Homological algebra (Oxford, fall 2016)

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Revision session on Thu. Jan. 19th, 3:00–4:30pm, in rooms C1 (3:00–3:30pm) and C4 (3:30–4:30pm).

Week 1

I recalled the notions of ring, modules, and the fact that $\mathbb{Z}$-modules are the same thing as abelian groups. The direct sum of R-modules M_i is defined by

$$\bigoplus_{i\in\mathcal{I}} M_i := \Big\{ f : \mathcal{I} \to \bigcup_{i\in\mathcal{I}} M_i \,\Big|\, f(i) \in M_i \text{ and } \#\{i \in \mathcal{I} : f(i) = 0_{M_i}\} < \infty \Big\}.$$

The product of modules M_i is given by

$$\prod_{i\in\mathcal{I}} M_i := \Big\{ f : \mathcal{I} \to \bigcup_{i\in\mathcal{I}} M_i \,\Big|\, f(i) \in M_i \Big\}.$$

The R-module structure is given by $(f+g)(i) = f(i) +_{M_i} g(i)$ and $(r\cdot f)(i) = r \cdot_{M_i} (f(i))$.

An inclusion of of R-modules $N \subset M$ is called *split* if there exists another submodule $N' \subset M$ such every element of M can be uniquely written as a sum of an element of N and an element of N'. *Example:* the inclusion $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$ is not split, but the inclusion $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/6$ is split.

Given a ring R, the tensor product over R of a right module M with a left module N is denoted $M \otimes_R N$. It is the abelian group generated by symbols $m_1 \otimes n_1 + \ldots + m_k \otimes n_k$, under the equivalence relation generated by

$$\begin{aligned} (m+m') \otimes n &= m \otimes n + m' \otimes n, \\ m \otimes (n+n') &= m \otimes n + m \otimes n', \\ \text{and} \qquad mr \otimes n &= m \otimes rn. \end{aligned}$$

A chain complex of R-modules $C_\bullet = (C_n, d_n)_{n\in\mathbb{Z}}$ is a collection of R-modules C_n and R-module maps $d_n : C_n \to C_{n-1}$, called 'differentials', subject to the axiom $d_n \circ d_{n+1} = 0$. This axiom is sometimes abusively abbreviated $d^2 = 0$. A chain complex is called *exact* if $\ker(d_n) = \operatorname{im}(d_{n+1})$.

A functor is called *exact* if it sends exact sequences to exact sequences.

Exercise 1. Let $R := \mathbb{R}[x]/x^2$. Prove that the obvious inclusion of R-modules $R/x \hookrightarrow R$ is not split.

Exercise 2. Let $R := M_2(\mathbb{Z})$ be the ring of two-by-two matrices with integer coefficients. Show that the R-module R can be written as a direct sum of two smaller R-modules.

Exercise 3. Show that if p and q are distinct prime numbers, then $\mathbb{Z}/p \otimes_{\mathbb{Z}} \mathbb{Z}/q = 0$.

Exercise 4. Prove that for any abelian group A, there is a canonical isomorphism $A \otimes_{\mathbb{Z}} \mathbb{Z}/2 \cong A/2A$.

Exercise 5. Let $R := \mathbb{Z}[x]$. Compute $Hom_R(R/2x, R/4)$ as an R-module. Show that it is isomorphic to R/I for some ideal $I \subset R$.

Exercise 6. Provide an example of a ring R and two modules M and N such that the abelian group $Hom_R(M, N)$ does not carry the structure of an R-module.

Week 2 A *zero object* is an object that admits exactly one morphism to it from any other object and exactly one morphism from it to any other object.

A *monomorphism* is a morphism f that satisfies $(f \circ g_1 = f \circ g_2) \Rightarrow (g_1 = g_2)$. Equivalently, it is a morphism $f : X \to Y$ with the property that whenever two morphisms $g_1, g_2 : Z \to X$ are distinct, they remain distinct after composing them with f. Dually, an *epimorphism* is a map f that satisfies $(g_1 \circ f = g_2 \circ f) \Rightarrow (g_1 = g_2)$.

The *direct sum* of two objects X_1 and X_2 is an object Z equipped with maps $i_1 : X_1 \to Z$, $i_2 : X_2 \to Z$, $p_1 : Z \to X_1$, $p_2 : Z \to X_2$ satisfying $p_1 \circ i_1 = \mathrm{id}$, $p_2 \circ i_2 = \mathrm{id}$, $p_1 \circ i_2 = 0$, $p_2 \circ i_1 = 0$, and $i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}$.

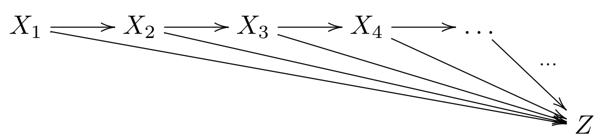
An pre-additive category is a category such that all the hom-sets are equipped with the structure of abelian groups and such that composition $\mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \to \mathrm{Hom}(x, z)$ is bilinear. An *additive category* is a category which is preadditive, admits a zero object, and admits all direct sums.

