now, since $E^\alpha v = 0$ for all $\alpha \in \Phi_+$, half of the terms in the expression $\sum' E^\alpha E^{-\alpha} v$ are zero. This gives us:

$$\begin{aligned} Cv &= \sum'_{\alpha \in \Phi_+} E^\alpha E^{-\alpha} v + \sum H_i H^i v \\ &= \sum'_{\alpha \in \Phi_+} [E^\alpha, E^{-\alpha}] v + \sum H_i H^i v \\ &= \sum_{\alpha \in \Phi_+} \langle \lambda, \alpha \rangle v + \|\lambda\|^2 v = (\langle \lambda, 2\rho \rangle + \langle \lambda, \alpha \rangle) v, \end{aligned}$$

where the third equality follows from (27). \square

Corollary: *Let $V \in \text{Rep}_{\text{pos. en.}}^{\text{integrable}}(\widetilde{L\mathfrak{g}}_k)$. Then the Segal-Sugawara operator L_0 acts on the lowest energy subspace of V by the scalar $\frac{\|\lambda + \rho\|^2 - \|\rho\|^2}{2(k+h^\vee)}$.*

Proof. The operators X_m for $m > 0$ lower energy by m , and thus annihilate the lowest energy subspace of V . Therefore, on that subspace, L_0 acts like $\frac{1}{2(k+h^\vee)} \cdot C$. \square

Let us go back to the main statements in Theorem 18. We begin by introducing the *un-normalised Segal-Sugawara operators*:

$$T_n := \sum_{X \in \mathcal{B}} \left(\sum_{m < 0} X_m X_{n-m} + \sum_{m \geq 0} X_{n-m} X_m \right).$$

Equivalently:

$$\begin{aligned}
T_0 &= \sum_{X \in \mathcal{B}} \left(X_0 X_0 + 2 \sum_{n>0} X_{-n} X_n \right) \\
T_n &= \sum_{X \in \mathcal{B}} \sum_{m \in \mathbb{Z}} X_m X_{n-m} \quad \text{for } n \neq 0
\end{aligned}$$

Our next main goal is to prove that the Segal-Sugawara operators satisfy the second equation listed in (30). Equivalently:

Proposition 22 *The un-normalised Segal-Sugawara operators satisfy*

$$[X_m, T_n] = 2m(k + h^\vee)X_{m+n}.$$

Proof. Step 1. We first prove the $n = 0$ case: $[X_m, T_0] = 2m(k + h^\vee)X_m$.

– *Case* $m = 0$. This holds because T_0 is G -invariant.

– *Case* $m = 1$ and $X \in \mathfrak{h}$. That’s the heart of the proof. We’ll do it later, in Lemma 23.

– *Case* $m = -1$ and $X \in \mathfrak{h}$, assuming the case $m = 1$. Let’s write H in place of X . Consider the involution $*$: $\mathfrak{g} \rightarrow \mathfrak{g}$ given by -1 on the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of the compact Lie group associated to \mathfrak{g} , and $+1$ on its orthogonal complement. Extend this to an involution $Y_m \mapsto (Y^*)_{-m}$ of $\widetilde{L}\mathfrak{g}$. Assuming w.l.o.g. that $H^* = H$, we have:

$$\begin{aligned}
[H_{-1}, T_0] &= [H_1^*, T_0^*] = -\underbrace{[H_1, T_0]^*}_{= 2(k+h^\vee)H_1 \text{ by assumption}} = -2(k + h^\vee)H_{-1}.
\end{aligned}$$

We now assemble the above three cases. As both $[-, T_0]$ and $X_m \mapsto 2m(k + h^\vee)X_m$ are derivations, and since they agree on a set of generators of $\widetilde{L}\mathfrak{g}_k$, they agree everywhere.

Step 2. Let us now assume that $n \neq 0$. Let $\mathbf{U}(\widetilde{L}\mathfrak{g})$ denote the version of the universal algebra of $\widetilde{L}\mathfrak{g}$ where we allow infinite sums, as long as they become finite when evaluated on any vector of any positive energy representation. For example, $T_0 = \sum_{X \in \mathcal{B}} (X_0 X_0 + 2 \sum_{m>0} X_{-m} X_m) \in \mathbf{U}(\widetilde{L}\mathfrak{g})$. The action $\ell_n \cdot X_m = -mX_{m+n}$ of the Witt algebra on $\widetilde{L}\mathfrak{g}$ extends to an action on $\mathbf{U}(\widetilde{L}\mathfrak{g})$. For $n \neq 0$, we have

$$\begin{aligned}
\ell_n \cdot T_0 &= 2 \sum_{X \in \mathcal{B}} \sum_{m>0} \left(\underbrace{(\ell_n \cdot X_{-m}) X_m}_{mX_n X_{-m}} + \underbrace{X_{-m} (\ell_n \cdot X_m)}_{-mX_{-m} X_{n+m}} \right) \\
&= 2 \sum_{X \in \mathcal{B}} \sum_{m \in \mathbb{Z}} m X_m X_{n-m} \\
&= \sum_{X \in \mathcal{B}} \sum_{m \in \mathbb{Z}} \left(m X_m X_{n-m} + \underbrace{m X_{n-m} X_m}_{(n-m')X_{m'} X_{n-m'}} \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{X \in \mathcal{B}} \sum_{m \in \mathbb{Z}} n X_m X_{-m} \\
&= n T_n
\end{aligned}$$

It follows that

$$\begin{aligned}
n[X_m, T_n] &= [X_m, \ell_n \cdot T_0] = \underbrace{[-\ell_n \cdot X_m, T_0]}_{m X_{m+n}} + \ell_n \cdot \underbrace{[X_m, T_0]}_{2m(k+h^\vee)X_m} \\
&= 2(m(m+n) - m^2)(k+h^\vee)X_{m+n} \\
&= 2mn(k+h^\vee)X_{m+n}
\end{aligned}$$

Now divide by n to obtain the desired equation. \square

At last, the following lemma contains the heart of the proof of Proposition 22:

Lemma 23 For $H \in \mathfrak{h}$, we have $[H_1, T_0] = 2(k+h^\vee)H_1$.

Proof. Let us assume, without loss of generality, that $\|H\|^2 = 1$. It will be convenient to use the following way of writing T_0 , analogous to the formula (31) for the Casimir:

$$T_0 = \sum'_{\alpha \in \Phi} E_0^\alpha E_0^{-\alpha} + \sum_i H_0^i H_0^i + 2 \sum_{n>0} \left(\sum'_{\alpha \in \Phi} E_{-n}^\alpha E_n^{-\alpha} + \sum_i H_{-n}^i H_n^i \right).$$

Here, H^1, \dots, H^r form an orthonormal basis of \mathfrak{h} , and the \sum' notation is as in (31). Let us also assume, without loss of generality, that $H^1 = H$. We may now compute:

$$\begin{aligned}
[H_1, T_0] &= \sum'_{\alpha \in \Phi} (\alpha, H) E_1^\alpha E_0^{-\alpha} - \sum'_{\alpha \in \Phi} (\alpha, H) E_0^\alpha E_1^{-\alpha} \\
&\quad + 2 \underbrace{\sum_{n>0} \sum'_{\alpha \in \Phi} (\alpha, H) E_{-n+1}^\alpha E_n^{-\alpha}}_{2 \sum_{n \geq 0} \sum'_{\alpha \in \Phi} (\alpha, H) E_{-n}^\alpha E_{n+1}^{-\alpha}} - 2 \sum_{n>0} \sum'_{\alpha \in \Phi} (\alpha, H) E_{-n}^\alpha E_{n+1}^{-\alpha} + 2[H_1, H_{-1}]H_1 \\
&\quad \leftarrow \text{the terms for } n > 0 \text{ cancel out} \\
&= \underbrace{\sum'_{\alpha \in \Phi} (\alpha, H) E_1^\alpha E_0^{-\alpha} + \sum'_{\alpha \in \Phi} (\alpha, H) E_0^\alpha E_1^{-\alpha}}_{\sum'_{\alpha \in \Phi} (\alpha, H) [E_1^\alpha, E_0^{-\alpha}]} + 2kH_1 \\
&= \underbrace{\sum'_{\alpha \in \Phi} (\alpha, H) H_1^\alpha}_{\text{By (27), this element of } \mathfrak{h} \text{ is equal to } \sum_{\alpha \in \Phi} (\alpha, H) \langle \alpha, - \rangle.} + 2kH_1 \\
&\quad \leftarrow \text{By symmetry reasons, it belongs to the line spanned by } H. \text{ We may thus replace each } \langle \alpha, - \rangle \text{ by its projection } (\alpha, H)H. \\
&= \left(\sum_{\alpha \in \Phi} (\alpha, H)^2 + 2k \right) H_1
\end{aligned}$$

