2.) Prove that for each $0<\epsilon<1$, the convergence of the series

$$\sum_{K=1}^{\infty} \frac{(-1)^K}{K}(X-1)^K$$

is uniform absolute on $[\epsilon, 2-\epsilon]$.

Pf

By definition, we need to show $\sum_{K=1}^{\infty} \sup_{X\in[\epsilon,2-\epsilon]} \left|\frac{(-1)^K}{K}(X-1)^K\right| < \infty$.

The left-hand side is equal to

$$\sum_{K=1}^{\infty} \frac{1}{K}(1-\epsilon)^K < \sum_{K=1}^{\infty} (1-\epsilon)^K < \infty \quad \text{since } |1-\epsilon|<1.$$

3.) Prove that the convergence of the same series on $(0,2)$:

a) is absolute

b) is not uniform

a) Pf $\sum_{K=1}^{\infty} \left|\frac{(-1)^K}{K}(x-1)^K\right| \leq \sum_{K=1}^{\infty} |X-1|^K < \infty$ for $X\in(0,2)$.

b) By 1.1, $\left(\sum_{K=1}^{n} \frac{(-1)^K(x-1)^K}{K}\right)_n$ converges pointwise to $\log(x)$.

We must show that this convergence is not uniform on $(0,2)$.

By definition, we need to show

$$\lim_{n\to\infty} \sup_{X\in(0,2)} \left|\log(x) - \sum_{K=1}^{n} \frac{(-1)^K}{K}(x-1)^K\right| \neq 0.$$

The left-hand side is $\lim_{n\to\infty} \sup_{X\in(0,2)} \left|\sum_{K=n+1}^{\infty} \frac{(-1)^K}{K}(X-1)^K\right| \geq \lim_{n\to\infty} \left|\sum_{K=n+1}^{\infty} \frac{1}{K}\right| = \infty,$

where the inequality comes from looking at $X=0$.

4. Show that the convergence of the same series on $[\epsilon, 2]$, $\epsilon > 0$ is

a) uniform

b) not absolute.

a) Pf: By 3. b) we must show $\lim_{n\to\infty} \sup_{x\in[\epsilon,2]} \left|\sum_{k=n+1}^{\infty} \frac{(-1)^k}{k}(x-1)^k\right| = 0.$

This supremum is achieved at $x=\epsilon$ and is equal to zero since the series converges pointwise at $x=\epsilon$.

b) The convergence is not absolute since at $x=2$,

$$\sum_{k=1}^{\infty} \left|\frac{(-1)^k}{k}(2-1)^k\right| = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

5. Let $\epsilon > 0$. Show that the convergence of the series on $[\epsilon, 2)$ is

a) uniform

b) absolute

c) not uniformly absolute.

Pf a) follows from 4a)

b) follows from 3 a)

For c): $$\sum_{k=1}^{\infty} \sup_{x\in[\epsilon,2)} \left|\frac{(-1)^k}{k}(x-1)^k\right| = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$