The *kernel* of a map $f : X \to Y$ is a morphism $i : K \to X$ which is universal w.r.t the property that $f \circ i = 0$. This means the following: it's an object K along with a morphism $i : K \to X$ satisfying $f \circ i = 0$, such that for every object $\tilde{K}$ and every morphism $\tilde{i} : \tilde{K} \to X$ satisfying $f \circ \tilde{i} = 0$, there exists a unique morphism $g : \tilde{K} \to K$ such that $\tilde{i} = i \circ g$.

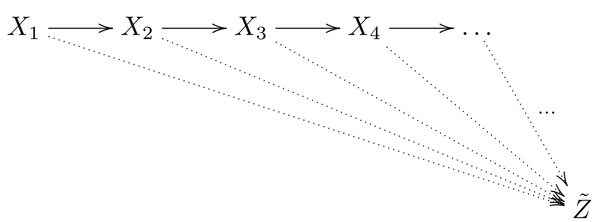
Dually, the *cokernel* of a map $f : X \to Y$ is a morhpism $q : Y \to C$ which is universal w.r.t the property that $q \circ f = 0$.

A *colimit* (also called *direct limit*) of a sequence of morphisms $X_1 \to X_2 \to X_3 \to \ldots$ is an object Z along with morphisms $X_i \to Z$ such that

all little triangles commute

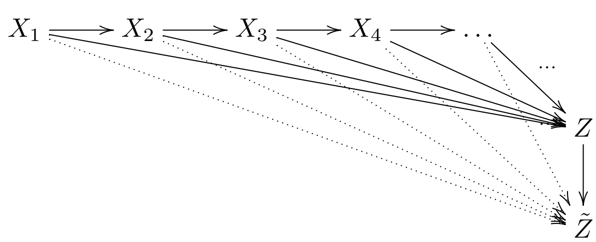


and such that for every other diagram



all of whose little triangles commute

there exists a unique morphism $Z \to \tilde{Z}$ such that all the triangles in this big diagram commute:



The colimit can be denoted colim X_i or $\varinjlim X_i$. Quite often 'colimit' means the same thing as 'union'. The dual notion is called a *limit*. It is denoted $\lim X_i$ or $\varprojlim X_i$.

An additive category is called *abelian* if every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel. By this, we mean that for every monomorphism $f : x \to y$, the canonical morphism from x to $\ker(y \to \mathrm{coker}(f))$ is an isomorphism (and similarly for the second condition, which concerns epimorphisms).

An additive functor between abelian categories is called *exact* if it sends exact sequences to exact sequences, equivalently, if it sends short exact sequences to short exact sequences. Note that the functor $- \otimes_{\mathbb{Z}} \mathbb{Z}/2$ is not exact: it sends the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0$$

to the sequence $0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{\simeq} \mathbb{Z}/2 \to 0$ which is not exact. Similarly, the functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -)$ sends the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0$$

to the sequence $0 \to 0 \to 0 \to \mathbb{Z}/2 \to 0$ which is not exact. Finally, the contravariant functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2)$ sends the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0$$

to the sequence $0 \leftarrow \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{\simeq} \mathbb{Z}/2 \leftarrow 0$ which is not exact. The functors $- \otimes_{\mathbb{Z}} \mathbb{Z}/2$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -)$ and $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2)$ are therefore not exact.

A functor F is *right exact* if for every short exact sequence $0 \to A \to B \to C \to 0$, the sequence $F(A) \to F(B) \to F(C) \to 0$ is exact. Similarly, a functor F is *left exact* if whenever $0 \to A \to B \to C \to 0$ is exact, then $0 \to F(A) \to F(B) \to F(C)$ is exact.

Lemma 1. *The functor* $\mathrm{Hom}_R(M, -)$ *is left exact.*

Lemma 2. *The functor* $- \otimes_R N$ *is right exact.*

Proof. Given a short exact sequence of right R-modules $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$, we need to show that $A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0$ is exact. The surjectivity of $B \otimes_R N \to C \otimes_R N$ is easy, so let us focus on the harder argument: given an element $\sum b_i \otimes n_i \in B \otimes_R N$ that goes to zero in $C \otimes_R N$, we need to show that it comes from $A \otimes_R N$.