The quantity $\sum_{\alpha \in \Phi} (\alpha, H)^2$ is equal to $\text{Tr}(\text{ad}(H)^2) = \|H\|_{\text{Killing}}^2 = \frac{\|H\|_{\text{Killing}}^2}{\|H\|^2} = 2h^\vee$. It follows that $[H_1, T_0] = 2(k+h^\vee)H_1$, as desired. \square

The results in Proposition 22 and Lemma 20 together imply that the Segal-Sugawara operators satisfy $[L_m, L_n] = (m - n)L_{m+n} + \text{cst}$. It remains to determine the constant (hence the central charge of the Virasoro algebra). Note that when $m + n \neq 0$ the constant is zero for degree reasons, so we may focus on the case $m + n = 0$:

Proposition 24 *The un-normalised Segal-Sugawara operators satisfy*

$$[T_n, T_{-n}] = 2(k + h^\vee) \left(2nT_0 + k \dim(\mathfrak{g}) \frac{n^3 - n}{6} \right).$$

Proof.

$$\begin{aligned} [T_n, T_{-n}] &= \left[\sum_{X \in \mathcal{B}} \left(\sum_{m < 0} X_m X_{n-m} + \sum_{m \geq 0} X_{n-m} X_m \right), T_{-n} \right] \\ &= 2(k + h^\vee) \sum_{X \in \mathcal{B}} \left(\underbrace{\sum_{m < 0} m X_{m-n} X_{n-m}}_{\sum_{m > n} (n-m) X_{-m} X_m} + \sum_{\substack{m < 0 \\ m > 0}} (n - m) X_m X_{-m} \right) \\ &\quad + \left(\sum_{m \geq 0} (n - m) X_{-m} X_m + \sum_{m \geq 0} m X_{n-m} X_{m-n} \right) \\ &= 2(k + h^\vee) \left(2nT_0 + \underbrace{\sum_{X \in \mathcal{B}} \left(\sum_{m=-n}^{-1} (m+n) X_{-m} X_m - \sum_{m=1}^n (n-m) X_{-m} X_m \right)}_{\sum_{m=1}^n (n-m) [X_m, X_{-m}] = k \cdot m} \right) \\ &= 2(k + h^\vee) \left(2nT_0 + k \dim(\mathfrak{g}) \underbrace{\sum_{m=1}^n m(n-m)}_{= \frac{n^3 - n}{6}} \right) \end{aligned}$$

That's a cubic in n . To check that it's $\frac{n^3 - n}{6}$, just evaluate it at four distinct values of n .

□

Corollary: *The Segal-Sugawara operators $L_n = \frac{1}{2(k+h^\vee)} T_n$ satisfy*

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n,0}$$

with $c = \frac{k \cdot \dim(\mathfrak{g})}{k + h^\vee}$.

This finishes the proof of Theorem 18.

□

Primary fields

Let $\Delta \in \mathbb{Z}$ be an integer.

Definition: A *primary field of conformal dimension Δ* is a gadget φ that assigns to every complex cobordism Σ equipped with:

- points $z_1, \dots, z_n \in \Sigma$, and
- tangent vectors $v_i \in T_{z_i}\Sigma$,

and to every object $\lambda \in \mathcal{C}(\partial_{in}\Sigma)$, a linear map

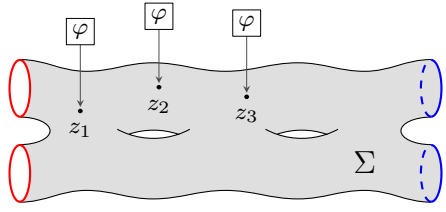
$$Z_{\Sigma, \varphi(z_1; v_1), \dots, \varphi(z_n; v_n)} : U(\lambda) \rightarrow U(F_{\Sigma}(\lambda)).$$

These maps are homogeneous of degree Δ in the v_i 's:

$$Z_{\Sigma, \varphi(z_1; v_1), \dots, \varphi(z_i; a v_i), \dots, \varphi(z_n; v_n)} = a^{\Delta} Z_{\Sigma, \varphi(z_1; v_1), \dots, \varphi(z_n; v_n)} \quad \forall a \in \mathbb{C}, \quad (32)$$

and agree with Z_{Σ} when $n = 0$. Moreover, they satisfy the same axioms that the Z_{Σ} satisfy (naturality in λ and in Σ , compatibility with disjoint union, and with composition of cobordisms).

The map $Z_{\Sigma, \varphi(z_1; v_1), \dots, \varphi(z_n; v_n)}$ is called the *evolution operator with field insertions*:



$$Z_{\Sigma, \varphi(z_1; v_1), \dots, \varphi(z_n; v_n)} : U(\lambda) \longrightarrow U(F_{\Sigma}(\lambda))$$

Example: The *vacuum field* Ω

$$Z_{\Sigma, \Omega(z_1; v_1), \dots, \Omega(z_n; v_n)} := Z_{\Sigma}$$

is a primary field of conformal dimension zero.

We will often suppress the vectors v_i from the notation, and write $Z_{\Sigma, \varphi(z_1), \dots, \varphi(z_n)}$ instead of $Z_{\Sigma, \varphi(z_1; v_1), \dots, \varphi(z_n; v_n)}$.

When Σ is a closed surface and $\lambda = 1_{\text{Vec}}$, then $Z_{\Sigma, \varphi(z_1), \dots, \varphi(z_n)}(1)$ is called a *correlator*, and denoted

$$\langle \varphi(z_1), \dots, \varphi(z_n) \rangle_{\Sigma} \in H_{\Sigma} := U(F_{\Sigma}(\mathbb{C})).$$

A linear functional $\mathcal{B} : H_{\Sigma} \rightarrow \mathbb{C}$ is called a *conformal block*³⁰, and we write

$$\langle \varphi(z_1), \dots, \varphi(z_n) \rangle_{\Sigma, \mathcal{B}} \in \mathbb{C} \quad (33)$$

for the image of $\langle \varphi(z_1), \dots, \varphi(z_n) \rangle_{\Sigma}$ under \mathcal{B} . When thought of as a function of the z_i 's, the expression (33) is also called a *correlation function*.

³⁰Some people would call the expression (33) (viewed as a function of the z_i) the 'conformal block'.

Theorem. (State-field correspondence) *There is a natural bijection*

$$\left\{ \begin{array}{l} \text{Primary fields of} \\ \text{conformal dimension } \Delta \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{States } \xi \in H_0 \text{ such that} \\ L_0(\xi) = \Delta\xi \text{ and } L_n(\xi) = 0 \forall n > 0 \end{array} \right\}.$$

Proof. Given a field φ , the corresponding state $\xi \in H_0$ is given by

$$\xi := \begin{array}{c} \boxed{\varphi} \\ \downarrow \\ \bullet \\ \text{0} \end{array} \mathbb{D} = Z_{\mathbb{D}, \varphi(0;1)}(1) \in H_0.$$

$\boxed{1 \in \mathbb{C} \in \text{Vec} = \mathcal{C}(\emptyset)}$
 $\boxed{1 \in T_0\mathbb{D} = \mathbb{C}}$

We need to show that

$$L_0(\xi) = \Delta\xi \quad \text{and} \quad L_n(\xi) = 0 \forall n > 0. \quad (34)$$

Let $\text{Univ}_0(\mathbb{D}) := \{f \in \text{Univ}(\mathbb{D}) \mid f(0) = 0\}$ be the semigroup associated to the Lie algebra $\text{Vir}_{\geq 0} := \text{Span}\{L_n\}_{n \geq 0}$. The conditions (34) are equivalent to

$$Z_A \xi = f'(0)^\Delta \xi \quad \forall f \in \text{Univ}_0(\mathbb{D}), A = \mathbb{D} \setminus f(\overset{\circ}{\mathbb{D}}).$$

We can then compute:

$$Z_A \xi = Z_A Z_{\mathbb{D}, \varphi(0;1)}(1) = Z_{A \cup \mathbb{D}, \varphi(0;1)}(1) = Z_{\mathbb{D}, \varphi(0;f'(0))}(1) = f'(0)^\Delta \xi.$$

Conversely, starting from a vector $\xi \in H_0$ that satisfies the equations (34), we proceed as follows. Given a complex cobordism Σ together with points z_1, \dots, z_n and tangent vectors $v_i \in T_{z_i}\Sigma$, choose disjoint embeddings $f_i : \mathbb{D} \rightarrow \Sigma$, $f_i(0) = z_i$, and let

$$\Sigma^0 := \Sigma \setminus (f_1(\overset{\circ}{\mathbb{D}}) \sqcup \dots \sqcup f_n(\overset{\circ}{\mathbb{D}})).$$

We then define

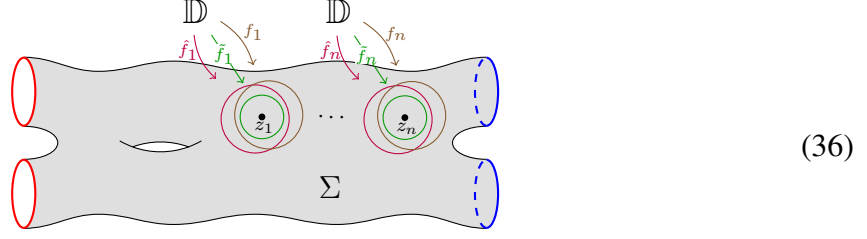
$$Z_{\Sigma, \varphi(z_1), \dots, \varphi(z_n)} := \prod \left(\frac{v_i}{f'_i(0)} \right)^\Delta Z_{\Sigma^0}(\xi \otimes \dots \otimes \xi \otimes -). \quad (35)$$

Let's check that this map lands in the right place:

$$\begin{aligned} Z_{\Sigma^0} : H_0 \otimes \dots \otimes H_0 \otimes U(\lambda) &= U(F_{\mathbb{D}}(1) \otimes \dots \otimes F_{\mathbb{D}}(1) \otimes \lambda) \\ \downarrow \quad \quad \quad \downarrow &\longrightarrow U(F_{\Sigma^0}(F_{\mathbb{D}}(1) \otimes \dots \otimes F_{\mathbb{D}}(1) \otimes \lambda)) \\ \xi \quad \quad \quad \xi &= U(F_{\Sigma^0 \cup (\mathbb{D} \sqcup \dots \sqcup \mathbb{D})}(1 \otimes \dots \otimes 1 \otimes \lambda)) = U(F_{\Sigma}(\lambda)). \quad \checkmark \end{aligned}$$

We need to show that the map (35) is independent of the choice of f_i . Let $\hat{f} : \mathbb{D} \rightarrow \Sigma$, $\hat{f}_i(0) = z_i$, be another set of embeddings, and let $\hat{Z}_{\Sigma, \varphi(z_1), \dots, \varphi(z_n)} : U(\lambda) \rightarrow U(F_{\Sigma}(\lambda))$ be the corresponding map, defined as in (35). We wish to show that $\hat{Z}_{\Sigma, \dots} = Z_{\Sigma, \dots}$. In order to do so, we introduce yet another set of embeddings $\tilde{f} : \mathbb{D} \rightarrow \Sigma$, $\tilde{f}_i(0) = z_i$, that

satisfy $\tilde{f}_i(\mathbb{D}) \subset f_i(\mathbb{D}) \cap \hat{f}_i(\mathbb{D})$. Let $\tilde{Z}_{\Sigma, \dots}$ be the corresponding map. We will show that $\hat{Z}_{\Sigma, \dots} = \tilde{Z}_{\Sigma, \dots} = Z_{\Sigma, \dots}$.



Clearly, it's enough to show that $\tilde{Z}_{\Sigma, \dots} = Z_{\Sigma, \dots}$ (the other equality follows by symmetry). Let

$$\psi_i := f_i^{-1} \circ \tilde{f}_i \quad \text{and let} \quad A_i := \mathbb{D} \setminus \psi_i(\mathring{\mathbb{D}}) = f_i(\mathbb{D}) \setminus \tilde{f}_i(\mathring{\mathbb{D}})$$

be the corresponding annuli. We then have $Z_{A_i}(\xi) = \psi_i'(0)^\Delta \xi = \left(\frac{\tilde{f}_i'(0)}{f_i'(0)}\right)^\Delta \xi$, from which we get:

$$\begin{aligned} \tilde{Z}_{\Sigma, \varphi_1(z_1), \dots, \varphi_n(z_n)} &= \prod \left(\frac{v_i}{\tilde{f}_i'(0)}\right)^\Delta Z_{\tilde{\Sigma}^0}(\xi \otimes \dots \otimes \xi \otimes -) \\ &= \prod \left(\frac{v_i}{\tilde{f}_i'(0)}\right)^\Delta Z_{\Sigma^0}(Z_{A_1}(\xi) \otimes \dots \otimes Z_{A_n}(\xi) \otimes -) \\ &= \prod \left(\frac{v_i}{\tilde{f}_i'(0)}\right)^\Delta \prod \left(\frac{\tilde{f}_i'(0)}{f_i'(0)}\right)^\Delta Z_{\Sigma^0}(\xi \otimes \dots \otimes \xi \otimes -) = Z_{\Sigma, \varphi_1(z_1), \dots, \varphi_n(z_n)}. \end{aligned} \quad (37)$$

□

Given a bunch of primary fields $\varphi_1, \dots, \varphi_n$, with corresponding vectors $\xi_1, \dots, \xi_n \in H_0$, it's now easy to adapt the definition (35):

$$Z_{\Sigma, \varphi_1(z_1; v_1), \dots, \varphi_n(z_n; v_n)} := \prod \left(\frac{v_i}{f_i'(0)}\right)^\Delta Z_{\Sigma^0}(\xi_1 \otimes \dots \otimes \xi_n \otimes -). \quad (38)$$

Here, as before, $\Sigma^0 = \Sigma \setminus (f_1(\mathring{\mathbb{D}}) \sqcup \dots \sqcup f_n(\mathring{\mathbb{D}}))$ for some $f_i : \mathbb{D} \rightarrow \Sigma$ satisfying $f_i(0) = z_i$.

It is also fruitful to allow the ξ_i to take their values in other sectors than the vacuum sector. The corresponding fields are called *charged fields*. Given irreducible objects $\mu_i \in \mathcal{C}(S^1)$, and vectors $\xi_i \in U(\mu_i)$ satisfying the same conditions (34) as before, the definition (38) still makes sense, even though it's no longer a map $U(\lambda) \rightarrow U(F_\Sigma(\lambda))$. Instead, it's a map:

$$Z_{\Sigma, \varphi_1(z_1), \dots, \varphi_n(z_n)} : U(\lambda) \rightarrow U(F_{\Sigma, \mu_1(z_1), \dots, \mu_n(z_n)}(\lambda)),$$

where $F_{\Sigma, \mu_1(z_1), \dots, \mu_n(z_n)}(\lambda) := F_{\Sigma^0}(\mu_1 \otimes \dots \otimes \mu_n \otimes \lambda)$.

As before, $Z_{\Sigma, \varphi_1(z_1), \dots, \varphi_n(z_n)}$ depends on the choice of tangent vectors $v_i \in T_{z_i \Sigma}$. Similarly, the functor

$$F_{\Sigma, \mu_1(z_1), \dots, \mu_n(z_n)} : \mathcal{C}(\partial_{in} \Sigma) \rightarrow \mathcal{C}(\partial_{out} \Sigma)$$

depends on some extra choices at the points z_i . But what it depends on is somewhat weaker than tangent vectors: the functor $F_{\Sigma, \dots}$ only depends on rays $\rho_i \subset T_{z_i \Sigma}$ [a ray in a

vector space V is an element of the quotient $(V \setminus \{0\})/\mathbb{R}_+$. Also, when defining Σ^0 , it was important to have used embeddings $f_i : \mathbb{D} \rightarrow \Sigma$ which satisfied $f_i'(0) \in \rho_i$.