Since $\sum \pi(b_i) \otimes n_i = 0$ in $A \otimes_R N$, there exist elements c'_α, c''_α, n_α, c_β, n'_β, n''_β, c_γ, r_γ, n_γ such that

$$\begin{aligned}\sum_i \pi(b_i) \otimes n_i + \sum_\alpha (c'_\alpha + c''_\alpha) \otimes n_\alpha - c'_\alpha \otimes n_\alpha - c''_\alpha \otimes n_\alpha \\ + \sum_\beta c_\beta \otimes (n'_\beta + n''_\beta) - c_\beta \otimes n'_\beta - c_\beta \otimes n''_\beta \\ + \sum_\gamma c_\gamma r_\gamma \otimes n_\gamma - c_\gamma \otimes r_\gamma n_\gamma\end{aligned}$$

is zero in the free abelian group on the set of symbols "$c \otimes n$". If we mod out that free abelian group by the first set of relations $(c' + c'') \otimes n = c' \otimes n + c'' \otimes n$, then we get the abelian group $\bigoplus_{n \in N} C$. So, another way of saying that $\sum \pi(b_i) \otimes n_i$ is zero in $A \otimes_R N$ is to say that there exist elements c_β, n'_β, n''_β, c_γ, r_γ, n_γ such that

$$\sum_i \pi(b_i) \otimes n_i + \sum_\beta c_\beta \otimes (n'_\beta + n''_\beta) - c_\beta \otimes n'_\beta - c_\beta \otimes n''_\beta + \sum_\gamma c_\gamma r_\gamma \otimes n_\gamma - c_\gamma \otimes r_\gamma n_\gamma = 0 \text{ in } \bigoplus_{n \in N} C,$$

where "$c \otimes n$" now stands for the element c put in the n-th copy of C.

Pick preimages b_β, $b_\gamma \in B$ of $c_\beta, c_\gamma \in C$, and consider the element

$$y := \sum_i b_i \otimes n_i + \sum_\beta b_\beta \otimes (n'_\beta + n''_\beta) - b_\beta \otimes n'_\beta - b_\beta \otimes n''_\beta + \sum_\gamma b_\gamma r_\gamma \otimes n_\gamma - b_\gamma \otimes r_\gamma n_\gamma \in \bigoplus_{n \in N} B.$$

This element goes to 0 in $\bigoplus_{n \in N} C$ and therefore comes from some $x \in \bigoplus_{n \in N} A$.

Let $[x]$ denote the image of x in $A \otimes_R N$ and let $[y]$ denote the image of y in $B \otimes_R N$. Since $x \mapsto y$, it follows that $[x] \mapsto [y]$. We are done since $[y] = \sum_i b_i \otimes n_i$ in $B \otimes_R N$. □

Exercise 7. Let C be a category and let $Z \in C$ be a zero object. Prove that any morphism $X \to Z$ is an epimorphism and that any morphism $Z \to X$ is an monomorphism.

Exercise 8. Let X_1 and X_2 be two objects in an additive category. Let $i_1 : X_1 \to Z$, $i_2 : X_2 \to Z$, $p_1 : Z \to X_1$, $p_2 : Z \to X_2$ be morphisms exhibiting Z as the direct sum of X_1 and X_2. Let $i'_1 : X_1 \to Z'$, $i'_2 : X_2 \to Z'$, $p'_1 : Z' \to X_1$, $p'_2 : Z' \to X_2$ be morphisms exhibiting Z' as the direct sum of X_1 and X_2. Show that there exists a unique isomorphism $f : Z \to Z'$ satisfying $f \circ i_1 = i'_1$, $f \circ i_2 = i'_2$, $p'_1 \circ f = p_1$, and $p'_2 \circ f = p_2$.

Exercise 9. Let X_i, $i \in \mathbb{N}$ be objects in an additive category. Show that $\varinjlim_n X_1 \oplus \ldots \oplus X_n \simeq \bigoplus_{i=1}^{\infty} X_i$. Show that $\varprojlim_n X_1 \oplus \ldots \oplus X_n \simeq \prod_{i=1}^{\infty} X_i$.

Exercise 10. Let R be a ring. Prove that the functor $R^2 \otimes_R - : \{R\text{-modules}\} \to \{\text{Abelian groups}\}$ is exact.

Exercise 11. Let C be the category whose objects are triples (A, B, f) where A and B are abelian groups and f a homomorphism from A to B, and whose morphisms are given by

$$Hom_C\big((A, B, f), (A', B', f')\big) := \{g : A \to A', h : B \to B' \mid hf = f'g\}.$$

Show that the functor $C \to \{\text{Abelian groups}\}$ which sends (A, B, f) to $\ker(f)$ is not exact.

Exercise 12. Let C be the category whose objects are free abelian groups and whose morphisms are group homomorphisms between free abelian groups. Show that C is not an abelian category.

Week 3

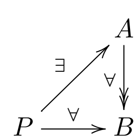
The *homology* of a chain complex of R-modules $C_\bullet = (C_n, d_n : C_n \to C_{n-1})_{n\in\mathbb{Z}}$ is defined by

$$H_n(C_\bullet) = \frac{\ker(d_n : C_n \to C_{n-1})}{\operatorname{im}(d_{n+1} : C_{n+1} \to C_n)}$$

If $C_\bullet$ is a chain complex in an arbitrary abelian category, the object $H_n(C_\bullet)$ can be defined in purely categorical terms, as the cokernel of the canonical map $C_{n+1} \to \ker(d_n : C_n \to C_{n-1})$.