Let us show that $F_{\Sigma, \mu_1(z_1), \dots, \mu_n(z_n)}$ doesn't depend of the choice of f_i (up to canonical iso). Let \hat{f} be another choice, and let $\hat{F}_{\Sigma, \dots}$ be the corresponding functor. To compare $\hat{F}_{\Sigma, \dots}$ and $F_{\Sigma, \dots}$, we pick a third set of maps \tilde{f} as in (36), and let $\tilde{F}_{\Sigma, \dots}$ be the corresponding functor. It's enough to identify $\tilde{F}_{\Sigma, \dots}$ with $F_{\Sigma, \dots}$. Let $\psi_i = f_i^{-1} \circ \tilde{f}_i$ and $A_i = \mathbb{D} \setminus \psi_i(\mathring{\mathbb{D}})$. Since $\psi_i' \in \mathbb{R}_+$, the annulus A_i comes with a canonical lift \tilde{A}_i to the universal cover of $\text{Univ}_0(\mathbb{D})$. The desired identification is then given by:

$$\begin{aligned} \tilde{F}_{\Sigma, \mu_1(z_1), \dots, \mu_n(z_n)} &= F_{\tilde{\Sigma}^0}(\mu_1 \otimes \dots \otimes \mu_n \otimes -) \\ &= F_{\Sigma^0}(F_{A_1}(\mu_1) \otimes \dots \otimes F_{A_n}(\mu_n) \otimes -) \\ &\quad \downarrow T_{\tilde{A}_1} \qquad \qquad \downarrow T_{\tilde{A}_n} \\ F_{\Sigma^0}(\mu_1 \otimes \dots \otimes \mu_n \otimes -) &= F_{\Sigma, \mu_1(z_1), \dots, \mu_n(z_n)}. \end{aligned}$$

Descendant fields

Our next goal is to generalize (38) to the case when the condition $\text{Vir}_{>0} \xi = 0$ is no longer satisfied. These are called *descendant fields*. More precisely, I'd call (the field associated to) $\xi \in H_0$ a *descendant* (of ξ_0) if $\xi = L_{m_1} \dots L_{m_k} \xi_0$ for some $m_1, \dots, m_k \leq 0$, and some primary ξ_0 . When dealing with descendant fields, we need to replace the vectors v_i by elements of the *jet space*:

Definition: Let Σ be a Riemann surface. The *jet space* of order d of Σ is given by:

$$\begin{aligned} J_z^d \Sigma &:= \left\{ j : \mathbb{C} \begin{array}{c} \dashrightarrow \\ \xrightarrow{z} \end{array} \Sigma \mid \begin{array}{l} j \text{ is holomorphic,} \\ j'(0) \neq 0 \end{array} \right\} / j_1 \sim j_2 \text{ if } j_1(z) = j_2(z) + o(z^d), \\ J^d \Sigma &:= \bigcup_{z \in \Sigma} J_z^d \Sigma. \end{aligned}$$

Here, the notation $j : \mathbb{C} \dashrightarrow \Sigma$ means that j is only defined in a neighbourhood of 0.

Consider the tower of Lie algebras

$$\text{Vir}_{[0]} \leftarrow \text{Vir}_{[0,1]} \leftarrow \text{Vir}_{[0,2]} \leftarrow \text{Vir}_{[0,3]} \leftarrow \dots \leftarrow \text{Vir}_{\geq 0},$$

where $\text{Vir}_{[0,d-1]} := \text{Vir}_{\geq 0} / \text{Vir}_{\geq d} = \text{Span}\{L_0, \dots, L_{d-1}\}$. They integrate to a tower of Lie groups:

$$\mathbb{C}^\times = G_1 \leftarrow G_2 \leftarrow G_3 \leftarrow G_4 \leftarrow \dots \leftarrow \text{Univ}_0(\mathbb{D}),$$

where $G_d = J_0^d \mathbb{C}$, with group operation given by composition of functions. In other words:

$$G_d = \left\{ \begin{array}{l} \text{changes of coordinate} \\ \text{defined up to degree } d \end{array} \right\}.$$

One can also describe that group more algebraically, as $\text{Aut}(\mathbb{C}[z]/z^{d+1})$.

Definition: A vector $\xi \in H_0$ is called a *finite energy vector* if it is a finite linear combination of eigenvectors of L_0 .

We now generalize (38) to the case when ξ_i are arbitrary finite energy vectors. By the positive energy condition, since the L_n for $n > 0$ are lowering operators, the action of $Vir_{\geq 0}$ on ξ_i generates a finite dimensional subspace. Call it $V_i \subset H_0$. The action of $Vir_{\geq 0}$ on V_i factors through a finite quotient $Vir_{[0, d_i-1]}$, and integrates to an action of G_{d_i} . [A priori, one might expect the action to only integrate to an action of the universal cover \tilde{G}_{d_i} of G_{d_i} . But the subalgebra $Vir_{[0]} \subset Vir_{[0, d_i-1]}$ integrates to a \mathbb{C}^\times . So the action of \tilde{G}_{d_i} on V_i descends to G_{d_i} .]

Instead of (38), we can then write:

$$Z_{\Sigma, \varphi_1(z_1; j_1), \dots, \varphi_n(z_n; j_n)} := Z_{\Sigma^0} (g_1 \xi_1 \otimes \dots \otimes g_n \xi_n \otimes -), \quad (39)$$

where $g_i := f_i^{-1} \circ j_i \in G_{d_i}$ and, as before, $\Sigma^0 = \Sigma \setminus (f_1(\mathbb{D}) \sqcup \dots \sqcup f_n(\mathbb{D}))$. Once again, we abbreviate things by writing $Z_{\Sigma, \varphi_1(z_1), \dots, \varphi_n(z_n)}$ instead of $Z_{\Sigma, \varphi_1(z_1; j_1), \dots, \varphi_n(z_n; j_n)}$.

In order to check that the map (39) is well defined (independent of the f_i), we proceed along the same lines as the previous proof. The analog of (37) (the most relevant part of the computation) is given by:

$$\begin{aligned} \tilde{Z}_{\Sigma, \varphi_1(z_1), \dots, \varphi_n(z_n)} &= Z_{\tilde{\Sigma}^0} (\tilde{g}_1 \xi_1 \otimes \dots \otimes \tilde{g}_n \xi_n \otimes -) \\ &= Z_{\Sigma^0} (Z_{A_1}(\tilde{g}_1 \xi_1) \otimes \dots \otimes Z_{A_n}(\tilde{g}_n \xi_n) \otimes -) \\ &= Z_{\Sigma^0} (\psi_1 \tilde{g}_1(\xi_1) \otimes \dots \otimes \psi_n \tilde{g}_n(\xi_n) \otimes -) \\ &= Z_{\Sigma^0} (g_1 \xi_1 \otimes \dots \otimes g_n \xi_n \otimes -) \\ &= Z_{\Sigma, \varphi_1(z_1), \dots, \varphi_n(z_n)}, \end{aligned}$$

where $\psi_i = f_i^{-1} \circ \tilde{f}_i$ and $A_i = f_i(\mathbb{D}) \setminus \tilde{f}_i(\mathring{\mathbb{D}})$.

Lemma 25 *Let $\xi \in H_0$ be a finite energy vector, with corresponding field φ . Let $g \in G_d$ be a group element, and let $g\varphi$ be the field that corresponds to $g\xi$. Then we have:*

$$\varphi(z; j \circ g) = (g\varphi)(z; j).$$

Proof. $Z_{\Sigma, \varphi(z; j \circ g), \dots} = Z_{\Sigma^0} ((f^{-1} j g) \xi \otimes \dots) = Z_{\Sigma^0} ((f^{-1} j)(g\xi) \otimes \dots) = Z_{\Sigma, (g\varphi)(z; j), \dots}$ where, as before, $\Sigma^0 = \Sigma \setminus (f(\mathring{\mathbb{D}}) \sqcup \dots)$ for some embeddings $f : \mathbb{D} \xrightarrow[0 \mapsto z]{}$ Σ, \dots \square

When ξ is an eigenvector of L_0 , one can also describe these more general types of fields axiomatically. In the definition of primary field, just replace the tangent vector v_i by a local coordinate $j_i : \mathbb{C} \xrightarrow[0 \mapsto z_i]{}$ Σ , and require the equation $\varphi(z; j \circ (z \mapsto az)) = a^\Delta \varphi(z; j)$ to hold:

$$Z_{\Sigma, \varphi(z_1; j_1), \dots, \varphi(z_i; j_i \circ (z \mapsto az)), \dots, \varphi(z_n; j_n)} = a^\Delta Z_{\Sigma, \varphi(z_1; j_1), \dots, \varphi(z_n; j_n)} \quad \forall a \in \mathbb{C}^\times.$$

We call such a thing a *field of conformal dimension Δ* . Similarly to the case of primary fields, we then have:

Theorem. (State-field correspondence) *There is a natural bijection:*

$$\left\{ \text{Fields of conformal dimension } \Delta \right\} \longleftrightarrow \left\{ \xi \in H_0 \mid L_0(\xi) = \Delta \xi \right\}.$$

The proof goes along the same lines as the one in the previous section.