Lemma: kernels are monomorphisms; cokernels are epimorphisms.

An R-module is called *projective* if it is a direct summand of a free module. An object P of an abelian category is called *projective* if for every epimorphism $A \to B$ and every morphism $P \to B$, there exists a morphism $P \to A$ such that the triangle commutes:



Let M be a right R-module and N a left R-module. Then:

$$\operatorname{Tor}_i^R(M, N) := H_i(P_\bullet \otimes_R N)$$

where $P_\bullet$ is a projective resolution of M. Implicit in the above definition is the fact that $\operatorname{Tor}_i^R(M, N)$ doesn't depend on the choice of projective resolution.

Let M and N be R-modules (either both right modules or both left modules). Then:

$$\operatorname{Ext}_R^i(M, N) := H^i(\operatorname{Hom}_R(P_\bullet, N))$$

Here, $P_\bullet$ is a projective resolution of M. Once again, the choice of resolution doesn't matter.

If $R = \mathbb{Z}$, then every module admits a resolution of length 1. This implies that $\operatorname{Tor}_i^{\mathbb{Z}}$ and $\operatorname{Ext}_{\mathbb{Z}}^i$ vanishes as soon as $i > 1$. This property is called '$\mathbb{Z}$ has cohomological dimension one'.

Exercise 13. Let M be an R-module. Prove that the functor $\mathrm{Hom}_R(-, M) : (R\text{-Mod})^{op} \to \mathrm{AbGp}$ is left exact.

Exercise 14 (exact functors)**.** Let R and S be rings, let $C := R$-Mod and $D := S$-Mod be the associated abelian categories of modules, and let $F : C \to D$ be an additive functor.

Assume that F sends short exact sequences to short exact sequences. Prove that it sends exact sequences (or any length) to exact sequences.

Exercise 15. Let k be a field, and let C be the abelian category of k-vector spaces. Let D be an arbitrary abelian category. Prove that every additive functor $C \to D$ is exact.

Exercise 16 (projective modules)**.** Let R be a ring. Prove that an R-module P is a direct summand of a free module iff for every surjective module map $p : A \to B$ and every morphism $f : P \to B$, there exists a factorisation of f through p.

Exercise 17 (this was done in class, but it all went pretty fast; the point of this exercise is to fill in the details)**.** Let $R := \mathbb{Z}[\sqrt{-5}]$. Prove that the ideal generated by 2 and $1 + \sqrt{-5}$ is a projective R-module which is not free.

Exercise 18. Let n and m be positive integers. Compute $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ and $\mathrm{Tor}^{\mathbb{Z}}_*(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$.

Week 4

A morphism of chain complexes $f_\bullet : C_\bullet \to D_\bullet$ induces a corresponding morphism at the level of cohomology groups $H_n(f_\bullet) : H_n(C_\bullet) \to H_n(D_\bullet)$.

Lemma 3. (snake lemma) *A short exact sequence of chain complexes* $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ *(which, by definition, means that for each n the sequence* $0 \to A_n \to B_n \to C_n \to 0$ *is exact) induces a long exact sequence in homology.* See p. 117 of Hatcher's book for a proof.

$\mathrm{Tor}^R_i(M, N)$ and $\mathrm{Ext}^i_R(M, N)$ are independent of the choice of resolution. They can be computed by resolving either M or N.

Let M be a right R-module and N a left R-module. Then:

$$H_i(P_\bullet \otimes_R N) \cong H_i(\mathrm{Tot}(P_\bullet \otimes_R Q_\bullet)) \cong H_i(M \otimes_R Q_\bullet)$$

where $P_\bullet$ is a projective resolution of M or $Q_\bullet$ is a projective resolution of N. The isomorphism $H_i(P_\bullet \otimes_R N) \cong H_i(\mathrm{Tot}(P_\bullet \otimes_R Q_\bullet))$ is the connecting homomorphism in the LES associated to the short exact sequence

$$0 \to P_\bullet \otimes_R N \to \mathrm{Tot}\big(P_\bullet \otimes_R Q_\bullet \to P_\bullet \otimes_R N\big) \to \mathrm{Tot}\big(P_\bullet \otimes_R Q_\bullet\big) \to 0.$$

The fact that the middle term is acyclic (the words 'acyclic' and 'exact' are synonyms) follows from the following lemma:

Lemma 4. *Let* $C_{\bullet\bullet}$ *be a double complex such that for every n there exists only finitely many pairs* (p, q) *such that* $p + q = n$ *and* $C_{p,q} \neq 0$*. Then we have*

$$\Big(C_{\bullet\bullet} \text{ has exact rows}\Big) \Rightarrow \Big(\mathrm{Tot}(C_{\bullet\bullet}) \text{ is exact}\Big)$$

Let now M and N be R-modules (either both right modules or both left modules). Then $\mathrm{Ext}^i_R(M, N)$ can be computed in any one of the following ways:

$$H^i(\mathrm{Hom}_R(P_\bullet, N)) \cong H^i(\mathrm{Tot}(\mathrm{Hom}_R(P_\bullet, I^\bullet))) \cong H^i(\mathrm{Hom}_R(M, I^\bullet)).$$

Here, $P_\bullet$ is a projective resolution of M and $I^\bullet$ is an injective resolution of N. Once again, the choice of resolution doesn't matter, neither does the choice of which of the two modules one decides to resolve.