One of the defining properties of chiral CFT is that the fields $\varphi(z)$ ‘depend holomorphically on z ’. We formalize this in the following proposition:

Proposition. *Let φ be a field. Then the map*

$$\begin{aligned} J^d(\Sigma \setminus \{\dots\}) &\longrightarrow \text{Hom}(U(\lambda), U(F_\Sigma(\lambda))) \\ j \in J_z^d \Sigma &\mapsto Z_{\Sigma, \varphi(z; j), \dots} \end{aligned} \quad (40)$$

is holomorphic. (Here, the “ $\{\dots\}$ ” refers to the finitely many points where the other field insertions take place.)

Proof. Holomorphicity is a local condition. For every disc $D \subset \Sigma \setminus \{\dots\}$, we have $Z_{\Sigma, \varphi(z; j), \dots} = Z_{\Sigma \setminus D, \dots} \circ Z_{D, \varphi(z; j)}$. So it’s enough to prove the statement when Σ is a disc.

When $\Sigma = \mathbb{D}$, we can identify a jet $j \in J_z^d(\mathbb{D})$ with an element $g \in G_d$. By Lemma 25, for every $r < 1$, and every z of norm at most $1 - r$, we have

$$Z_{\mathbb{D}, \varphi(z; j)} = Z_{\mathbb{D}, (g\varphi)(z; w \mapsto w+z)} = Z_{\mathbb{D} \setminus (r\mathbb{D}+z)} \circ Z_{r\mathbb{D}, (g\varphi)(0; \text{id})}.$$

$Z_{\mathbb{D} \setminus (r\mathbb{D}+z)}$ depends holomorphically on z because $\mathbb{D} \setminus (r\mathbb{D} + z)$ does, and $Z_{r\mathbb{D}, (g\varphi)(0; \text{id})}$ depends holomorphically on j because $g\varphi$ does. So $Z_{\mathbb{D}, \varphi(z; j)}$ depends holomorphically on z and on j . \square

Examples of fields

- If our conformal field theory is a chiral WZW model, or anything that admits affine Lie algebra symmetries, then, for every $X \in \mathfrak{g}$, we can consider the element $X_{-1}\Omega \in H_0$. The associated field is called a **current** and is denoted $J^X(z)$. This is a primary field of conformal dimension one:

$$L_1 X_{-1} \Omega = X_{-1} \underbrace{L_1 \Omega}_{= 0 \text{ because } \Omega \text{ is } \mathfrak{g}\text{-invariant}} + \underbrace{X_0 \Omega}_{= 0 \text{ because that's in degree } -1} = 0.$$

- If our conformal field theory is a chiral minimal model, or anything that contains Virasoro algebra symmetries (which is to say... any chiral CFT), then we can consider the element $\omega := L_{-2}\Omega \in H_0$, the so-called “conformal vector”. The associated field is called the **stress-energy tensor** and is denoted $T(z)$. This field is also called the ‘energy-momentum tensor’ or ‘Virasoro field’, and it is *not* primary, unless $c = 0$:

$$\begin{aligned}
L_1\omega &= L_1L_{-2}\Omega = 3\underbrace{L_{-1}\Omega}_{=0} = 0 \\
L_2\omega &= L_2L_{-2}\Omega = 4\underbrace{L_0\Omega}_{=0} + \frac{c}{12} \cdot 6 \cdot \Omega = \frac{c}{2} \cdot \Omega
\end{aligned}$$

= 0 because Ω is $PSU(1,1)$ -invariant

The action of $Vir_{\geq 0}$ on ω generates a two dimensional subspace $\text{Span}\{\Omega, \omega\} \subset H_0$, on which the Lie algebra $Vir_{[0,2]}$ acts by

n	0	1	2	
$L_n\Omega$	0	0	0	(41)
$L_n\omega$	2ω	0	$\frac{c}{2}\Omega$	

Claim: At the Lie group level, this integrates to the action of $G_3 = \text{Aut}(\mathbb{C}[z]/z^4)$ given by:

$$\begin{cases} g \cdot \Omega = \Omega \\ g \cdot \omega = g'(0)^2\omega + \frac{c}{12} \left(\frac{g'''(0)}{g'(0)} - \frac{3}{2} \left(\frac{g''(0)}{g'(0)} \right)^2 \right) \Omega. \end{cases} \quad (42)$$

In terms of the basis $\{\Omega, \omega\}$, that representation can be equivalently described as:

$$g \mapsto \left(\begin{array}{cc} 1 & \frac{c}{12} \left(\frac{g'''}{g'} - \frac{3}{2} \left(\frac{g''}{g'} \right)^2 \right) \\ 0 & g'^2 \end{array} \right)_{z=0}$$

Here, the expression $\frac{c}{12} \left(\frac{g'''}{g'} - \frac{3}{2} \left(\frac{g''}{g'} \right)^2 \right)$ is called the *Schwarzian derivative*, and is often denoted by the symbol $\{g, z\}$.

Now, if someone hands you a representation of a Lie group and says “that’s the representation which integrates the following Lie algebra rep”, how do you check it? One way to proceed is as follows:

(1) You compute the formula to first order for an element close to the identity, and check that it agrees with the given Lie algebra representation. Computing modulo ε^2 , we get:

$$\begin{aligned}
\text{If } g(z) &= z + \varepsilon z, \text{ then } \{g, z\} = 0 & \rightsquigarrow & g \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1+2\varepsilon \end{pmatrix}. & \checkmark \\
\text{If } g(z) &= z + \varepsilon z^2, \text{ then } \{g, z\} = 0 & \rightsquigarrow & g \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. & \checkmark \\
^{31}\text{If } g(z) &= z + \varepsilon z^3, \text{ then } \{g, z\} = 6\varepsilon & \rightsquigarrow & g \mapsto \begin{pmatrix} 1 & \frac{c}{2}\varepsilon \\ 0 & 1 \end{pmatrix}. & \checkmark
\end{aligned}$$

(2) You check that it’s indeed a representation:

$$\left(\begin{array}{cc} 1 & \frac{c}{12} \left(\frac{g'''}{g'} - \frac{3}{2} \left(\frac{g''}{g'} \right)^2 \right) \\ 0 & g'^2 \end{array} \right)_{z=0} \left(\begin{array}{cc} 1 & \frac{c}{12} \left(\frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 \right) \\ 0 & f'^2 \end{array} \right)_{z=0} \stackrel{?}{=} \left(\begin{array}{cc} 1 & \frac{c}{12} \left(\frac{(g \circ f)'''}{(g \circ f)'} - \frac{3}{2} \left(\frac{(g \circ f)''}{(g \circ f)'} \right)^2 \right) \\ 0 & (g \circ f)'^2 \end{array} \right)_{z=0}$$

³¹We write $z + \varepsilon z^3$ as opposed to $z - \varepsilon z^3$ for the infinitesimal transformation corresponding to L_2 because of the issue pointed out in the remark on page 13.

This looks like an annoying computation, but it's actually not too bad. To begin with, we compute the first, second, and third derivatives of $g \circ f$ at zero:

$$\begin{aligned}(g \circ f)' &=_{z=0} g' f' \\ (g \circ f)'' &=_{z=0} g'' f'^2 + g' f'' \\ (g \circ f)''' &=_{z=0} g''' f'^3 + 3g'' f' f'' + g' f'''.\end{aligned}$$

Ignoring the $\frac{c}{12}$, the upper right corners of the two sides of the above equation are given by:

$$\frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 + \frac{g''' f'^2}{g'} - \frac{3}{2} \left(\frac{g'' f'}{g'} \right)^2$$

and

$$\begin{aligned}& \frac{g''' f'^3 + 3g'' f' f'' + g' f'''}{g' f'} - \frac{3}{2} \left(\frac{g'' f' + g' f''}{g' f'} \right)^2 \\ &= \frac{g''' f'^2}{g'} + \frac{3g'' f''}{g'} + \frac{f'''}{f'} - \frac{3}{2} \left(\frac{g'' f'}{g'} \right)^2 - 3 \frac{g'' f''}{g'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2.\end{aligned}$$

Those two expressions are indeed equal.

This finishes the proof that the action (41) of $Vir_{[0,2]}$ integrates to the action (42) of the group G_3 .