Exercise 19. Let $a, b \leq n$. Compute $\mathrm{Ext}^*_{\mathbb{C}[x]/x^n}(\mathbb{C}[x]/x^a, \mathbb{C}[x]/x^b)$ and $\mathrm{Tor}_*^{\mathbb{C}[x]/x^n}(\mathbb{C}[x]/x^a, \mathbb{C}[x]/x^b)$.

Exercise 20. Let a, b divide n. Compute $\mathrm{Ext}^*_{\mathbb{Z}/n\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z})$ and $\mathrm{Tor}_*^{\mathbb{Z}/n\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}/b\mathbb{Z})$.

Exercise 21. Compute $\mathrm{Ext}^*_{\mathbb{C}[x,y]/(x^3,xy,y^3)}(\mathbb{C}, \mathbb{C})$ and $\mathrm{Tor}_*^{\mathbb{C}[x,y]/(x^3,xy,y^3)}(\mathbb{C}, \mathbb{C})$.

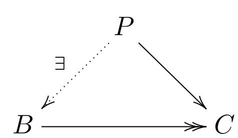
Exercise 22. Prove that $\mathbb{Z}$ has cohomological dimension one. I.e., prove that every $\mathbb{Z}$-module (not necessarily finitely generated) admits a projective resolution of length 1.

Exercise 23. Provide an example of a ring R and a module M such that there does not exist a projective resolution of M of finite length.

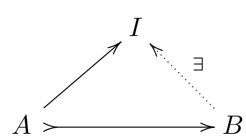
Exercise 24. Show that a short exact sequence $0 \to A \to B \to C \to 0$ has an associated invariant in $\mathrm{Ext}^1(C, A)$.

Week 5

A module P is *projective* if $\mathrm{Hom}_R(P, -)$ is exact. A module I is *injective* if $\mathrm{Hom}_R(-, I)$ is exact. A module F is *flat* if $- \otimes_R F$ is exact. Equivalently, P is projective if for every solid arrow diagram there exists a dotted arrow making the diagram commute:



In the same vein, I is injective if for every solid arrow diagram there exists a dotted arrow making the diagram commute:

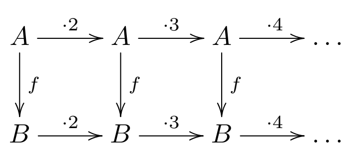


Every projective module is flat. Indeed, if $M = M' \oplus M''$, then we have (M is flat) $\Leftrightarrow$ (M' is flat and M'' is flat). Starting from the obvious fact that free modules are flat, we conclude that every projective module is flat.

$\mathbb{Q}$ is a flat $\mathbb{Z}$-module. That's because $\mathbb{Q} = \mathrm{colim}(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \xrightarrow{\cdot 5} \ldots)$ and for every abelian group A we have

$$\mathbb{Q} \otimes_{\mathbb{Z}} A = \mathrm{colim}(A \xrightarrow{\cdot 2} A \xrightarrow{\cdot 3} A \xrightarrow{\cdot 4} A \xrightarrow{\cdot 5} \ldots).$$

In order to check that $\mathbb{Q}$ is flat, one needs to check that an injective map $f : A \to B$ remains injective after applying the functor $\mathbb{Q} \otimes_{\mathbb{Z}} -$. This is a diagram chase in the diagram:



The *pullback* of a diagram of modules $A \xrightarrow{f} C \xleftarrow{g} B$ is the set $\{(a, b) \in A \oplus B : f(a) = g(b)\}$. It is also the limit of the diagram $A \to C \leftarrow B$. The *pushout* of a diagram of modules $A \xleftarrow{f} C \xrightarrow{g} B$ is the quotient $A \oplus B/\{(f(c), -g(c)) : c \in C\}$. It is also the colimit of the diagram $A \leftarrow C \to B$.

A *diagram* of R-modules indexed by a poset P is just a functor $P \to R$-Mod. Concretely, this is the data of R-modules M_α indexed by P, and maps $f_{\alpha\beta} : M_\alpha \to M_\beta$ for all $\alpha < \beta \in P$, satisfying $f_{\beta\gamma} f_{\alpha\beta} = f_{\alpha\gamma}$.