As an upshot of the above computation, we get the following special case of Lemma 25, known as the '*anomalous transformation of the stress-energy tensor*': For $f : \mathbb{C} \dashrightarrow \mathbb{C}$, $f(0) = 0$, we have

$$T(w; j \circ f) = f'(0)^2 T(w; j) + \frac{c}{12} \{f, z\}_{z=0} \Omega.$$

For $w \in \mathbb{C}$, this becomes:

$$T(w; z \mapsto f(z)) = f'(0)^2 T(w; z \mapsto z + w) + \frac{c}{12} \{f, z\}_{z=0} \Omega.$$

In physics lingo, this is usually expressed in the following terms: "Under the map $z \rightarrow f(z)$, the stress-energy tensor transforms as $T(z) \rightarrow (\partial f)^2 T(f(z)) + \frac{c}{12} \{f, z\}$."

Remark 26 *The Schwarzian derivative also appears in the formulas which describe the action of $\varphi \in \text{Diff}(S^1)$ on the universal central extensions of $\mathfrak{X}_{\mathbb{C}}(S^1)$. Namely, for $(f \frac{\partial}{\partial z}, a) \in {}^{\mathbb{C}}\mathfrak{X}_{\mathbb{C}}(S^1)$, we have $\varphi^*(f \frac{\partial}{\partial z}, a) = (\frac{f \circ \varphi}{\varphi'} \frac{\partial}{\partial z}, a + \frac{1}{12} \int_{S^1} \frac{f \circ \varphi(z)}{\varphi'(z)} \{ \varphi, z \} \frac{dz}{2\pi i})$.*

Because the Schwarzian derivative vanishes on all infinitesimal coordinate transformations of the form $z \rightarrow z + \varepsilon$, $z \rightarrow z + \varepsilon z$ and $z \rightarrow z + \varepsilon z^2$, it vanishes on the group generated by them. Namely, on the group $PSL(2, \mathbb{C})$ of fractional linear transformations.

Slogan: The Schwarzian derivative $\{f, z\}$ is a version of the third derivative which measures the failure of f being a fractional linear transformation (just like f''' measures the failure of f being a quadratic polynomial).

Let $\varphi(z)$ be a field of conformal dimension Δ . One way to say that $\varphi(z)$ is primary is to say that it satisfies

$$\varphi(z; j \circ f) = f'(0)^\Delta \varphi(z; j) \quad \forall f : \mathbb{C} \dashrightarrow \mathbb{C} \\ 0 \mapsto 0$$

That's just a complicated way of saying that $\varphi(z)$ only depends on the vector $j'(0) \in T_z \Sigma$. An arbitrary field $\varphi(z)$ of conformal dimension Δ only satisfies

$$\varphi(z; j \circ (z \mapsto az)) = a^\Delta \varphi(z; j) \quad \forall a \in \mathbb{C}^\times.$$

There's also an intermediate condition which is useful:

Definition: A field $\varphi(z)$ of conformal dimension Δ is called *quasi-primary* if

$$\varphi(z; j \circ f) = f'(0)^\Delta \varphi(z; j) \quad \forall f \in PSL(2, \mathbb{C}), f(0) = 0.$$

The stress-energy tensor $T(z)$ is a quasi-primary field of conformal dimension 2.

The state-field correspondence for quasi-primary fields reads as follows:

Theorem. (State-field correspondence) *There is a natural bijection*

$$\left\{ \begin{array}{l} \text{Quasi-primary fields} \\ \text{of conformal dimension } \Delta \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{States } \xi \in H_0 \text{ s.t.} \\ L_0(\xi) = \Delta \xi \text{ and } L_1(\xi) = 0 \end{array} \right\}.$$

The Segal commutation relations

Let $\xi \in H_0$ be a finite energy vector, let φ be the associated field, let $V \subset H_0$ be the $Vir_{\geq 0}$ -module generated by ξ , and let $d \in \mathbb{N}$ be such that the action of $Vir_{\geq 0}$ on V descends to an action of $Vir_{[0, d-1]}$ and hence to an action of the group G_d . Let Σ be a complex cobordism and let $\lambda \in \mathcal{C}(\partial_{in} \Sigma)$. By Lemma 25, we have a commutative diagram:

$$\begin{array}{ccc} J^d \overset{\circ}{\Sigma} & \longrightarrow & \text{Hom}\left(U(\lambda), U(F_\Sigma(\lambda))\right) \\ & \searrow \downarrow \scriptstyle j \mapsto Z_{\Sigma, \varphi(j(0); j)} & \\ & \downarrow \scriptstyle [(j, \xi)] & \nearrow \scriptstyle \Phi_\Sigma \\ V_\Sigma := J^d \overset{\circ}{\Sigma} \times_{G_d} V & & \end{array} \quad (43)$$

The dotted map is given by $\Phi_\Sigma : [(j, \eta)] \mapsto Z_{\Sigma, \varphi_\eta(j(0); j)}$, where φ_η is the field associated to η . The map Φ_Σ is holomorphic, and linear on the fibers of the vector bundle $V_\Sigma \rightarrow \overset{\circ}{\Sigma}$.

Our next goal is to generalize the above picture to allow the insertion point $z = j(0)$ to be on the boundary $\partial \Sigma$. Assuming Σ is equipped with *collars* (see the picture (46) below), we'll upgrade (43) to a map

$$\Phi_\Sigma : V_\Sigma := J^d \Sigma \times_{G_d} V \rightarrow \text{Hom}\left(\widehat{U(\lambda)}, \widehat{U(F_\Sigma(\lambda))}\right) \quad (44)$$

Here, the cech on $U(\lambda)$ means that we make the space a little bit ‘thinner’, in a way that we’ll explain below, and the hat on $U(F_\Sigma(\lambda))$ means that we make the space a bit ‘fatter’.

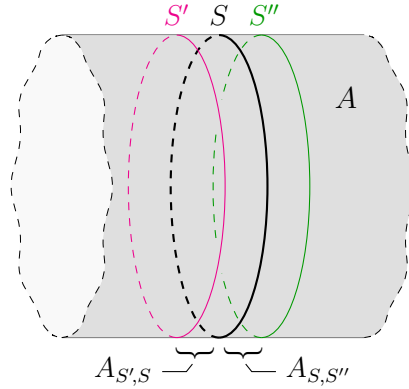
Remark: If φ is primary of conformal dimension Δ , then the vector space V is one dimensional, and $V_\Sigma = T^{\otimes \Delta} \Sigma$.

Let S be a circle. A *collar* is a piece of Riemann surface A in which S is analytically embedded. Two collars $S \hookrightarrow A$ and $S \hookrightarrow B$ are equivalent if there exist open subsets $U \subset A$ and $V \subset B$ containing S and an isomorphism $U \cong V$ that restricts to the identity on S (by analytic continuation, such an isomorphism is unique provided it exists.) An equivalence class of collars is the same thing as an *analytic structure* on S , i.e., a subsheaf $\mathcal{O}_S^{an} \subset \mathcal{O}_S$ of the sheaf of smooth functions on S which is locally isomorphic to the sheaf of analytic functions on \mathbb{R} .

Let $S \hookrightarrow A$ be a collar. Given two circles $S_1, S_2 \subset A$ that are isotopic to S and such that S_2 is ‘in the future’ of S_1 , we write A_{S_1, S_2} for the part of A which lies between S_1 and S_2 (A_{S_1, S_2} is an annulus). For convenience, we abbreviate $F_{A_{S_1, S_2}}$ by F_{S_1, S_2} and $Z_{A_{S_1, S_2}}$ by Z_{S_1, S_2} . For any object $\lambda \in \mathcal{C}(S)$, we can then define

$$\check{H}_\lambda := \varinjlim_{S' \text{ in the past of } S} U(F_{S', S}^{-1}(\lambda)) \quad \text{and} \quad \hat{H}_\lambda := \varprojlim_{S'' \text{ in the future of } S} U(F_{S, S''}(\lambda))$$

where the maps used to define the limits are given by $Z_{S', S_2} : U(F_{S', S}^{-1}(\lambda)) \rightarrow U(F_{S_2, S}^{-1}(\lambda))$ and $Z_{S_1, S''} : U(F_{S, S''}(\lambda)) \rightarrow U(F_{S_1, S''}(\lambda))$, respectively.