The *limit* of a a diagram $P \to R$-Mod (where P is a poset) can be described concretely as $\{(m_\alpha) \in \prod_{\alpha\in P} M_\alpha : f_{\alpha\beta}(m_\alpha) = m_\beta, \forall \alpha < \beta \in P\}$. The *colimit* of a a diagram $P \to R$-Mod is given by $\bigoplus_{\alpha\in P} M_\alpha / \mathrm{Span}\{m - f_{\alpha\beta}(m) : m \in M_\alpha\}$. Limits and colimits can alternatively be defined by means of a universal property.

A poset is called *directed* if for every $x, y \in P$, there exists $z \in P$ such that $z \geq x$ and $z \geq y$. If P is a directed poset, then every element of $\mathrm{colim}_{\alpha\in P} M_\alpha$ is represented by some element m of some M_α. Moreover, if P is a direct poset, then an element $m \in M_\alpha$ represents the zero element in $\mathrm{colim}_{\alpha\in P} M_\alpha$ iff there exists some $\beta \geq \alpha$ in P such that m becomes zero in M_β.

The latter fails miserably for e.g. pushout($\mathbb{Z}/2 \leftarrow \mathbb{Z} \to \mathbb{Z}/3$).

Exercise 25. Prove that, in the category of abelian groups, an abelian group A is *flat* if and only if it is *torsion-free*. (The argument is essentially the same as the one which I presented in class to show that $\mathbb{Q}$ is flat.)

Exercise 26. Prove that, in the category of abelian groups, if an abelian group A is *injective* then it is *divisible* (here, divisible means $\forall a \in A, \forall n \in \mathbb{N}, \exists x \in A$ s.t. $nx = a$).

For the next exercise, you may assume without proof the fact that an abelian group is injective if and only if it is divisible.

Exercise 27. Compute $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$, $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$, and $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ using injective resolutions.

Exercise 28. Compute $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ using the formula $H^*(\mathrm{Tot}(\mathrm{Hom}_R(P_\bullet, I^\bullet)))$.

Exercise 29. Compute $\mathrm{Tor}_*^{\mathbb{C}[x]/x^2}(\mathbb{C}, \mathbb{C})$ using the formula $H_*(\mathrm{Tot}(P_\bullet \otimes_R Q_\bullet))$.

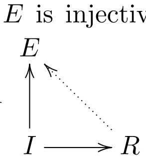
Exercise 30. Write an example of a bigraded chain complex $C_{\bullet\bullet}$ which fails the condition *"for every n there exists only finitely many pairs (p, q) such that $p + q = n$ and $C_{p,q} \neq 0$"*, and which also fails the condition

$$\Big(C_{\bullet\bullet} \text{ has exact rows}\Big) \Rightarrow \Big(\mathrm{Tot}(C_{\bullet\bullet}) \text{ is exact}\Big).$$

In other words, you must find a bigraded chain complex $C_{\bullet\bullet}$ which has has exact rows, but such that $\mathrm{Tot}(C_{\bullet\bullet})$ is not exact.

<u>**Week 6**</u>

Theorem (Baer's criterion)

An R-module E is injective if and only if every left ideal $I < R$ and any map $I \to E$, the extension problem  admits a solution.

See e.g. https://ncatlab.org/nlab/show/Baer's+criterion for a proof.

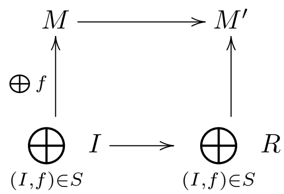
Corollary of Baer's criterion: if R is a PID, then a module M is injective iff it is *divisible*, i.e. iff for every $x \in M$ and every non-zero $r \in R$ there exists $y \in M$ such that $ry = x$.

An abelian category is said to *have enough projectives* if for every object X, there exists a projective object P and an epimorphism $P \to X$. Dually, an abelian category is said to *have enough injectives* if for every object X, there exists an injective object I and a monomorphism $X \to I$.

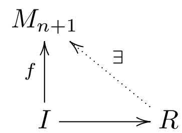
It is easy to see that for any ring R, the category of R-modules has enough projectives: take P to be free R module on the underlying set of X (any generating set would also do).

Showing the R-mod has enough injectives is much harder. Given an R-module M, let S denote the set of all pairs (I, f), where I is an ideal of R, and $f : I \to M$ is an R-module homomorphism.

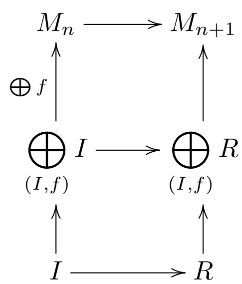
We write M' for the following pushout:



Write $M_0 := M$ and $M_{n+1} := (M_n)'$. If every ideal is finitely generated, then $M_\infty := \mathrm{colim}(M_0 \to M_1 \to M_2 \to \ldots)$ is an injective module. It obviousely contains M as a submodule. To show that M_∞ is injective, we use Baer's criterion. Using the fact that every ideal is finitely generated, every map $f : I \to M_\infty$ factors through some finite stage of the colimit, let's say $f : I \to M_n$. The extension problem will then admit a solution at the next stage:



. Here, the map $R \to M_{n+1}$ comes from



, where the bottom vertical maps $I \to \bigoplus_{(I,f)} I$ and $R \to \bigoplus_{(I,f)} R$ are the inclusions of the summands indexed by (I, f).

For general rings, i.e. without the condition that every ideal is finitely generated, then a similar construction can be made to work, provided one replaces $\mathrm{colim}_{n\in\mathbb{N}} M_n$ by a colimit indexed over all ordinals which are small than a suitably chosen cardinal. Let λ be the smallest cardinal which is bigger than the cardinality of R. For every ordinal α with $|\alpha| < \lambda$, define inductively $M_0 := M$, $M_\alpha := (M_\beta)'$ if $\alpha = \beta + 1$, and $M_\alpha := \mathrm{colim}_{\beta<\alpha} M_\beta$ if α is a limit ordinal. Then $\mathrm{colim}_{|\alpha|<\lambda} M_\alpha$ is an injective that contains M as a submodule.

Week 7

Let A and B be abelian categories. Assume that A has enough projectives. Let $F : A \to B$ be a right exact functor. The nth *left derived functor* of F, denoted $L_n F : A \to B$ is defined by $X \to H_n(F(P_\bullet))$, where $P_\bullet \to X$ is a projective resolution. Let us now assume that A admits functorial projective resolutions. The *total left derived functor* of F, denoted $LF : A \to Ch(B)$ is defined by $X \to F(P_\bullet)$. Here, $Ch(B)$ denotes the category of chain complexes in B.

Assume now that A has enough injectives and that $F : A \to B$ is a left exact functor. The nth *right derived functor* of F, denoted $R^n F : A \to B$ is defined by $X \to H^n(F(I^\bullet))$, where $X \to I^\bullet$ is an injective resolution. Let us now assume that A admits functorial injective resolutions. The *total right derived functor* of F, denoted $RF : A \to Ch(B)$ is defined by $X \to F(I^\bullet)$.

Two chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are *chain homotopic* if there exists a degree -1 map $h : C_\bullet \to D_\bullet$ satisfying $hd + dh = f - g$. The notion of chain homotopy is made so that whenever $f_\bullet : C_\bullet \to D_\bullet$ and $g_\bullet : C_\bullet \to D_\bullet$ are chain homotopic maps, then $H_*(f_\bullet) = H_*(g_\bullet) : H_*(C_\bullet) \to H_*(D_\bullet)$.

If $I_\bullet$ denotes the "interval" chain complex $I_\bullet = (\mathbb{Z}^2 \xleftarrow{\binom{1}{-1}} \mathbb{Z})$, then a chain homotopy between two maps $C_\bullet \to D_\bullet$ is the same thing as a map $\mathrm{tot}(C_\bullet \otimes I_\bullet) \to D_\bullet$. This is reminiscent to the situation in topology, where a homotopy between two continuous maps $X \to Y$ is defined to be a map $X \times [0, 1] \to Y$.

Exercise 31. Prove that $\{(m_\alpha) \in \prod_{\alpha\in P} M_\alpha : f_{\alpha\beta}(m_\alpha) = m_\beta, \forall \alpha < \beta \in P\}$ satisfies the universal property of a limit, and that $\bigoplus_{\alpha\in P} M_\alpha / \mathrm{Span}\{m - f_{\alpha\beta}(m) : m \in M_\alpha\}$ satisfies the universal property of a colimit.

Exercise 32. Consider the following two proofs (you may take $P = \mathbb{N}$ if you want):

Lemma A: Let $\{M_\alpha\}_{\alpha\in P}$ be a diagram of R-modules indexed by some directed poset P. Then we have a canonical isomorphism $$\mathrm{Tor}_n(\mathrm{colim}\, M_\alpha, N) = \mathrm{colim}\, \mathrm{Tor}_n(M_\alpha, N).$$ *Proof.* Let $F_\bullet^\alpha$ be the canonical free resolution of M_α. Then $F_\bullet := \mathrm{colim}_{\alpha\in P} F_\bullet^\alpha$ is the canonical free resolution of $M := \mathrm{colim}_{\alpha\in P} M_\alpha$. We then have $$\begin{aligned}&\mathrm{colim}_{\alpha\in P}\, \mathrm{Tor}_n^R(M_\alpha, N)\\ &= \mathrm{colim}_{\alpha\in P}\, H_n(F_\bullet^\alpha \otimes_R N)\\ &= H_n(\mathrm{colim}_{\alpha\in P}(F_\bullet^\alpha \otimes_R N))\\ &= H_n((\mathrm{colim}_{\alpha\in P}\, F_\bullet^\alpha) \otimes_R N)\\ &= H_n(F_\bullet \otimes_R N) = \mathrm{Tor}_n^R(M, N).\end{aligned}$$ □	**Lemma B:** Let $\{M_\alpha\}_{\alpha\in P}$ be a diagram of R-modules indexed by (the opposite of) a directed poset. Then we have a canonical isomorphism $$\mathrm{Ext}^n(\mathrm{colim}\, M_\alpha, N) = \lim \mathrm{Ext}^n(M_\alpha, N).$$ *Proof.* Let $F_\bullet^\alpha$ be the canonical free resolution of M_α. Then $F_\bullet := \mathrm{colim}_{\alpha\in P} F_\bullet^\alpha$ is the canonical free resolution of $M := \mathrm{colim}_{\alpha\in P} M_\alpha$. We then have $$\begin{aligned}&\lim_{\alpha\in P}\, \mathrm{Ext}_n^R(M_\alpha, N)\\ &= \lim_{\alpha\in P}\, H^n(\mathrm{Hom}_R(F_\bullet^\alpha, N))\\ &= H^n(\lim_{\alpha\in P}(\mathrm{Hom}_R(F_\bullet^\alpha, N)))\\ &= H^n(\mathrm{Hom}_R(\mathrm{colim}_{\alpha\in P} F_\bullet^\alpha, N))\\ &= H^n(\mathrm{Hom}_R(F_\bullet, N)) = \mathrm{Ext}_R^n(M, N)\end{aligned}$$ □