We then have dense inclusions $\check{H}_\lambda \subset H_\lambda \subset \hat{H}_\lambda$.³²

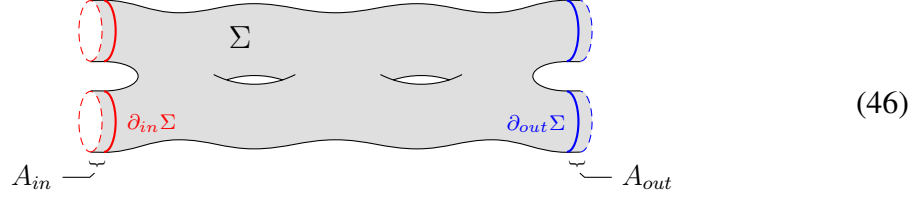
The advantage of working with the $\check{}$ and $\hat{}$ versions is that the map

$$Z_{\Sigma, \varphi_1(z_1), \dots, \varphi_n(z_n)} : \check{U}(\lambda) \longrightarrow U(\widehat{F_\Sigma(\lambda)}) \quad (45)$$

is now well defined for all z_1, \dots, z_n , including on the boundary of Σ . It is induced by the maps $Z_{\Sigma^+, \varphi_1(z_1), \dots, \varphi_n(z_n)} : U(F_{A_{in}}^{-1}(\lambda)) \rightarrow U(F_{A_{out}}(F_\Sigma(\lambda)))$, where $\Sigma^+ = A_{in} \cup \Sigma \cup A_{out}$,

³²G. Segal suggests to add the axiom $H_\lambda = \hat{H}_\lambda$ to the definition of a chiral CFT. Unfortunately, I think that this is incompatible with the requirement that the representations be smooth in the sense of Remark 13.

and A_{in} and A_{out} are thin collars on the outside of Σ :



We also have maps

$$\check{Z}_{\Sigma, \dots} : \widehat{U(\lambda)} \rightarrow U(\widehat{F_{\Sigma}(\lambda)}) \quad \text{and} \quad \widehat{Z}_{\Sigma, \dots} : \widehat{U(\lambda)} \rightarrow U(\widehat{F_{\Sigma}(\lambda)}),$$

defined in the obvious way.

As a particular case of (45), for every circle with collar S , and every point with local coordinate $z \in S$, we have a map

$$\varphi(z) : \check{H}_{\lambda} \rightarrow \widehat{H}_{\lambda}$$

given by $Z_{id_S, \varphi(z)}$.

Let $V_S := J_{\mathbb{C}}^d S \times_{G_d} V$ be as in (43), where $J_{\mathbb{C}}^d$ denotes the complexified jet space of S , and let $\Phi_S : V_S \rightarrow \text{Hom}(\check{H}_{\lambda}, \widehat{H}_{\lambda})$ be as in (44). Given a smooth section $f \in \Gamma(\Omega_S^1 \otimes V_S)$, we define the **smearred field** $\varphi[f] : \check{H}_{\lambda} \rightarrow \widehat{H}_{\lambda}$ to be the image of f under the map

$$\Gamma(\Omega_S^1 \otimes V_S) \xrightarrow{\Phi_S} \Gamma(\Omega_S^1 \otimes \text{Hom}(\check{H}_{\lambda}, \widehat{H}_{\lambda})) \xrightarrow{J_S} \text{Hom}(\check{H}_{\lambda}, \widehat{H}_{\lambda}).$$

Remark: When φ is primary of conformal dimension Δ , then f is just a section of $T^{\otimes(\Delta-1)}\Sigma$ (in particular, when φ is a current f is just a function). In that case, we can rewrite $\varphi[f]$ in the following more intuitive form:

$$\varphi[f] = \int_S f(z) \varphi(z) dz.$$

A priori, $\varphi[f]$ is only a map from \check{H}_{λ} to \widehat{H}_{λ} . However, as far as I understand, when working with nuclear Fréchet spaces, this always extends by continuity to a map

$$\varphi[f] : H_{\lambda} \rightarrow H_{\lambda}$$

In that sense, quantum fields are *operator valued distributions*. They are things which take a test function f as input and produce an operator $H_{\lambda} \rightarrow H_{\lambda}$ as output.

Remark. When working with Hilbert spaces, a smearred field $\varphi[f]$ is typically not an operator $H_{\lambda} \rightarrow H_{\lambda}$. It is only a map $\check{H}_{\lambda} \rightarrow H_{\lambda}$, as well as a map $H_{\lambda} \rightarrow \widehat{H}_{\lambda}$. In other words, it is an *unbounded operator* on H_{λ} .

Proposition. (Segal commutation relations) *Let Σ be a complex cobordism, and let φ be a field. Then, for every holomorphic section $f \in \Gamma_{\text{hol}}(\Omega_{\Sigma}^1 \otimes V_{\Sigma})$, letting $f_{in} := f|_{\partial_{in}\Sigma}$ and $f_{out} := f|_{\partial_{out}\Sigma}$, we have:*

$$\varphi[f_{out}] \circ \check{Z}_{\Sigma} = \widehat{Z}_{\Sigma} \circ \varphi[f_{in}]. \quad (47)$$

Proof. Consider the image of f under the map

$$\Gamma_{\text{hol}}(\Omega_{\Sigma}^1 \otimes V_{\Sigma}) \xrightarrow{\Phi_{\Sigma}} \Gamma_{\text{hol}}\left(\Omega^1 \otimes \text{Hom}\left(\widehat{U(\lambda)}, U(\widehat{F_{\Sigma}(\lambda)})\right)\right).$$

Then $\Phi_{\Sigma}(f)$ is a $\text{Hom}\left(\widehat{U(\lambda)}, U(\widehat{F_{\Sigma}(\lambda)})\right)$ -valued 1-form which is holomorphic on all of Σ . The integrals $\widehat{Z}_{\Sigma} \circ \varphi[f_{in}] = \int_{\partial_{in}\Sigma} \Phi_{\Sigma}(f)$ and $\varphi[f_{out}] \circ \check{Z}_{\Sigma} = \int_{\partial_{out}\Sigma} \Phi_{\Sigma}(f)$ are therefore equal by Cauchy's theorem. \square

Assuming that all smeared fields extend to maps $H_{\lambda} \rightarrow H_{\lambda}$, the commutation relations (47) simplify to:

$$\varphi[f_{out}] \circ Z_{\Sigma} = Z_{\Sigma} \circ \varphi[f_{in}]$$

It is expected that, when $\partial\Sigma \neq \emptyset$, the functor $F_{\Sigma} : \mathcal{C}(\partial_{in}\Sigma) \rightarrow \mathcal{C}(\partial_{out}\Sigma)$ is universal with respect to the Segal commutation relations:

Conjecture 27 Fix a functorial chiral CFT. Let Σ be a complex cobordism with non-empty boundary, and let $\lambda \in \mathcal{C}(\partial_{in}\Sigma)$. Then, for every object $\mu \in \mathcal{C}(\partial_{out}\Sigma)$ and every linear map $\zeta : U(\lambda) \rightarrow U(\mu)$,

if for every field φ of the CFT and every holomorphic section $f \in \Gamma_{\text{hol}}(\Omega_{\Sigma}^1 \otimes V_{\Sigma})$, the equation

$$\varphi[f_{out}] \circ \zeta = \zeta \circ \varphi[f_{in}]$$

holds,

then there exists a unique morphism $\kappa : F_{\Sigma}(\lambda) \rightarrow \mu$ such that

$$\zeta = U(\kappa) \circ Z_{\Sigma}.$$

2d chiral CFT as a boundary of 3d TQFT

We can generalize the moduli space (19) by replacing the disc \mathbb{D} by an arbitrary complex cobordism. Given a complex cobordism Σ , let:

$$\mathcal{D}_{\Sigma}(n) := \left\{ \begin{array}{l} n \text{ holomorphic embeddings } \mathbb{D} \rightarrow \Sigma \\ \text{with non-overlapping images.} \end{array} \right\}$$

Then, for every $P \in \mathcal{D}_{\Sigma}(n)$, we get a functor

$$F_{\Sigma, P} : \mathcal{C}(\partial_{in}\Sigma) \otimes \mathcal{C}^n \longrightarrow \mathcal{C}(\partial_{out}\Sigma).$$

And for every path $\gamma : [0, 1] \rightarrow \mathcal{D}_\Sigma(n)$ from P_1 to P_2 we get an isomorphism $T_\gamma : F_{\Sigma, P_1} \rightarrow F_{\Sigma, P_2}$. Moreover, the isomorphism T_γ only depends on the path γ up to homotopy.

If we fix objects $\mu_1, \dots, \mu_n \in \mathcal{C}$ and interior points $z_1, \dots, z_n \in \overset{\circ}{\Sigma}$ together with rays $\rho_i \subset T_{z_i}\Sigma$, we can always find embeddings $f_i : \mathbb{D} \rightarrow \Sigma$ satisfying $f_i(0) = z_i$ and $f_i'(0) \in \rho_i$. This yields a point $P \in \mathcal{D}_\Sigma(n)$, well defined up to contractible choice. The functor $F_{\Sigma, P}(- \otimes \mu_1 \otimes \dots \otimes \mu_n)$ then agrees with what we had denoted

$$F_{\Sigma, \mu_1(z_1; \rho_1), \dots, \mu_n(z_n; \rho_n)} : \mathcal{C}(\partial_{in}\Sigma) \rightarrow \mathcal{C}(\partial_{out}\Sigma).$$

Moreover, given finite energy vectors $\xi_i \in H_{\mu_i}$, the corresponding charged fields φ_i are maps

$$Z_{\Sigma, \varphi_1(z_1; j_1), \dots, \varphi_n(z_n; j_n)} : U(\lambda) \rightarrow U(F_{\Sigma, \mu_1(z_1; \rho_1), \dots, \mu_n(z_n; \rho_n)}(\lambda)), \quad (48)$$

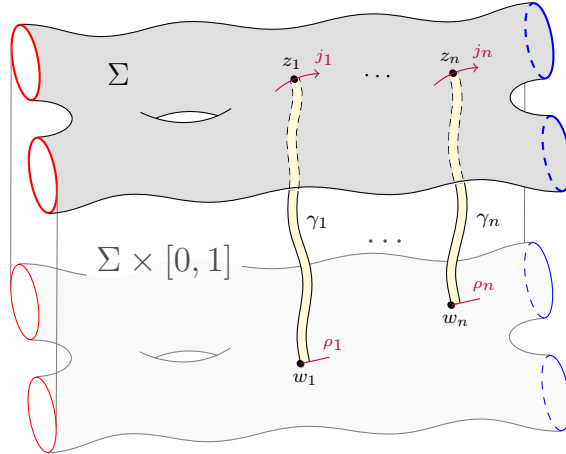
provided $j_i'(0) \in \rho_i$.

We'd like to say that $\varphi_i(z_i)$ depends holomorphically on z_i (and on j_i). This is a little bit tricky to formulate given that the place in which (48) takes its values depends on z_i (and on j_i). But it depends in a *flat* way, i.e., it's a vector bundle with flat connection over the moduli space of z_i 's and ρ_i 's. So we can locally trivialize the right hand side of (48) and pretend that all the $\varphi_i(z_i)$ take their values in the same space, at least locally in z_i and ρ_i .

Thinking more globally, we can make sense of the statement

$$Z_{\Sigma, \varphi_1(z_1; j_1), \dots, \varphi_n(z_n; j_n)}(\eta) \in U(F_{\Sigma, \mu_1(w_1; \rho_1), \dots, \mu_n(w_n; \rho_n)}(\lambda)) \quad \text{for } \eta \in U(\lambda) \quad (49)$$

whenever we are provided with a (homotopy class of) path from $(z_i; j_1'(0)), \dots, (z_n; j_n'(0))$ to $(w_1; \rho_1), \dots, \mu_n(w_n; \rho_n)$. The picture which I wish to associate to (49) is the following:



Here, the path $\gamma : [0, 1] \rightarrow \mathcal{D}(n)$ is interpreted as a ribbon braid $\gamma = (\gamma_1, \dots, \gamma_n)$ inside $\Sigma \times [0, 1]$, connecting the points $(w_1, \dots, w_n) \subset \Sigma \times \{0\}$ to the points $(z_1, \dots, z_n) \subset \Sigma \times \{1\}$. The fact that we could draw things as we did is a manifestation of the fact that *a chiral conformal field theory sits at the boundary of a 3d topological field theory*. The surface on which we drew the z_i 's carries the chiral CFT, whereas the bulk, namely $\Sigma \times [0, 1)$, is where the 3d TQFT lives.

In the special case $\partial\Sigma = \emptyset$, $\lambda = \mathbb{C}$, $\eta = 1$, one can rewrite (49) as follows:

$$\langle \varphi_1(z_1), \dots, \varphi_n(z_n) \rangle_{\Sigma, \gamma_1, \dots, \gamma_n} \in H_{\Sigma, \mu_1(w_1), \dots, \mu_n(w_n)}. \quad (50)$$

Here, $H_{\Sigma, \mu_1(w_1), \dots, \mu_n(w_n)}$ is just an equivalent name for the finite dimensional vector space $U(F_{\Sigma, \mu_1(w_1), \dots, \mu_n(w_n)}(\mathbb{C}))$. For charged fields, the correlator $\langle \varphi_1(z_1), \dots, \varphi_n(z_n) \rangle$ doesn't just depend on the points z_i (and the local coordinates j_i), but also on the ribbon braid γ . (And if you were to try to think of it as a function of just the z_i and the j_i , then it would become a multivalued function.)

As before, a *conformal block* is a linear map $\mathcal{B} : H_{\Sigma, \mu_1(w_1), \dots, \mu_n(w_n)} \rightarrow \mathbb{C}$, and the correlation function

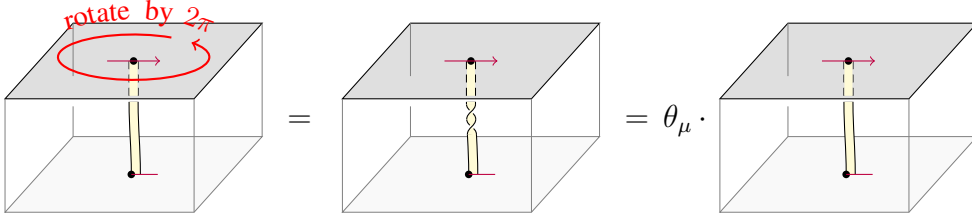
$$\langle \varphi_1(z_1), \dots, \varphi_n(z_n) \rangle_{\Sigma, \mathcal{B}, \gamma_1, \dots, \gamma_n} \in \mathbb{C}.$$

is the image of (50) under that map.

Let $\varphi(z)$ be a field of charge μ which is primary. Then we have

$$\varphi(z; av) = a^\Delta \varphi(z; v) \quad \text{for } v \in T_z \Sigma \quad (51)$$

as in (32). But the point z now has a ribbon attached to it, so a lives no longer in \mathbb{C}^\times but instead in the universal cover of \mathbb{C}^\times . Correspondingly, the conformal dimension Δ is no longer an integer. It is an eigenvalue of the action of L_0 on H_μ . Namely, it is a number of the form $h_\mu + n$, where h_μ is the minimal energy of H_μ and $n \in \mathbb{N}$ is a natural number. For example, if $a = e^{2\pi i}$ then $a^\Delta = (e^{2\pi i})^\Delta = \theta_\mu$ is the conformal spin of μ (which is equal to the twist of μ , coming from the fact that $\mathcal{C}(S^1)$ is balanced). Equation (51) becomes:



Let $\mathcal{M}_{g,n}$ be the moduli space of closed Riemann surfaces Σ of genus g with n marked points w_1, \dots, w_n together with rays $\rho_i \subset T_{z_i} \Sigma$. The fiber of the map $\mathcal{M}_{g,n} \rightarrow \mathcal{M}_g := \mathcal{M}_{g,0}$ [Note that \mathcal{M}_g is not a space but something slightly more general, called a stack] over a Riemann surface $\Sigma \in \mathcal{M}_g$ is the configuration space $\text{Conf}_\Sigma(n)$. The latter is a space which is homotopy equivalent to the space $\mathcal{D}_\Sigma(n)$ defined above. So we have a fiber bundle

$$\begin{array}{ccc} \text{Conf}_\Sigma(n) & \rightarrow & \mathcal{M}_{g,n} \\ & & \downarrow \\ & & \mathcal{M}_g \end{array} \quad (52)$$

Fix objects $\mu_1, \dots, \mu_n \in \mathcal{C}$. Then

$$(\Sigma, (w_1, \dots, w_n)) \mapsto H_{\Sigma, \mu_1(w_1), \dots, \mu_n(w_n)}$$

is a vector bundle of finite rank over $\mathcal{M}_{g,n}$ called (the dual of) the *bundle of conformal blocks*. Its fibers are called the (dual) *spaces of conformal blocks*.

The trivializations $T_{\tilde{A}}$ (see Table 1 on page 20) equip the bundle of conformal blocks with a flat projective connection. What this means is that **the mapping class group $\pi_1(\mathcal{M}_{g,n})$ acts projectively on spaces of conformal blocks**. Moreover, by the construction described in (20), that flat projective connection admits a lift to an honest (i.e. non-projective) flat connection in the direction of the fibers of (52). What this means is the above action of the mapping class group **restricts to an honest action of the surface ribbon braid group $\pi_1(\text{Conf}_{\Sigma}(n))$** .