Lemma A is correct and its proof is correct, but Lemma B is wrong and its proof is flawed. Find the error in the proof of Lemma B, and illustrate the mistake by providing a counterexample. Explain (by filling the details of the proof) why the corresponding step in the proof of Lemma A is ok.

Exercise 33. Prove that a $\mathbb{Z}$-module is injective iff it is a (possibly infinite) direct sum of modules of the form $\mathbb{Q}$ and $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$.

Exercise 34. Let K be an algebraically closed field. Prove that a $K[x]$-module is injective iff it is a (possibly infinite) direct sum of modules of the form $K[x]$ and $K[y, y^{-1}]/K[y]$ for $y = x - a$ and $a \in K$.

Exercise 35. Let P be a directed poset. Prove that the category whose objects are functors $P \to \mathrm{AbGp}$ and whose morphisms are natural transformations between such functors has enough projectives.

Exercise 36. Prove that the relation of chain homotopy is an equivalence relation.

<u>Week 8</u>

The total (co)chain complex of a bigraded (co)chain complex comes in two flavours: $\mathrm{tot}_\prod$ and $\mathrm{tot}_\oplus$.

If $A \leftarrow P_\bullet$ and $B \leftarrow Q_\bullet$ are projective resolutions, then the cochain complex $\mathrm{tot}_\prod\Big(\mathrm{Hom}(P_\bullet, Q_\bullet)\Big)$ computes $\mathrm{Ext}(A, B)$.

Using this fact, the composition of homomorphisms $\mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ induces a well-defined map $\mathrm{Ext}^i(A, B) \otimes \mathrm{Ext}^j(B, C) \to \mathrm{Ext}^{i+j}(A, C)$. In particular,

$$\mathrm{Ext}^*(A, A) := \bigoplus_{i=0}^{\infty} \mathrm{Ext}^i(A, A)$$

is a ring.

Writing $\underline{\mathrm{Hom}}(C_\bullet, D_\bullet) := \mathrm{tot}_\prod\big(\mathrm{Hom}(C_\bullet, D_\bullet)\big)$, we have a canonical isomorhpism

$$H^n\Big(\underline{\mathrm{Hom}}(C_\bullet, D_\bullet)\Big) = \frac{\text{degree } (-n) \text{ chain maps } C_\bullet \to D_\bullet}{\text{maps which are chain-homotopic to zero}}$$

We performed the following Ext-ring computations in class:

- $\text{Ext}_{k[x]}(k,k) = k[y]/y^2$, with y in degree 1.
- $\text{Ext}_{k[x]/(x^2)}(k,k) = k[y]$, with y in degree 1.
- $\text{Ext}_{k[x]/(x^3)}(k,k) = k[y,z]/(y^2)$, with y in degree 1 and z in degree 2.
- $\text{Ext}_{k[x,y]}(k,k) = k\langle y,z\rangle/y^2, z^2, yz = -zy$, with both y and z in degree 1.

Exercise 37. Compute the Ext-ring $\text{Ext}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2)$.

Exercise 38. Compute the Ext-ring $\text{Ext}_{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2)$.

Exercise 39. Compute the Ext-ring $\text{Ext}_{\mathbb{Z}/8}(\mathbb{Z}/2, \mathbb{Z}/2)$.

Exercise 40. Compute the Ext-ring $\text{Ext}_{\mathbb{Z}[x]}(\mathbb{Z}/2, \mathbb{Z}/2)$.

Exercise 41. Compute the Ext-ring $\text{Ext}_{\mathbb{Z}[x]}(\mathbb{Z}/3, \mathbb{Z}/3)$.

Exercise 42. Compute the Ext-ring $\text{Ext}_{\mathbb{Z}[x]/(x^2)}(\mathbb{Z}/2, \mathbb{Z}/2)$.